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ON CHAOTIC ORDER OF TWO OPERATORS ON HILBERT SPACE

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Dedicated to
Professor J. Winfield Poole
on his retirement

(*communicated by T. Furuta*)

Abstract. In this article we characterize an inequality resulted from the chaotic order of two operators on a Hilbert space in terms of Heinz-Kato-Furuta-type inequalities. Consequently, some new characterizations of the weaker version for the Löwner-Heinz inequality are obtained. Finally we introduce the notion of high-order Cauchy-Schwarz inequalities, and its applications are given.

1. Notations and introduction

In what follows the capital letters mean bounded linear operators on a Hilbert space H. By a positive operator T (written $T \geqslant O$) we mean $(Tx, x) \geqslant 0$ for all $x \in H$. If S and T are selfadjoint, we write $T \geqslant S$ in case $T - S \geqslant O$. $T = U|T|$ is the polar decomposition of T with U the partial isometry, and $|T|$ the positive square root of the positive operator T^*T. We write $T > O$ if T is invertible and $T \geqslant O$. For $A, B > O$ we denote as usual the chaotic order $\log\ A \geqslant \log\ B$ of A and B by $A \gg B$ in short, which is clearly weaker than the condition that $A \geqslant B$. It is known that Ando's characterization of the chaotic order [1] was generalized in [4], and a different proof without using Ando's result was shown in [7]. In this article we present characterizations of the chaotic order of A and B in terms of Heinz-Kato-Furuta-type inequalities. Consequently, we obtain new characterizations of the Löwner-Heinz inequality under a weaker condition. Related log-hyponormal operators and p-hyponormal operators are discussed. Finally, we introduce the notion of high-order Cauchy-Schwarz inequalities, and its applications are given.

2. Basic results

We shall start with three lemmas below as basic tools.

LEMMA 1. *Let $T = U|T|$ be the polar decomposition of T and $r > 0$. Then*

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(i) $U^*U|T|^r = |T|^r$.
(ii) $|T^*|^r = U|T|^rU^*$.

Proof. The proof can be found in [2] by a straightforword verification.

LEMMA 2. *For any $x,y,z \in H$ the following statements are equivalent:*
(1) $|(x,y)| \leqslant \| x \| \| y \|$ *(Cauchy-Schwarz inequality);*
(2) $2|(z,x)(x,y)| \leqslant \| x \|^2 [\| y \| \| z \| + |(z,y)|]$.

Proof. $(1) \Rightarrow (2)$. In (1) replace x by $\nabla = 2(z,x)x - \| x \|^2 z$ and notice that $\| \nabla \| = \| x \|^2 \| z \|$, since

$$\begin{aligned}\| \nabla \|^2 &= (2(z,x)x - \| x \|^2 z, 2(z,x)x - \| x \|^2 z) \\ &= 4|(z,x)|^2 \| x \|^2 - 2|(z,x)|^2 \| x \|^2 - 2|(z,x)|^2 \| x \|^2 + \| x \|^4 \| z \|^2 \\ &= \| x \|^4 \| z \|^2 .\end{aligned}$$

Hence, from (1) we have

$$\begin{aligned}|(2(z,x)x - \| x \|^2 z, y)| &= |(2(z,x)(x,y) - \| x \|^2 (z,y)| \\ &\leqslant \| x \|^2 \| y \| \| z \|,\end{aligned}$$

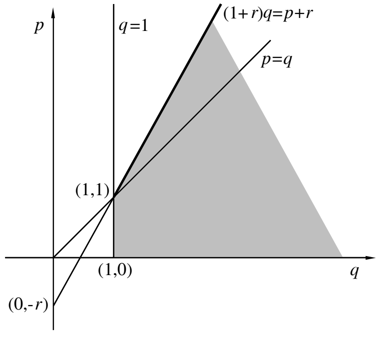
and (2) follows.
$(2) \Rightarrow (1)$. Just let $z = x$ in (2).

LEMMA 3. *For $A,B > O$, $A \gg B$ if and only if $B^{-1} \gg A^{-1}$.*

Proof. $(\Rightarrow)$ First we consider the case $\log A > \log B$. Equivalently, there exists $\alpha \in (0,1]$ such that $A^\alpha > B^\alpha$ by [7, Corollary 2]. So, $B^{-\alpha} > A^{-\alpha}$. It follows that $B^{-\alpha} > A^{-\alpha} + \delta$ for some $\delta > 0$. Therefore,

$$\alpha \log B^{-1} = \log B^{-\alpha} \geqslant \log(A^{-\alpha} + \delta) > \log A^{-\alpha} = \alpha \log A^{-1}.$$

Hence, $\log B^{-1} > \log A^{-1}$. The case $\log A = \log B$ is now trivial.
$(\Leftarrow)$ By similar arguments, and the proof is finished.



Recall the Furuta inequality [3]: If $A \geqslant B \geqslant O$, then for each $r \geqslant 0$,
(i) $A^{\frac{p+r}{q}} \geqslant (A^{r/2}B^pA^{r/2})^{\frac{1}{q}}$

and

(ii) $(B^{r/2}A^pB^{r/2})^{\frac{1}{q}} \geqslant B^{\frac{p+r}{q}}$

hold for $p \geqslant 0$ and $q \geqslant 1$ with $(1+r)q \geqslant p+r$.

