Mathematical
Inequalities
& Applications
Volume 7, Number 1 (2004), 1–6

COMPARISON THEOREMS BETWEEN SEVERAL QUASI–ARITHMETIC MEANS

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(*communicated by Th. Rassias*)

Abstract. Comparison between quasi-arithmetic means are established by which we prove Jensen's type inequalities.

1. Introduction

In [2] Y – H Kim proved the following (Theorems 1 and 2 there).

THEOREM A. *Let* $a_1, \dots, a_n$, *an be a set of nonnegative quantities,* $n \geqslant 1$, $x \geqslant 1$, $y \geqslant 0$; *Then*

$$n\left(\sum_{i=1}^{n}\frac{1}{n}a_i\right)^{x+y} \leqslant n\left(\sum_{i=1}^{n}\frac{1}{n}a_i^x\right)^{\frac{x+y}{x}} \leqslant \sum_{i=1}^{n}a_i^{x+y} \quad , \tag{1}$$

with all equalities holding if and only if a_i *are the same.*

Let $a_1, a_2, \dots, a_n$ *be a set of positive quantities,* $n \geqslant 1$, $0 < x < 1$, $y \geqslant 0$; *then*

$$\left(\prod_{i=1}^{n}a_i\right)^{\frac{x+y}{n}} \leqslant \left(\sum_{i=1}^{n}\frac{1}{n}a_i^x\right)^{\frac{x+y}{x}} \leqslant \left(\sum_{i=1}^{n}\frac{1}{n}a_i\right)^{x+y} \quad , \tag{2}$$

with equality holding if and only if all a_i *are the same.*

THEOREM B. *Let* $a_1, a_2, \dots, a_n$ *be a set of* n *nonnegative quantities,* $n \geqslant 1$, $0 < x \leqslant 1$, $-x \leqslant y \leqslant 0$; *then*

$$n\left(\prod_{i=1}^{n}a_i\right)^{\frac{x+y}{n}} \leqslant \sum_{i=1}^{n}a_i^{x+y} \leqslant n\left(\sum_{i=1}^{n}\frac{1}{n}a_i^x\right)^{\frac{x+y}{x}} \leqslant n\left(\sum_{i=1}^{n}\frac{1}{n}a_i\right)^{x+y} \quad ,$$

with all equalities holding if and only if all a_i *are the same.*

In this paper we present generalizations of Kim's results. We also show that his results are special cases of inequalities proved in [1] involving convex and concave functions which extend Hölder's inequality.

Mathematics subject classification (2000): 26A51, 26B25, 26D07, 26D15, 26D20.
Key words and phrases: Convex and concave functions, inequalities, means, Quasi-Arithmetic means.

Paper MIA-07-01

2. Main results

THEOREM 1. *Let $a_i \in [A,B]$, $i = 1,\dots,n$ be real numbers. Let $F_k, k = 1,\dots,m$ be one to one functions defined on $[A,B]$ where $F_k : [A,B] \to [C,D]$, $k = 1,\dots,m$ and let $f_1 = F_1$, $f_{k+1} = F_{k+1} \circ F_k^{-1}, k = 1,\dots,m-1$ be concave increasing functions. Let $F_0(a) = a$ and $0 \leqslant \alpha_i \leqslant 1$, $i = 1,\dots,n$, $\sum_{i=1}^{n} \alpha_i = 1$; then we get the following comparsion between the quasi arithmetic mean $F_k^{-1}\left(\sum_{i=1}^{1} \alpha_i F_k(a_i)\right)$ and the quasi arithmetic mean $F_{k-1}^{-1}\left(\sum_{i=1}^{n} \alpha_i F_{k-1}(a_i)\right)$:*

$$F_k^{-1}\left(\sum_{i=1}^{n} \alpha_i F_k(a_i)\right) \leqslant F_{k-1}^{-1}\left(\sum_{i=1}^{n} \alpha_i F_{k-1}(a_i)\right), \quad k = 1,\dots,m \tag{3}$$

Equality holds iff $F_k \circ F_{k-1}^{-1} = L \cdot a + R$, where L, R are constant, or if all a_i, $i = 1,\dots,n$ are equal.

