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EXPLICIT BOUND ON A RETARDED INTEGRAL INEQUALITY

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Abstract. The aim of the present note is to establish a new retarded integral inequality which provides explicit bound on unknown function, and can be used as a tool in the study of certain general integral equation.

1. Introduction

Over the years many papers on integral inequalities which provide explicit bounds on unknown functions have appeared in the literature. A detailed account related to such inequalities and their applications can be found in [2, 6]. Recently, in [8] the present author has established the following useful integral inequality.

LEMMA. *Let* $u(t) \in C(I, \mathbf{R}_+)$, $a(t,s)$, $b(t,s) \in C(D, \mathbf{R}_+)$, $a(t,s)$, $b(t,s)$ *are nondecreasing in* t *for each* $s \in I$, *where* $I = [\alpha, \beta]$, $\mathbf{R}_+ = [0, \infty)$, $D = \{(t,s) \in I^2 : \alpha \leqslant s \leqslant t \leqslant \beta\}$ *and suppose that*

$$u(t) \leqslant k + \int_\alpha^t a(t,s)u(s)\,ds + \int_\alpha^\beta b(t,s)u(s)\,ds,$$

for $t \in I$, *where* $k \geqslant 0$ *is a constant. If*

$$c(t) = \int_\alpha^\beta b(t,s)\exp\left(\int_\alpha^s a(s,\sigma)\,d\sigma\right)ds < 1,$$

for $t \in I$, *then*

$$u(t) \leqslant \frac{k}{1-c(t)}\exp\left(\int_\alpha^t a(t,s)\,ds\right),$$

for $t \in I$.

The main purpose of this note is to establish a general version of the above inequality which can be used as a tool to study the qualitative behavior of solutions of a general retarded Volterra-Fredholm integral equation.

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2. Main result

In what follows, let $I = [\alpha, \beta]$, $\mathbf{R}_+ = [0, \infty)$ be the given subsets of $\mathbf{R}$, the set of real numbers. We denote by $D = \{(t,s) \in I^2 : \alpha \leqslant s \leqslant t \leqslant \beta\}$, $\mathbf{R}^n$ the n-dimensional Euclidean space with norm $|\cdot|$, $C(M,N)$ and $C^1(M,N)$ are the class of continuous functions and class of continuosly differentiable functions from M to N.

Our main result is given in the following theorem.

THEOREM 1. *Let $u(t) \in C(I, \mathbf{R}_+)$, $a(t,s)$, $b(t,s)$, $c(t,s) \in C(D, \mathbf{R}_+)$, $a(t,s)$, $b(t,s)$ are nondecreasing in t for each $s \in I$, $h(t) \in C^1(I,I)$ be nondecreasing, with $h(t) \leqslant t$ on I, $k \geqslant 0$ be a constant and*

$$u(t) \leqslant k + \int_{h(\alpha)}^{h(t)} a(t,s)\left[f(s)u(s) + \int_{h(\alpha)}^{s} c(s,\sigma)u(\sigma)\,d\sigma\right] ds + \int_{h(\alpha)}^{h(\beta)} b(t,s)u(s)\,ds, \tag{2.1}$$

for $t \in I$. If

$$p(t) = \int_{h(\alpha)}^{h(\beta)} b(t,s)\exp(A(s))\,ds < 1, \tag{2.2}$$

for $t \in I$, where

$$A(t) = \int_{h(\alpha)}^{h(t)} a(t,\xi)\left[f(\xi) + \int_{h(\alpha)}^{\xi} c(\xi,\sigma)\,d\sigma\right] d\xi, \tag{2.3}$$

for $t \in I$, then

$$u(t) \leqslant \frac{k}{1-p(t)}\exp(A(t)), \tag{2.4}$$

for $t \in I$.

Proof. From the hypotheses, we observe that $h'(t) \geqslant 0$ for $t \in I$. Let $k > 0$ and fix $T \in I$, then for $\alpha \leqslant t \leqslant T$, from (2.1) we have

$$u(t) \leqslant k + \int_{h(\alpha)}^{h(t)} a(T,s)\left[f(s)u(s) + \int_{h(\alpha)}^{s} c(s,\sigma)u(\sigma)\,d\sigma\right] ds + \int_{h(\alpha)}^{h(\beta)} b(T,s)u(s)\,ds. \tag{2.5}$$

Define a function $z(t)$, $t \in [\alpha, T]$ by the right hand side of (2.5). Then for $t \in [\alpha, T]$, $u(t) \leqslant z(t)$, $z(t)$ is positive and nondecreasing in t,

$$z(\alpha) = k + \int_{h(\alpha)}^{h(\beta)} b(T,s)u(s)\,ds, \tag{2.6}$$

and

$$\begin{aligned} z'(t) &= a(T,h(t))\left[f(h(t))u(h(t)) + \int_{h(\alpha)}^{h(t)} c(h(t),\sigma)u(\sigma)\,d\sigma\right] h'(t) \\ &\leqslant a(T,h(t))\left[f(h(t))z(h(t)) + \int_{h(\alpha)}^{h(t)} c(h(t),\sigma)z(\sigma)\,d\sigma\right] h'(t). \end{aligned} \tag{2.7}$$

From (2.7) it is easy to observe that

$$\frac{z'(t)}{z(t)} \leqslant a(T,h(t))\Big[f(h(t)) + \int_{h(\alpha)}^{h(t)} c(h(t),\sigma)\,d\sigma\Big]h'(t). \tag{2.8}$$

By setting $t = s$ in (2.8) and integrating it with respect to s from α to T we get

$$z(T) \leqslant z(\alpha)\exp\Big(\int_{\alpha}^{T} a(T,h(s))\Big[f(h(s)) + \int_{h(\alpha)}^{h(s)} c(h(s),\sigma)\,d\sigma\Big]h'(s)\,ds\Big). \tag{2.9}$$