In fact, without loss of generality we may assume that $A, B > O$ with $A \geqslant B > O$ in the Furuta inequality. Also, assuming that $p \geqslant 1$ is sufficient, since (i) and (ii) hold for $1 \geqslant p \geqslant 0$ by the Löwner-Heinz inequality, namely, if $A \geqslant B \geqslant O$, then $A^\alpha \geqslant B^\alpha$ for $\alpha \in [0,1]$; but the inequality does not hold in general for $\alpha > 1$. Yet, a weaker condition (use $A \gg B$, instead of $A \geqslant B \geqslant O$) for the Furuta inequality is possible as shown next.

Recall a result with weaker condition due to Fujii et al. [7, Theorem 5]: For $A, B > O$, $A \gg B$ if and only if (i) in the Furuta inequality holds for $p \geqslant 0$, $q \geqslant 1$ and $r \geqslant 0$ such that $rq \geqslant p+r$. It is essential to distingush conditions between the two results above; one is $(1+r)q \geqslant p+r$ in the original Furuta inequality and the other is $rq \geqslant p+r$. Correspondingly, for (ii) of the Furuta inequality we have the next result.

COROLLARY 1. *For $A, B > O$, $A \gg B$ if and only if (ii) in the Furuta inequality holds for $p \geqslant 0$, $q \geqslant 1$ and $r \geqslant 0$ such that $rq \geqslant p+r$.*

Proof. By hypothesis and Lemma 3 we have $B^{-1} \gg A^{-1}$. It follows from above remark and (i) that

$$B^{-\frac{p+r}{q}} \geqslant (B^{-r/2}A^{-p}B^{-r/2})^{\frac{1}{q}}.$$

The required inequality follows by taking inverses of the inequality above. The converse is trivial now.

COROLLARY 2. *For $A, B > O$, if $A \gg B$, then*

$$(B^{r/2}A^pB^{r/2})^{\frac{\alpha r}{p+r}} \geqslant B^{\alpha r}$$

holds for $r > 0$, $p \geqslant 0$ and $\alpha \in [0,1]$.

Proof. If $A \gg B$, then by Corollary 1 the inequality $(B^{r/2}A^pB^{r/2})^{\frac{1}{q}} \geqslant B^{\frac{p+r}{q}}$ holds for $p \geqslant 0$, $q \geqslant 1$ and $r \geqslant 0$ such that $rq \geqslant p+r$. Let $r > 0$, $\alpha \in (0,1]$ and $q = \frac{p+r}{\alpha r}$. Then $q \geqslant 1$ and $rq = \frac{p+r}{\alpha} \geqslant p+r > 0$. It follows that the inequality $(B^{r/2}A^pB^{r/2})^{\frac{\alpha r}{p+r}} \geqslant B^{\alpha r}$ holds for $r > 0$, $p \geqslant 0$ and $\alpha \in (0,1]$. But if $\alpha = 0$, our inequality becomes a trivial case $I \geqslant I$, the identity operator. So, we conclude that $\alpha \in [0,1]$.

3. Characterizations of inequality resulted from chaotic order

The next result is characterizations of the inequality from Corollary 2 in terms of Heinz-Kato-Furuta-type inequalities [5], i.e., inequalities of types (2), (3) and (4) in Theorem 1 below.

THEOREM 1. *Assume $A, B > O$, then the following statements hold and they are equivalent:*

(1) *If* $\log\ A \geqslant \log\ B$, *then*

$$(B^{r/2}A^pB^{r/2})^{\frac{\alpha r}{p+r}} \geqslant B^{\alpha r}$$

for $r > 0,\ p \geqslant 0$ *and* $\alpha \in [0,1]$;

(2) *For an invertible operator* T *such that* $\log\ A^2 \geqslant \log\ |T|^2$ *and* $\log\ B^2 \geqslant \log|T^*|^2$,

$$|(T|T|^{\alpha r+\beta s-1}x,y)|^2 \leqslant ((|T|^rA^{2p}|T|^r)^{\frac{\alpha r}{p+r}}x,x)((|T^*|^sB^{2q}|T^*|^s)^{\frac{\beta s}{q+s}}y,y)$$

for all $x,y \in H,\ p,q,r,s > 0$ *and* $\alpha,\beta \in [0,1]$ *with* $\alpha r + \beta s \geqslant 1$;

(3) *For an invertible operator* T *such that* $\log\ A^2 \geqslant \log\ |T|^2$ *and* $\log\ B^2 \geqslant \log|T^*|^2$,

$$\begin{aligned}&|(|T|^{2\alpha r}z,x)(T|T|^{\alpha r+\beta s-1}x,y)|\\ &\quad\leqslant ((|T|^rA^{2p}|T|^r)^{\frac{\alpha r}{p+r}}x,x)((|T^*|^sB^{2q}|T^*|^s)^{\frac{\beta s}{q+s}}y,y)^{1/2}\cdot((|T|^rA^{2p}|T|^r)^{\frac{\alpha r}{p+r}}z,z)^{1/2}\end{aligned}$$

for all $x,y,z \in H,\ p,q,r,s > 0$ *and* $\alpha,\beta \in [0,1]$ *with* $\alpha r + \beta s \geqslant 1$;

(4) *For an invertible operator* T *such that* $\log\ A^2 \geqslant \log\ |T|^2$ *and* $\log\ B^2 \geqslant \log|T^*|^2$,

$$\begin{aligned}&|(|T|^{2\alpha r}x,x)(T|T|^{\alpha r+\beta s-1}x,y)|\\ &\quad\leqslant ((|T|^rA^{2p}|T|^r)^{\frac{\alpha r}{p+r}}x,x)^{3/2}((|T^*|^sB^{2q}|T^*|^s)^{\frac{\beta s}{q+s}}y,y)^{1/2}\end{aligned}$$

for all $x,y \in H,\ p,q,r,s > 0$ *and* $\alpha,\beta \in [0,1]$ *with* $\alpha r + \beta s \geqslant 1$.