Proof. Since $f_1(a) = F_1(a)$ is increasing, it follows that $F_k(a), k = 1,\dots,m$ are increasing too, and therefore the inequality (3) is equivalent to

$$\sum_{i=1}^{n} \alpha_i f_k(F_{k-1}(a_i)) \leqslant f_k\left(\sum_{i=1}^{n} \alpha_i F_{k-1}(a_i)\right) \quad , k = 1,\dots,m, \tag{4}$$

and this follows because $f_k = F_k \circ F_{k-1}^{-1}$ is concave. Equality in (3) holds obviously for all choices of a_i, $i = 1,\dots,n$, if $F_k\left(F_{k-1}^{-1}(a)\right) = L \cdot a + R$ where L, R are constant. $\square$

COROLLARY 1. *Choosing $f_1(a) = F_1(a) = a^{\frac{x}{x+y}}$, $F_2(a) = a^{\frac{1}{x+y}}$, $1 \leqslant x \leqslant x+y$, $0 \leqslant \alpha_i \leqslant 1$, $i = 1,\dots,n$, $\sum_{i=1}^{n} \alpha_i = 1$ we get that $f_2(a) = F_2\left(F_1^{-1}(a)\right) = a^{\frac{1}{x}}$, and $F_1(a) = a^{\frac{x}{x+y}}$, are concave increasing. Therefore we get from (3) that*

$$\begin{aligned} F_2^{-1}\left(\sum_{i=1}^{n} \alpha_i F_2\left(F_2^{-1}(a_i)\right)\right) &\leqslant F_1^{-1}\left(\sum_{i=1}^{n} \alpha_i F_1\left(F_2^{-1}(a_i)\right)\right) \\ &\leqslant F_0^{-1}\left(\sum_{i=1}^{n} \alpha_i F_0\left(F_2^{-1}(a_i)\right)\right), \end{aligned}$$

in other words we get a Jensen's type inequality [3]

$$\left(\sum_{i=1}^{n} \alpha_i a_i\right)^{x+y} \leqslant \left(\sum_{i=1}^{n} \alpha_i a_i^x\right)^{\frac{x+y}{x}} \leqslant \sum_{i=1}^{n} \alpha_i a_i^{x+y} \quad .$$

When we choose $\alpha_i = 1/n$, $i = 1,\dots,n$ the last inequality in (1) in Theorem A.

COROLLARY 2. *Choosing* $F_1(a) = a^x$, $F_2(a) = \log(a^x)$, $0 < x \leqslant 1$, *we get that* $F_2\left(F_1^{-1}(a)\right) = \log(a)$, *and* $F_1(a) = a^x$ *are concave increasing. Therefore we get from (3) that*

$$F_2^{-1}\left(\sum_{i=1}^{n} \alpha_i F_2(a_i)\right) \leqslant F_1^{-1}\left(\sum_{i=1}^{n} \alpha_i F_1(a_i)\right) \leqslant \sum_{i=1}^{n} \alpha_i a_i$$

which in our case is

$$\exp\left(\frac{1}{x}\left(\sum_{i=1}^{n} \alpha_i \log a_i^x\right)\right) = \prod_{i=1}^{n} a_i^{\alpha_i} \leqslant \left(\sum_{i=1}^{n} \alpha_i a_i^x\right)^{\frac{1}{x}} \leqslant \sum_{i=1}^{n} \alpha_i a_i.$$

Taking the last inequality to the power $x + y$ *and* $\alpha_i = 1/n$, $i = 1,...,n$, *we get Inequality (2) in Theorem A, where instead of* $0 < x \leqslant 1$, $y > 0$, *the inequality is satisfied for* $0 < x \leqslant 1$ *and* $x + y > 0$.

COROLLARY 3. *Choosing* $F_1(a) = a^x$, $F_2(a) = a^{x+y}$, $F_3(a) = \log(a^{x+y})$, $0 < x \leqslant 1$, $0 < x + y \leqslant x$ *and inserting in (3) we get that for* $\alpha_i = \frac{1}{n}$

$$n\left(F_3^{-1}\left(\sum_{i=1}^{n} \frac{1}{n} F_3(a_i)\right)\right)^{x+y} \leqslant n\left(F_2^{-1}\left(\sum_{i=1}^{n} F_2(a_i)\right)\right)^{x+y}$$
$$\leqslant n\left(F_1^{-1}\left(\sum_{i=1}^{n} \frac{1}{n} F_1(a_i)\right)\right)^{x+y} \leqslant n\left(\sum_{i=1}^{n} \frac{1}{n} a_i\right)^{x+y}$$

and this is the result of Theorem B.

