Mathematical
Inequalities
& Applications
Volume 6, Number 4 (2003), 713–714

SIMPLE PROOF OF JOINTLY CONCAVITY OF THE RELATIVE OPERATOR ENTROPY $S(A|B) = A^{\frac{1}{2}}\log(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}}$

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Dedicated to Professor Hisaharu Umegaki
on his 77th birthday
with respect and affection

(*communicated by J. Pečarić*)

Abstract. A capital letter means a bounded linear and *strictly positive* operator on a Hilbert space H. Here we shall give a simple proof of the result in [2] [3] that the relative operator entropy $S(A|B) = A^{\frac{1}{2}}\log(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}}$ is subadditive and jointly concave.

THEOREM A. *The relative operator entropy is subadditive and jointly concave:*
(i) $S(A_1+A_2|B_1+B_2) \geqslant S(A_1|B_1)+S(A_2|B_2)$.
(ii) $S\big(\alpha A_1+(1-\alpha)A_2|\alpha B_1+(1-\alpha)B_2\big) \geqslant \alpha S(A_1|B_1)+(1-\alpha)S(A_2|B_2)$ *for* $0 \leqslant \alpha \leqslant 1$.

The following result is well known by [Theorem 3.5, 5].

$$(A+B)\sigma(C+D) \geqslant A\sigma C + B\sigma D \quad \text{for every mean} \quad \sigma. \tag{1}$$

Proof of Theorem A. We recall the following obvious formula:

$$\lim_{n\to\infty}(T^{\frac{1}{n}}-I)n = \log T \quad \text{for any} \quad T > 0. \tag{2}$$

(i). As $f(A|B) = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\frac{1}{n}}A^{\frac{1}{2}}$ is a mean for natural number n, so that $f(A_1+A_2|B_1+B_2) \geqslant f(A_1|B_1)+f(A_2|B_2)$ by (1), namely,

$$\begin{aligned}(A_1+A_2)^{\frac{1}{2}}&\left\{(A_1+A_2)^{\frac{-1}{2}}(B_1+B_2)(A_1+A_2)^{\frac{-1}{2}}\right\}^{\frac{1}{n}}(A_1+A_2)^{\frac{1}{2}}\\ &\geqslant A_1^{\frac{1}{2}}(A_1^{\frac{-1}{2}}B_1A_1^{\frac{-1}{2}})^{\frac{1}{n}}A_1^{\frac{1}{2}}+A_2^{\frac{1}{2}}(A_2^{\frac{-1}{2}}B_2A_2^{\frac{-1}{2}})^{\frac{1}{n}}A_2^{\frac{1}{2}}\end{aligned}$$

we have the following by slightly modification

$$\begin{aligned}(A_1+A_2)^{\frac{1}{2}}&\left[\{(A_1+A_2)^{\frac{-1}{2}}(B_1+B_2)(A_1+A_2)^{\frac{-1}{2}}\}^{\frac{1}{n}}-I\right]n(A_1+A_2)^{\frac{1}{2}}\\ &\geqslant A_1^{\frac{1}{2}}\left[(A_1^{\frac{-1}{2}}B_1A_1^{\frac{-1}{2}})^{\frac{1}{n}}-I\right]nA_1^{\frac{1}{2}}+A_2^{\frac{1}{2}}\left[(A_2^{\frac{-1}{2}}B_2A_2^{\frac{-1}{2}})^{\frac{1}{n}}-I\right]nA_2^{\frac{1}{2}}\end{aligned}$$

Mathematics subject classification (2000): 47A63.
Key words and phrases: relative operator entropy, subadditive, jointly concave.

Paper MIA-06-64

for natural number n, tending $n \to \infty$, we obtain the following by (2)

$$(A_1+A_2)^{\frac{1}{2}}\left[\log\{(A_1+A_2)^{\frac{-1}{2}}(B_1+B_2)(A_1+A_2)^{\frac{-1}{2}}\}\right](A_1+A_2)^{\frac{1}{2}}$$
$$\geqslant A_1^{\frac{1}{2}}\left[\log(A_1^{\frac{-1}{2}}B_1A_1^{\frac{-1}{2}})\right]A_1^{\frac{1}{2}}+A_2^{\frac{1}{2}}\left[\log(A_2^{\frac{-1}{2}}B_2A_2^{\frac{-1}{2}})\right]A_2^{\frac{1}{2}}$$

whence the proof of (i) is complete.

(ii). (ii) follows by (i) since $S(A|B)$ is homogeneous: $S(tA|tB) = tS(A|B)$ for any $t \geqslant 0$.

This proof is along the simple one in [4] based on (2), in which there given a simple proof of concavity of the usual operator entoropy $A\log A^{-1}$ in [1] and [6].

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(Received September 2, 2002)

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