Proof. Note that (1) holds true due to Corollary 2. We shall use the following substitutions: In Lemma 2 replace x by $U|T|^{\alpha r}x$, y by $|T^*|^{\beta s}y$, and z by $U|T|^{\alpha r}z$, and then use Lemma 1 to simplify relations. More precisely,

$$|(U|T|^{\alpha r}x,|T^*|^{\beta s}y)| = |(U|T|^{\alpha r+\beta s}x,y)| = |(T|T|^{\alpha r+\beta s-1}x,y)|.$$

$(1) \Rightarrow (2)$. Since, by (1) in Lemma 2 and equalities above,

$$\begin{aligned}|(T|T|^{\alpha r+\beta s-1}x,y)|^2 &= |(U|T|^{\alpha r}x,|T^*|^{\beta s}y)|^2\\ &\leqslant \| \,|T|^{\alpha r}x\,\|^2 \| \,|T^*|^{\beta s}y\,\|^2\\ &\leqslant ((|T|^rA^{2p}|T|^r)^{\frac{\alpha r}{p+r}}x,x)((|T^*|^sB^{2q}|T^*|^s)^{\frac{\beta s}{q+s}}y,y).\end{aligned}$$

The last inequality is due to (1).

$(2) \Rightarrow (1)$. In (2) put $T = B,\ \beta = \alpha,\ s = r,\ q = p$ and $y = x$. It follows that

$$(B^{2\alpha r}x,x) \leqslant ((B^rA^{2p}B^r)^{\frac{\alpha r}{p+r}}x,x)$$

holds for $A^2 \gg B^2$, and so we have (1) if $A \gg B$.

$(1) \Rightarrow (3)$. Since, by (2) in Lemma 2,

$$
\begin{aligned}
&2|(|T|^{2\alpha r}z,x)(T|T|^{\alpha r+\beta s-1}x,y)| \\
&\quad \leqslant \| \, |T|^{\alpha r}x \, \|^2 \, [\| \, |T^*|^{\beta s}y \, \| \| \, |T|^{\alpha r}z \, \| + |(T|T|^{\alpha r+\beta s-1}z,y)|] \\
&\quad \leqslant ((|T|^r A^{2p}|T|^r)^{\frac{\alpha r}{p+r}}x,x) \cdot [((|T^*|^s B^{2q}|T^*|^s)^{\frac{\beta s}{q+s}}y,y)^{1/2}((|T|^r A^{2p}|T|^r)^{\frac{\alpha r}{p+r}}z,z)^{1/2} \\
&\qquad + ((|T|^r A^{2p}|T|^r)^{\frac{\alpha r}{p+r}}z,z)^{1/2}((|T^*|^s B^{2q}|T^*|^s)^{\frac{\beta s}{q+s}}y,y)^{1/2}] \\
&\quad = 2((|T|^r A^{2p}|T|^r)^{\frac{\alpha r}{p+r}}x,x)((|T^*|^s B^{2q}|T^*|^s)^{\frac{\beta s}{q+s}}y,y)^{1/2} \cdot ((|T|^r A^{2p}|T|^r)^{\frac{\alpha r}{p+r}}z,z)^{1/2}.
\end{aligned}
$$

$(3) \Rightarrow (4)$. Let $z = x$ in (3).

$(4) \Rightarrow (1)$. The proof is the same as that $(2) \Rightarrow (1)$, i.e., let $T = B$, $\beta = \alpha$, $s = r$, $q = p$, and $y = x$ in (4).

Remark that (2) in Theorem 1 has been already given in [6]. In the proof of Theorem 1 we may use different substitutions and get the same result. For example, replacing x by $|T|^{\alpha r}x$, y by $U^*|T^*|^{\beta s}y$, and z by $|T|^{\alpha r}z$ in Lemma 2 and use Lemma 1 to simplify relations. Also remark that when $A \geqslant B \geqslant O$ we can talk about the Löwner-Heinz inequality $A^\alpha \geqslant B^\alpha$ for $\alpha \in [0,1]$. Indeed, the inequality is obtained from (ii) in the Furuta inequality by letting $r = 0$ there. Correspondingly, when $A \gg B$ for $A, B > O$, we don't get exactly the same inequality but we have $(B^{1/2}A^pB^{1/2})^{\frac{\alpha}{p+1}} \geqslant B^\alpha$ for $p \geqslant 0$ and $\alpha \in [0,1]$. This is obtained by setting $r = 1$ in Corollary 2. In fact, we have the following characterizations of this inequality.

COROLLARY 3. *Assume $A, B > O$, then the following statements hold and they are equivalent:*

(1) *If* $\log\ A \geqslant \log\ B$, *then*

$$(B^{1/2}A^pB^{1/2})^{\frac{\alpha}{p+1}} \geqslant B^\alpha$$

holds for $p \geqslant 0$ and $\alpha \in [0,1]$;

(2) *For an invertible operator T such that* $\log\ A^2 \geqslant \log\ |T|^2$ *and* $\log\ B^2 \geqslant \log |T^*|^2$,

$$|(T|T|^{\alpha+\beta-1}x,y)|^2 \leqslant ((|T|A^{2p}|T|)^{\frac{\alpha}{p+1}}x,x)((|T^*|B^{2q}|T^*|)^{\frac{\beta}{q+1}}y,y)$$

for all $x, y \in H$, $p, q \geqslant 0$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta \geqslant 1$;