THEOREM 2. *Let* $b_i \in [A_m, B_m]$, $i = 1,...,n$ *be real numbers. Let* g_k $k = 1,...,m$ *be real functions defined on a real interval* $g_k : [A_k, B_k] \to [A_{k-1}, B_{k-1}]$, $k = 1,...,m$ *which are ordered as follows:*

$g_{j_{2s}+1},..., g_{j_{2s+1}}$ *are concave functions,* $g_{j_{2s+1}+1},..., g_{j_{2s+2}}$ *are convex functions,* $s = 0,..., s_0$, $j_0 = 0$

g_{j_s} *are decreasing and all the other* g_k-th *are increasing. Only* g_m, *no matter if it is convex or concave is not confined to be monotone. Then the inequality*

$$\begin{aligned}
&\sum_{i=1}^{n} \alpha_i g_1(g_2(\dots(g_m(b_i)))) \leqslant g_1\left(\sum_{i=1}^{n} \alpha_i g_2(\dots(g_m(b_i)))\right) \\
&\quad \leqslant \dots \leqslant g_1\left(g_2 \dots \left(g_{k-1}\left(\sum_{i=1}^{n} \alpha_i g_k(g_{k+1} \dots (g_m(b_i)))\right)\right)\right) \\
&\quad \leqslant g_1\left(g_2 \dots \left(g_k\left(\sum_{i=1}^{n} \alpha_i g_{k+1}(\dots g_m(b_i))\right)\right)\right) \\
&\quad \leqslant \dots \leqslant g_1\left(g_2\left(\dots\left(g_{m-1}\left(\sum_{i=1}^{n} \alpha_i g_m(b_i)\right)\right)\right)\right) \\
&\quad \leqslant g_1\left(g_2\left(\dots\left(g_{m-1}\left(g_m\left(\sum_{i-1}^{n} \alpha_i b_i\right)\right)\right)\right)\right)
\end{aligned} \tag{5}$$

holds. Moreover, under the above conditions equality holds in

$$g_1\left(g_2 \dots \left(g_{k-1}\left(\sum_{i=1}^{n} \alpha_i g_k(g_{k+1} \dots (g_m(b_i)))\right)\right)\right)$$

$$\leqslant g_1\left(g_2 \ldots\left(g_k\left(\sum_{i=1}^n \alpha_i g_{k+1}\left(\ldots g_m\left(b_i\right)\right)\right)\right)\right) \tag{6}$$

iff $g_k(a) = La+R$, *L, R, are constants, or if all* $b_i, i=1, \ldots, n$ *are equal.*

3. More on inequalities involving concave and convex functions

DEFINITION 1. Let $g_i, i=1,\ldots,m-1$ be positive functions on $x>0$ and let $x_i>0$, $i=1,\ldots,m$. For $r<s$, $r,s \in \{1,2,\ldots,m-1\}$ we denote $G_{rs}\left(x_r, x_{r+1}, \ldots, x_{s+1}\right) = x_r g_r\left(\frac{x_{r+1}}{x_r} g_{r+1}\left(\frac{x_{r+2}}{x_{r+1}} \ldots g_s\left(\frac{x_{s+1}}{x_s}\right)\right)\right)$, and $G_r\left(a_r, a_{r+1}\right) = a_r g_r\left(\frac{a_{r+1}}{a_r}\right)$, $1 \leqslant r \leqslant m-1$, $m>1$.

The following theorem is proved in [1] (Theorem 1 there).

THEOREM C. *Let* $g_i \in A$ *where* A *is a set of monotone functions defined on* $x>0$, *which are nonlinear on any subinterval of* $x>0$ *and which consists of concave increasing functions and convex functions. Then*

$$\begin{aligned}
&\sum_{k=1}^n G_{1,m-1}\left(a_{1k}, a_{2k}, \ldots, a_{m,k}\right) \\
&\quad \geqslant G_{1,1}\left(\sum_{k=1}^n a_{1,k}, \sum_{k=1}^n G_{2,m-1}\left(a_{2,k}, \ldots, a_{m,k}\right)\right) \\
&\quad \geqslant \ldots \geqslant G_{1,j}\left(\sum_{k=1}^n a_{1k}, \sum_{k=1}^n a_{2k}, \ldots, \sum_{k=1}^n a_{j,k}, \sum_{k=1}^n G_{j+1,m-1}\left(a_{j+1,k}, \ldots, a_{m,k}\right)\right) \\
&\quad \geqslant \ldots \geqslant G_{1,m-1}\left(\sum_{k=1}^n a_{1,k}, \sum_{k=1}^n a_{2,k,\ldots,} \sum_{k=1}^n a_{m,k}\right)
\end{aligned} \tag{7}$$