(3) *For an invertible operator T such that* $\log\ A^2 \geqslant \log\ |T|^2$ *and* $\log\ B^2 \geqslant \log |T^*|^2$,

$$
\begin{aligned}
&|(|T|^{2\alpha}z,x)(T|T|^{\alpha+\beta-1}x,y)| \\
&\qquad \leqslant ((|T|A^{2p}|T|)^{\frac{\alpha}{p+1}}x,x)((|T^*|B^{2q}|T^*|)^{\frac{\beta}{q+1}}y,y)^{1/2} \cdot ((|T|A^{2p}|T|)^{\frac{\alpha}{p+1}}z,z)^{1/2}
\end{aligned}
$$

for all $x, y, z \in H$, $p, q \geqslant 0$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta \geqslant 1$;

(4) *For an invertible operator* T *such that* $\log\ A^2 \geqslant \log\ |T|^2$ *and* $\log\ B^2 \geqslant \log|T^*|^2$,

$$|(|T|^{2\alpha}x,x)(T|T|^{\alpha+\beta-1}x,y)| \leqslant ((|T|A^{2p}|T|)^{\frac{\alpha}{p+1}}x,x)^{3/2}((|T^*|B^{2q}|T^*|)^{\frac{\beta}{q+1}}y,y)^{1/2}$$

for all $x,y \in H,\ p,q \geqslant 0,$ *and* $\alpha,\beta \in [0,1]$ *with* $\alpha+\beta \geqslant 1$;

(5) *For an invertible operator* T *such that* $\log\ A^2 \geqslant \log\ |T|^2$ *and* $\log\ B^2 \geqslant \log|T^*|^2$,

$$|(Tx,y)|^2 \leqslant ((|T|A^{2p}|T|)^{\frac{\alpha}{p+1}}x,x)((|T^*|B^{2q}|T^*|)^{\frac{1-\alpha}{q+1}}y,y)$$

for all $x,y \in H,\ p,q \geqslant 0,$ *and* $\alpha \in [0,1]$;

(6) *For an invertible operator* T *such that* $\log\ A^2 \geqslant \log\ |T|^2$ *and* $\log\ B^2 \geqslant \log|T^*|^2$,

$$|(|T|^{2\alpha}z,x)(Tx,y)| \leqslant ((|T|A^{2p}|T|)^{\frac{\alpha}{p+1}}x,x)((|T|A^{2p}|T|)^{\frac{\alpha}{p+1}}z,z)^{1/2}\cdot((|T^*|B^{2q}|T^*|)^{\frac{1-\alpha}{q+1}}y,y)^{1/2}$$

for all $x,y,z \in H,\ p,q \geqslant 0,$ *and* $\alpha \in [0,1]$;

(7) *For an invertible operator* T *such that* $\log\ A^2 \geqslant \log\ |T|^2$ *and* $\log\ B^2 \geqslant \log|T^*|^2$,

$$|(|T|^{2\alpha}x,x)(Tx,y)| \leqslant ((|T|A^{2p}|T|)^{\frac{\alpha}{p+1}}x,x)^{3/2}((|T^*|B^{2q}|T^*|)^{\frac{1-\alpha}{q+1}}y,y)^{1/2}$$

for all $x,y \in H,\ p,q \geqslant 0,$ *and* $\alpha \in [0,1]$.

Proof. Let $s=r=1$ in Theorem 1. Then we have equivalence of statements (1), (2), (3) and (4). Moreover, if we put $s=r=1$ and $\alpha+\beta=1$ in Theorem 1, then equivalence of statements (1), (5), (6) and (7) follows, since (7) ⇒ (1) holds by putting $T=B,\ q=p$ and $y=B^{2\alpha-1}x$.

Recall that T is a log-hyponormal operator if T is invertible and $\log\ T^*T \geqslant \log TT^*$, and denoted as usual by $T^*T \gg TT^*$ in short. Thus, T is log-hyponormal if and only if $|T|^2 \gg |T^*|^2$. Let $A=B=|T|$ in (2), (3) and (4) of Theorem 1, so that T is log-hyponormal. This leads to the next result and we shall omit the proof.

COROLLARY 4. *Let* T *be a log-hyponormal operator. Then the following inequalities hold, and are equivalent to each other:*

(1) $|(T|T|^{\alpha r+\beta s-1}x,y)| \leqslant \| \,|T|^{\alpha r}x\, \| \,((|T^*|^s|T|^{2q}|T^*|^s)^{\frac{\beta s}{q+s}}y,y)^{1/2}$ *for all* $x,y \in H,\ q \geqslant 0,\ r,s > 0,$ *and* $\alpha,\beta \in [0,1]$ *with* $\alpha r+\beta s \geqslant 1$;

(2) $|(|T|^{2\alpha r}z,x)(T|T|^{\alpha r+\beta s-1}x,y)| \leqslant \||T|^{\alpha r}x\|^2\||T|^{\alpha r}z\|((|T^*|^s|T|^{2q}|T^*|^s)^{\frac{\beta s}{q+s}}y,y)^{1/2}$ *for all* $x,y,z \in H,\ q \geqslant 0,\ r,s > 0,$ *and* $\alpha,\beta \in [0,1]$ *with* $\alpha r+\beta s \geqslant 1$.

Recall that T is a p-hyponormal operator if $(T^*T)^p \geqslant (TT^*)^p$ for $p>0$. Evidently a hyponormal operator is 1-hyponormal, and a hyponormal operator is p-hyponormal for $1 \geqslant p > 0$ by the Löwner-Heinz inequality. Obviously, it is superfluous to consider a log-p-hyponormal operator, since a log-p-hyponormal operator is a log-hyponormal operator and vice versa. Now, let $s=r=1$ and $\alpha,\beta \in [0,1]$ with $\alpha+\beta=1$ in Corollary 4. Then we have the next result and we shall omit the proof.