holds for all $a_{i,k}>0$, $i=1,\ldots,m$, $j \leqslant y \leqslant m-1$, $k=1,\ldots,n$ *if* $g_i(x)$ $i=1,\ldots m-2$ *are convex increasing and* $g_{m-1}(x)$ *is convex. Moreover, if the compound function* $H = g_1 \circ \ldots \circ g_{m-1}$ *is an increasing function, then*

$$\sum_{k=1}^n G_{1,m-1}\left(a_{1,k}, \ldots, a_{m,k}\right) \geqslant G_{1,m-1}\left(\sum_{k=1}^n a_{1,k}, \ldots, \sum_{k=1}^n a_{m,k}\right) \tag{8}$$

holds for all $a_{i,k}>0$, $i=1,\ldots,m, k=1,\ldots,n$ *if and only if* $g_i(x), i=1,\ldots,m-1$ *are convex increasing. Equality holds in the last inequality iff* $\frac{a_{i,1}}{a_{j,1}} = \frac{a_{i,2}}{a_{j,2}} = \ldots = \frac{a_{i,n}}{a_{j,n}}$, $i,j=1,\ldots,m$ *holds.*

COROLLARY 4. *Let* $a_{1,k} = a_{2,k} = \ldots = a_{m-1,k} = \alpha_k$, $\sum_{k=1}^n \alpha_k = 1$, $k=1,\ldots,n$ *and* $\frac{a_{m,k}}{a_{m-1,k}} = b_k$, $k=1,\ldots,n$. *Then we get from Theorem C inequalities (7) that*

$$\begin{aligned}
&\sum_{k=1}^n \alpha_k g_1\left(\ldots\left(g_{m-1}\left(b_k\right)\right)\right) \\
&\qquad \geqslant g_1\left(\sum_{k=1}^n \alpha_k g_2\left(\ldots\left(g_{m-1}\left(b_k\right)\right)\right)\right)
\end{aligned}$$

$$\geqslant \sum_{k=1}^{n} g_1 g_2 \left(\sum_{k=1}^{n} \alpha_k g_3 \left(\ldots \left(g_{m-1} \left(b_k \right) \right) \right) \right)$$

$$\geqslant \ldots. \geqslant g_1 \left(g_2 \left(\ldots \left(g_{m-1} \left(\sum_{k=1}^{n} \alpha_k b_k \right) \right) \right) \right) \quad .$$

Equality holds iff $b_1 = b_2 = \ldots = b_n$.

This is a special case of Theorem 2 in Chapter 2 here for convex functions.
We may extend theorems 3 and 4 from [2] as follows

THEOREM 3. *Let* $p_1 > 1$ *and* $p_2 < 0$. *Let* $0 \leqslant \alpha_i \leqslant 1$, $\sum_{i=1}^{n} \alpha_i = 1$ *and let* $a_i > 0$, $i = 1, \ldots, n$ *be real numbers. Then*

$$\sum_{i=1}^{n} \alpha_i a_i^{p_1 p_2} \geqslant \prod_{i=1}^{n} a_i^{\alpha_i p_1 p_2} \geqslant \left(\sum_{i=1}^{n} \alpha_i a_i \right)^{p_1 p_2} \geqslant \left(\sum_{i=1}^{n} \alpha_i a_i^{p_1} \right)^{p_2} \quad , \tag{9}$$

and if also p_2 *<-1 we get*

$$\sum_{i=1}^{n} \alpha_i a_i^{p_1 p_2} \geqslant \left(\sum_{i=1}^{n} \alpha_i a_i^{p_2} \right)^{p_1} \geqslant \prod_{i=1}^{n} a_i^{\alpha_i p_1 p_2} \geqslant \left(\sum_{i=1}^{n} \alpha_i a_i \right)^{p_1 p_2} \geqslant \left(\sum_{i=1}^{n} \alpha_i a_i^{-p_2} \right)^{-p_1} . \tag{10}$$