COROLLARY 5. *Let T be a log-hyponormal operator. Then the following inequalities hold, and are equivalent to each other:*

(1) $|(Tx,y)| \leqslant \| |T|^{\alpha} x \| \, ((|T^*||T|^{2q}|T^*|)^{\frac{1-\alpha}{q+1}} y, y)^{1/2}$
for all $x, y \in H$, $q \geqslant 0$ *and* $\alpha \in [0,1]$;

(2) $|(|T|^{2\alpha} z, x)(Tx,y)| \leqslant \| |T|^{\alpha} x \|^2 \| |T|^{\alpha} z \| \, ((|T^*||T|^{2q}|T^*|)^{\frac{1-\alpha}{q+1}} y, y)^{1/2}$
for all $x, y, z \in H$, $q \geqslant 0$ *and* $\alpha \in [0,1]$.

4. Further characterizations

In this section we shall present further characterizations of results obtained from the previous section. We require Lemma 4 below which is known in [10, Lemma] and the proof should be omitted. First recall that the Selberg inequality is

$$\sum_{i=1}^{n} \frac{|(x,x_i)|^2}{\sum_{j=1}^{n} |(x_i,x_j)|} \leqslant \| x \|^2$$

for $x \in H$ and nonzero vectors $\{x_i\}_{i=1}^n \subseteq H$, which is a generalization of the Bessel's inequality: $\sum_{i=1}^{n} |(x,x_i)|^2 \leqslant \| x \|^2$ for $x \in H$ with an orthonormal set $\{x_i\}_{i=1}^n \subseteq H$. Notice that (4) in Lemma 4 below is known as a refined form of the Selberg inequality [8, Lemma 1]. Nevertheless, we shall give a different type of generalized Selberg inequality in the last section. Remark that the original definition of the set $\{u_i\}_{i=1}^n$ in (3) of Lemma 4 below can be found in [9].

LEMMA 4. *For any $x, y, z \in H$ the following are equivalent:*
(1) $|(x,y)| \leqslant \| x \| \| y \|$ *(Cauchy-Schwarz inequality);*
(2) *If* $(z,y) = 0$, *then*

$$\| z \|^2 \, |(x,y)|^2 \leqslant \| y \|^2 \, [\| z \|^2 \| x \|^2 - |(x,z)|^2];$$

(3) *Let $\{e_i\}_{i=1}^n$ be a set of unit vectors and let the set $\{u_i\}_{i=1}^n \subseteq H$ be defined as follows: $u_0 = x$, and $u_i = u_{i-1} - (u_{i-1}, e_i)e_i$, $i = 1, 2, \ldots, n$. If $(e_i, y) = 0$ for all i, then*

$$|(x,y)|^2 + \| y \|^2 \sum_{i=1}^{n} |(u_{i-1}, e_i)|^2 \leqslant \| x \|^2 \| y \|^2;$$

(4) *Given nonzero vectors $\{x_i\}_{i=1}^n$, if $(x_i, y) = 0$ for all i, then*

$$|(x,y)|^2 + \sum_{i=1}^{n} \frac{|(x,x_i)|^2 \, \| y \|^2}{\sum_{j=1}^{n} |(x_i,x_j)|} \leqslant \| x \|^2 \| y \|^2 \, .$$

COROLLARY 6. *For any T, the following are equivalent and each statement holds true:*

(1) $|(T|T|^{\alpha r+\beta s-1}x,y)| \leqslant \| \, |T|^{\alpha r}x \, \| \| \, |T^*|^{\beta s}y \, \|$

for all $x,y \in H$, $r,s > 0$, and $\alpha,\beta \in [0,1]$ with $\alpha r+\beta s \geqslant 1$.

(2) $|(T|T|^{\alpha r+\beta s-1}x,y)|^2 + \frac{\| |T^*|^{\beta s}y\|^2 |(|T|^{2\alpha r}x,z)|^2}{\| |T|^{\alpha r}z\|^2} \leqslant \| \, |T|^{\alpha r}x \, \|^2 \| \, |T^*|^{\beta s}y \, \|^2$

for all $x,y,z \in H$, $r,s > 0$, and $\alpha,\beta \in [0,1]$ with $\alpha r+\beta s \geqslant 1$. Also for $(T|T|^{\alpha r+\beta s-1}z,y) = 0$ and $|T|^{\alpha r}z \neq 0$;

(3) $|(T|T|^{\alpha r+\beta s-1}x,y)|^2 + \frac{\| |T|^{\alpha r}x\|^2 |(T|T|^{\alpha r+\beta s-1}z,y)|^2}{\| |T|^{\alpha r}z\|^2} \leqslant \| \, |T|^{\alpha r}x \, \|^2 \| \, |T^*|^{\beta s}y \, \|^2$

for all $x,y,z \in H$, $r,s > 0$, and $\alpha,\beta \in [0,1]$ with $\alpha r+\beta s \geqslant 1$. Also for $(|T|^{2\alpha r}z,x) = 0$ and $|T|^{\alpha r}z \neq 0$;