Proof. In the proof of (9) we will use the facts that $\exp(a)$ is increasing, $\log(a)$ is concave, $a^{p_1 p_2}$ is decreasing and $a^{\frac{1}{p_1}}$ is concave and we get that

$$\begin{aligned}
\sum_{i=1}^{n} \alpha_i a_i^{p_1 p_2} &= \exp \left(\log \left(\sum_{i=1}^{n} \alpha_i a_i^{p_1 p_2} \right) \right) \geqslant \exp \left(\sum_{i=1}^{n} \alpha_i \left(\log a_i^{p_1 p_2} \right) \right) \\
&= \exp \left(\log \left(\left(\prod_{i=1}^{n} a_i^{\alpha_i} \right)^{p_1 p_2} \right) \right) = \left(\prod_{i=1}^{n} a_i^{\alpha_i} \right)^{p_1 p_2} \\
&= \left(\exp \left(\log \left(\prod_{i=1}^{n} a_i^{\alpha_i} \right) \right) \right)^{p_1 p_2} = \left(\exp \left(\sum_{i=1}^{n} \alpha_i \left(\log a_i \right) \right) \right)^{p_1 p_2} \\
&\geqslant \left(\exp \log \left(\sum_{i=1}^{n} \alpha_i a_i \right) \right)^{p_1 p_2} = \left(\sum_{i=1}^{n} \alpha_i a_i \right)^{p_1 p_2} \\
&= \left(\sum_{i=1}^{n} \alpha_i \left(a_i^{p_1} \right)^{\frac{1}{p_1}} \right)^{p_1 p_2} \geqslant \left(\sum_{i=1}^{n} \alpha_i a_i^{p_1} \right)^{p_2} \quad .
\end{aligned}$$

□

The proof of (10) is similar.

COROLLARY 5. *Replacing in (9)* $p_1 = x > 1$, $p_2 = (x+y)/x < 0$, $\alpha_i = 1/n$ *we get Theorem 3 of [2] (corrected) :*

$$\sum_{i=1}^{n} \frac{a_i^{x+y}}{n} \geqslant \prod_{i=1}^{n} a_i^{\frac{x+y}{n}} \geqslant \left(\sum_{i=1}^{n} \frac{a_i}{n} \right)^{x+y} \geqslant \left(\sum_{i=1}^{n} \frac{a_i^x}{n} \right)^{\frac{x+y}{x}} \quad .$$

Replacing in (10) $p_1 = -(x+y)/x > 1$, $p_2 = -x < -1$, $a_i = 1/n$ we get Theorem 4 of [2] (corrected):

$$\sum_{i=1}^{n} \frac{a_i^{x+y}}{n} \geqslant \left(\sum_{i=1}^{n} \frac{1}{n} a_i^{-x} \right)^{\frac{-(x+y)}{x}} \geqslant \prod_{i=1}^{n} a_i^{\frac{x+y}{n}} \geqslant \left(\sum_{i=1}^{n} a_i \right)^{x+y} \geqslant \left(\sum_{i=1}^{n} \frac{1}{n} a_i^{x} \right)^{\frac{x+y}{x}} .$$

The proof of the following theorem is immediate:

THEOREM 4. *Let* $1 < p_1 < p_2$. *Let* $0 \leqslant \alpha_i \leqslant 1$, $\sum_{i=1}^{n} \alpha_i = 1$ *and* $a_i \geqslant 0$, $i = 1, \ldots, n$. *Then*

$$\sum_{i=1}^{n} \alpha_i a_i^{p_1 p_2} \geqslant \left(\sum_{i=1}^{n} \alpha_i a_i^{p_2} \right)^{p_1} \geqslant \left(\sum_{i=1}^{n} \alpha_i a_i^{p_1} \right)^{p_2} \geqslant \left(\sum_{i=1}^{n} \alpha_i a_i \right)^{p_1 p_2} .$$

COROLLARY 6. *Let* $x \geqslant 1$, $y \geqslant 0$. *If* $\frac{x+y}{x} \geqslant x$, *then replacing in Theorem 4* $\alpha_i = \frac{1}{n}, i = 1, \ldots, n$, $p_1 = x$, $p_2 = \frac{x+y}{x}$ *we get a refined version of (1)*

$$\sum_{i=1}^{n} \frac{a_i^{x+y}}{n} \geqslant \left(\sum_{i=1}^{n} \frac{a_i^{\frac{x+y}{x}}}{n} \right)^{x} \geqslant \left(\sum_{i=1}^{n} \frac{a_i^{x}}{n} \right)^{\frac{x+y}{x}} \geqslant \left(\sum_{i=1}^{n} \frac{a_i}{n} \right)^{x+y} .$$

REMARK. In corollaries 1, 2, 3 and theorems 3, 4 Jensen's type inequalities are established.

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(Received March 1, 2002)

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