(4) $|(T|T|^{\alpha r+\beta s-1}x,y)|^2 + \| |T^*|^{\beta s}y\|^2 \sum_{i=1}^{n} |(u_{i-1}, U|T|^{\alpha r}x_i)|^2 \leqslant \| |T|^{\alpha r}x\|^2 \| |T^*|^{\beta s}y\|^2$

for all $x,y \in H$, $r,s > 0$, and $\alpha,\beta \in [0,1]$ with $\alpha r+\beta s \geqslant 1$. Also for nonzero vectors $\{x_i\}_{i=1}^n$ such that $(T|T|^{\alpha r+\beta s-1}x_i,y) = 0$ and $\| \, |T|^{\alpha r}x_i \, \| = 1$, $i = 1,2,\ldots,n$; and the set $\{u_i\}_{i=1}^n$ is defined as follows: $u_0 = U|T|^{\alpha r}x$, and $u_i = u_{i-1} - (u_{i-1}, U|T|^{\alpha r}x_i)U|T|^{\alpha r}x_i$, $i = 1,2,\ldots,n$;

(5) $|(T|T|^{\alpha r+\beta s-1}x,y)|^2 + \| \, |T^*|^{\beta s}y \, \|^2 \sum_{i=1}^{n} \frac{|(|T|^{2\alpha r}x,x_i)|^2}{\sum_{j=1}^{n} |(|T|^{2\alpha r}x_i,x_j)|} \leqslant \| \, |T|^{\alpha r}x \, \|^2 \| \, |T^*|^{\beta s}y \, \|^2$

for all $x,y \in H$, $r,s > 0$, and $\alpha,\beta \in [0,1]$ with $\alpha r+\beta s \geqslant 1$. Also for nonzero vectors $\{x_i\}_{i=1}^n$ such that $\sum_{j=1}^{n} |(|T|^{2\alpha r}x_i,x_j) \neq 0$ and $(T|T|^{\alpha r+\beta s-1}x_i,y) = 0$, $i = 1,2,\ldots,n$.

Proof. In Lemma 4 replace x by $U|T|^{\alpha r}x$, y by $|T^*|^{\beta s}y$, and z by $U|T|^{\alpha r}z$. Then (1) and (2) are easily obtained from (1) and (2) in Lemma 4, and they are mutually equivalent.

(3) Interchange of the vector x with y in (2) of Lemma 4, and use the same replacements as above.

(4) Besides the replacements as in above, let e_i be $U|T|^{\alpha r}x_i$, $i = 1,2,\ldots,n$, in (3) of Lemma 4 so that $\| \, |T|^{\alpha r}x_i \, \| = 1$ and $(T|T|^{\alpha r+\beta s-1}x_i,y) = 0$, $i = 1,2,\ldots,n$.

(5) The same replacements as in above, and let x_i be $U|T|^{\alpha r}x_i$, $i = 1,2,\ldots,n$, in (4) of Lemma 4 so that $(T|T|^{\alpha r+\beta s-1}x_i,y) = 0$, $i = 1,2,\ldots,n$.

Now we are ready for further characterizations. The next result can be easily done due to Corollary 6 and by similar arguments as in the proof of Theorem 1.

THEOREM 2. *For $A,B > O$, the following are equivalent and each statement holds true:*

(1) *If* $\log A \geqslant \log B$, *then*

$$(B^{r/2}A^pB^{r/2})^{\frac{\alpha r}{p+r}} \geqslant B^{\alpha r}$$

for $p \geqslant 0$, $r > 0$ and $\alpha \in [0,1]$;

(2) *For an invertible operator T such that* $\log A^2 \geqslant \log |T|^2$ *and* $\log B^2 \geqslant \log |T^*|^2$,

$$|(T|T|^{\alpha r+\beta s-1}x,y)|^2 + \frac{\| \, |T^*|^{\beta s} y \, \|^2 \, |(|T|^{2\alpha r}x,z)|^2}{\| \, |T|^{\alpha r} z \, \|^2} \leqslant ((|T|^r A^{2p}|T|^r)^{\frac{\alpha r}{p+r}}x,x)((|T^*|^s B^{2q}|T^*|^s)^{\frac{\beta s}{q+s}}y,y)$$

for all $x,y,z \in H$, $p,q \geqslant 0$, $r,s > 0$ *and* $\alpha,\beta \in [0,1]$ *with* $\alpha r + \beta s \geqslant 1$. *Also for* $|T|^{\alpha r} z \neq 0$ *and* $(T|T|^{\alpha r+\beta s-1}z,y) = 0$;

(3) *For an invertible operator T such that* $\log A^2 \geqslant \log |T|^2$ *and* $\log B^2 \geqslant \log |T^*|^2$,

$$|(T|T|^{\alpha r+\beta s-1}x,y)|^2 + \frac{\| \, |T^*|^{\beta s} x \, \|^2 \, |(T|T|^{\alpha r+\beta s-1}z,y)|^2}{\| \, |T|^{\alpha r} z \, \|^2} \leqslant ((|T|^r A^{2p}|T|^r)^{\frac{\alpha r}{p+r}}x,x)((|T^*|^s B^{2q}|T^*|^s)^{\frac{\beta s}{q+s}}y,y)$$

for all $x,y,z \in H$, $p,q \geqslant 0$, $r,s > 0$ *and* $\alpha,\beta \in [0,1]$ *with* $\alpha r + \beta s \geqslant 1$. *Also for* $|T|^{\alpha r} z \neq 0$ *and* $(|T|^{2\alpha r}z,x) = 0$;

(4) *For an invertible operator T such that* $\log A^2 \geqslant \log |T|^2$ *and* $\log B^2 \geqslant \log |T^*|^2$,

$$|(T|T|^{\alpha r+\beta s-1}x,y)|^2 + \| \, |T^*|^{\beta s} y \, \|^2 \sum_{i=1}^{n} |(u_{i-1}, U|T|^{\alpha r}x_i)|^2 \leqslant ((|T|^r A^{2p}|T|^r)^{\frac{\alpha r}{p+r}}x,x)((|T^*|^s B^{2q}|T^*|^s)^{\frac{\beta s}{q+s}}y,y)$$

for all $x,y \in H$, $p,q \geqslant 0$, $r,s > 0$ *and* $\alpha,\beta \in [0,1]$ *with* $\alpha r + \beta s \geqslant 1$. *Also for nonzero vectors* $\{x_i\}_{i=1}^n$ *such that* $(T|T|^{\alpha r+\beta s-1}x_i,y) = 0$ *and* $\| \, |T|^{\alpha r}x_i \, \| = 1$, $i = 1,2,\ldots,n$. *And the set* $\{u_i\}_{i=1}^n$ *is defined as in (4) of Corollary 6;*

(5) *For an invertible operator T such that* $\log A^2 \geqslant \log |T|^2$ *and* $\log B^2 \geqslant \log |T^*|^2$,

$$|(T|T|^{\alpha r+\beta s-1}x,y)|^2 + \| \, |T^*|^{\beta s} y \, \|^2 \sum_{i=1}^{n} \frac{|(|T|^{2\alpha r}x,x_i)|^2}{\sum_{j=1}^{n} |(|T|^{2\alpha r}x_i,x_j)|} \leqslant ((|T|^r A^{2p}|T|^r)^{\frac{\alpha r}{p+r}}x,x)((|T^*|^s B^{2q}|T^*|^s)^{\frac{\beta s}{q+s}}y,y)$$

for all $x,y \in H$, $p,q \geqslant 0$, $r,s > 0$ *and* $\alpha,\beta \in [0,1]$ *with* $\alpha r + \beta s \geqslant 1$. *Also for nonzero vectors* $\{x_i\}_{i=1}^n$ *such that* $\sum_{j=1}^{n} |(T|T|^{2\alpha r}x_i,x_j)| \neq 0$ *and* $(T|T|^{\alpha r+\beta s-1}x_i,y) = 0$, $i = 1,2,\ldots,n$.

Proof. $(1)\Rightarrow(2)$. By (2) in Corollary 6, we have

$$|(T|T|^{\alpha r+\beta s-1}x,y)|^2+\frac{\|\,|T^*|^{\beta s}y\,\|^2\,|(|T|^{2\alpha r}x,z)|^2}{\|\,|T|^{\alpha r}z\,\|^2}\leqslant\|\,|T|^{\alpha r}x\,\|^2\|\,|T^*|^{\beta s}y\,\|^2$$
$$\leqslant((|T|^rA^{2p}|T|^r)^{\frac{\alpha r}{p+r}}x,x)((|T^*|^sB^{2q}|T^*|^s)^{\frac{\beta s}{q+s}}y,y).$$

The last inequality is due to (1).

Using (3), (4) and (5) of Corollary 6, a similar proof shows that (1) implies (3), (4) and (5), respectively.

Now, clearly (2), (3), (4) or (5) implies (2) in Theorem 1, and hence (1) follows.

5. High-order Cauchy-Schwarz inequalities

Remark that the substitution of x by $\nabla=2(z,x)x-\|\,x\,\|^2\,z$ in the proof of Lemma 2 is important and useful tool in generalization of the Cauchy-Schwarz inequality as we are going to show in this last section. Let us call the usual Cauchy-Schwarz inequality, i.e., (1) in Lemma 2, the order-two inequality, because it involves two vectors (the order-one is, if we wish, the trivial expression: $|(x,x)|=(x,x)=\|\,x\,\|^2$). Accordingly, (2) in Lemma 2 is the order-three inequality. Moreover, if y is replaced by $\nabla'=2(u,y)y-\|\,y\,\|^2\,u$ in the order-three inequality, then we have the order-four inequality, which is

$$4|(z,x)(x,y)(y,u)|$$
$$\leqslant 2\,\|\,y\,\|^2\,|(z,x)(x,u)|+\|\,x\,\|^2\,[\|\,y\,\|^2\|\,z\,\|\|\,u\,\|+|2(z,y)(y,u)-\|\,y\,\|^2\,(z,u)|].$$

A similar further process can be used for the order-five inequality, etc. We may easily show that high-order inequalities are all equivalent to the order-two inequality by keeping x fixed and letting all other vectors equal to y, as we did in the proof of Lemma 2.

The so called Bessel's inequality (in one variable x) is the relation

$$\sum_{i=1}^{n}|(x,x_i)|^2\leqslant\|\,x\,\|^2$$

for any $x\in H$, and $\{x_i\}_{i=1}^n\subseteq H$ is an orthonormal set. the next result is a generalized Bessel's inequalitu (in two variables x and z) in terms of substitution by ∇.

PROPOSITION 1. *Let $x,z\in H$ and $\{x_i\}_{i=1}^n\subseteq H$ be an orthonormal set. Then*
(1) $\sum\limits_{i=1}^{n}|2(z,x)(x,x_i)-\|\,x\,\|^2\,(z,x_i)|^2\leqslant\|\,x\,\|^4\|\,z\,\|^2$.
Moreover, inequality (1) implies the following:
(2) *Bessel's inequality.*
(3) *The order-two inequality.*
(4) *The order-three inequality.*

Proof. (1) All we have to do is in Bessel's inequality let x be replaced as before by $2(z,x)x-\|\,x\,\|^2\,z$.

(2) Let $z = x$ in (1).
(3) In (1) let $n = 1$, $x_1 = \frac{y}{\|y\|}$, $0 \neq y \in H$, and $z = x$.
(4) In (1) let $n = 1$ and $x_1 = \frac{y}{\|y\|}$, $0 \neq y \in H$.

For Bessel's inequality, it is known that the equality holds if and only if the orthonormal set $\{x_i\}_{i=1}^n$ is also a basis, and in this case it is called Parseval's identity. Likewise, we have a generalized Parseval's identity as follows.

PROPOSITION 2. *Let $x, z \in H$ and $\{x_i\}_{i=1}^n \subseteq H$ be an orthonormal set. Then the equality*

$$\sum_{i=1}^{n} |2(z,x)(x,x_i) - \| x \|^2 \ (z,x_i)|^2 = \| x \|^4 \| z \|^2$$

holds if and only if $\{x_i\}_{i=1}^n$ is also a basis.

Note that the equality in Proposition 2 implies Parseval's identity if $z = x$. Next, by the same proof as in Proposition 1 we have a generalized Selberg's inequality (cf. Section 4) in two variables x and z.

PROPOSITION 3. *Let $x, z \in H$ and $\{x_i\}_{i=1}^n \subseteq H$. Then*

$$\sum_{i=1}^{n} \frac{|2(z,x)(x,x_i) - \| x \|^2 \ (z,x_i)|^2}{\sum\limits_{j=1}^{n} |(x_i,x_j)|} \leqslant \| x \|^4 \| z \|^2 .$$

In conclusion we remark that in literature it is shown that the order-two inequality is a simple consequence of Bessel's inequality. More precisely, if $y \neq 0$, then the set consisting of the vector $\frac{y}{\|y\|}$ alone is orthonormal, and so $|(x, \frac{y}{\|y\|})|^2 \leqslant \| x \|^2$ by Bessel's inequality. Thus the order-two inequality follows. It is the author's belief that we have not seen the proof of the converse in literature, i.e., the order-two inequality implies Bessel's inequality. So we shall include a simple proof as follows.

Let the set $\{u_i\}_{i=1}^n$ be similar to (3) of Lemma 4, i.e., $u_0 = x$, and $u_i = u_{i-1} - (u_{i-1}, x_i)x_i$, $i = 1, 2, \ldots, n$, where $\{x_i\}_{i=1}^n \subseteq H$ is an orthonormal set. Then we have the following system:

$$\begin{aligned}
u_1 &= x - (x, x_1)x_1 \\
u_2 &= u_1 - (u_1, x_2)x_2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
u_{n-1} &= u_{n-2} - (u_{n-2}, x_{n-1})x_{n-1} \\
u_n &= u_{n-1} - (u_{n-1}, x_n)x_n.
\end{aligned}$$

It follows that $(u_{i-1}, x_i) = (x, x_i)$, $i = 1, 2, \ldots, n$. By adding n equalities above the term u_n becomes independent of u_i, $i \neq n$, which is essential in the proof. More precisely we obtain the relation

$$u_n = x - \sum_{i=1}^{n} (u_{i-1}, x_i)x_i = x - \sum_{i=1}^{n} (x, x_i)x_i.$$

Therefore,

$$\| u_n \|^2 = \left(x - \sum_{i=1}^{n} (x, x_i) x_i , x - \sum_{i=1}^{n} (x, x_i) x_i \right) = \| x \|^2 - \sum_{i=1}^{n} |(x, x_i)|^2 .$$

Now, in particular if we use the order-two inequality $|(u_n, e)|^2 \leqslant \| u_n \|^2$ for any unit vector $e \in H,$ then we obtain

$$\left| (x, e) - \sum_{i=1}^{n} (x, x_i)(x_i, e) \right|^2 + \sum_{i=1}^{n} |(x, x_i)|^2 \leqslant \| x \|^2,$$

which is sharper than Bessel's inequality.

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REFERENCES

[1] T. ANDO, *On some operator inequalities*, Math. Ann., **279** (1987), 157–159.
[2] T. FURUTA, *A simplified proof of Heinz inequality and scrutiny of its equality*, Proc. Amer. Math. Soc., **97** (1986), 751–753.
[3] T. FURUTA, *$A \geqslant B \geqslant O$ assures $(B^r A^p B^r)^{1/q} \geqslant B^{(p+2r)/q}$ for $r \geqslant 0,$ $p \geqslant 0,$ $q \geqslant 1$ with $(1+2r)q \geqslant p+2r,$* Proc. Amer. Math. Soc., **101** (1987), 85–88.
[4] M. FUJII, T. FURUTA AND E. KAMEI, *Furuta's inequality and its application to Ando's theorem*, Lin. Alge. Appl., **179** (1993), 161–169.
[5] M. FUJII, *Around the Furuta inequality*, preprint.
[6] M. G. FUJII, E. KAMEI, C. KOTANI AND H. YAMADA, *Furuta's determinant type generalizations of Heinz-Kato inequality*, Math. Japon., **40** (1994), 259–267.
[7] M. FUJII, J. F. JIANG AND E. KAMEI, *Characterization of chaotic order and its application to Furuta inequality*, Proc. Amer. Math. Soc., **125** (1997), 3655–3658.
[8] M. FUJII AND R. NAKAMOTO, *Simultaneous extensions of Selberg inequality and Heinz-Kato-Furuta inequality*, Nihonkai Math. J., **9** (1998), 219–225.
[9] C.-S. LIN, *Heinz's inequality and Bernstein's inequality*, Proc. Amer. Math. Soc., **125** (1997), 2319–2325.
[10] C.-S. LIN, *Heinz-Kato-Furuta-type inequalities with bounds and equality conditions*, Math. Inequal. Appl., **5** (2002), 735–743.

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