Since T is arbitrary, from (2.9), (2.6) with T replaced by t we have

$$z(t) \leqslant z(\alpha)\exp\Big(\int_{\alpha}^{t} a(t,h(s))\Big[f(h(s)) + \int_{h(\alpha)}^{h(s)} c(h(s),\sigma)\,d\sigma\Big]h'(s)\,ds\Big). \tag{2.10}$$

$$z(\alpha) = k + \int_{h(\alpha)}^{h(\beta)} b(t,s)u(s)\,ds. \tag{2.11}$$

By making the change of variable on the right hand side of (2.10) and using $u(t) \leqslant z(t)$, $t \in I$ we get

$$u(t) \leqslant z(\alpha)\exp(A(t)), \tag{2.12}$$

for $t \in I$. Using (2.12) in (2.11) and in view of (2.2), it is easy to observe that

$$z(\alpha) \leqslant \frac{k}{1-p(t)}. \tag{2.13}$$

The required inequality in (2.4) follows by using (2.13) in (2.12). If $k \geqslant 0$, we carry out the above procedure with $k+\varepsilon$ instead of k, where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\varepsilon \to 0$ to obtain (2.4).

3. An application

Consider the following retarded Volterra-Fredholm integral equation

$$\begin{aligned} x(t) = f(t) &+ \int_{\alpha}^{t} F\Big(t,s,x(s-h(s)),\int_{\alpha}^{s} g(s,\sigma,x(\sigma-h(\sigma)))\,d\sigma\Big)\,ds \\ &+ \int_{\alpha}^{\beta} G(t,s,x(s-h(s)))\,ds, \end{aligned} \tag{3.1}$$

for $t \in I$, where $x(t)$ is an unknown function, $h \in C^1(I,I)$ be nonincreasing with $t-h(t) \geqslant 0$, $h(\alpha) = 0$, $t-h(t) \in C^1(I,I)$, $h'(t) < 1$, $f \in C(I,\mathbf{R}^n)$, $F \in C(D\times\mathbf{R}^n\times\mathbf{R}^n,\mathbf{R}^n)$, $g, G \in C(D\times\mathbf{R}^n,\mathbf{R}^n)$. For the qualitative behaviour of solutions of equations of the form (3.1) without retarded arguments, see [1, 4, 5] and the references cited therein.

The following theorem gives the bound on the solution of equation (3.1).

THEOREM 2. *Suppose that the functions* f, g, G, F, h *in (3.1) satisfy the conditions*

$$|f(t)| \leqslant k, \tag{3.2}$$

$$|g(t,s,x)| \leqslant c(t,s)|x|, \tag{3.3}$$

$$|G(t,s,x)| \leqslant b(t,s)|x|, \tag{3.4}$$

$$|F(t,s,x,y)| \leqslant a(t,s)[|x| + |y|], \tag{3.5}$$

$$L = \max_{t\in I} \frac{1}{1-h'(t)}, \tag{3.6}$$

where $a(t,s)$, $b(t,s)$, $c(t,s)$, k *are as in Theorem 1. Let*

$$\overline{p}(t) = \int_{\alpha}^{\beta-h(\beta)} Lb(t,\xi+h(s))\exp(\overline{A}(\xi))\,d\xi < 1, \tag{3.7}$$

for $t \in I$, *where*

$$\overline{A}(t) = \int_{\alpha}^{t-h(t)} La(t,\xi+h(s))\left[1+\int_{\alpha}^{\xi} Lc(\xi+h(s),\tau+h(\sigma))\,d\tau\right]d\xi, \tag{3.8}$$

for t, s, σ *in* I. *If* $x(t)$ *is any solution of (3.1), then*

$$|x(t)| \leqslant \frac{k}{1-\overline{p}(t)}\exp(\overline{A}(t)), \tag{3.9}$$

for $t \in I$.

Proof. Let $x(t)$ be a solution of (3.1). From (3.1)–(3.6) and making the change of variables we have

$$\begin{aligned}
|x(t)| &\leqslant k + \int_{\alpha}^{t} a(t,s)\left[|x(s-h(s))| + \int_{\alpha}^{s} c(s,\sigma)|x(\sigma-h(\sigma))|\,d\sigma\right]ds \\
&\quad + \int_{\alpha}^{\beta} b(t,s)|x(s-h(s))|\,ds \\
&\leqslant k + \int_{\alpha}^{t} a(t,s)\left[|x(s-h(s))| + \int_{\alpha}^{s-h(s)} Lc(s,\tau+h(\sigma))|x(\tau)|\,d\tau\right]ds \\
&\quad + \int_{\alpha}^{\beta} b(t,s)|x(s-h(s))|\,ds \\
&\leqslant k + \int_{\alpha}^{t-h(t)} La(t,\xi+h(s))\left[|x(\xi)| + \int_{\alpha}^{\xi} Lc(\xi+h(s),\tau+h(\sigma))|x(\tau)|\,d\tau\right]d\xi \\
&\quad + \int_{\alpha}^{\beta-h(\beta)} Lb(t,\xi+h(s))|x(\xi)|\,d\xi,
\end{aligned} \tag{3.10}$$

for t, s, σ in I. Now a suitable application of Theorem 1 to (3.10) in view of (3.7) yields the required estimate in (3.9).

For some other applications of the special version of the inequality given in Theorem 1 with $b(t,s) = 0$ and $I = [\alpha, T)$, we refer the interested readers to [7].

In concluding we note that, if we take in Theorem 1,

(i) $c(t,s) = 0$, $b(t,s) = 0$, $a(t,s) = a(s)$, then it reduces to the inequality given in [3, Corollary on p. 391], for $t \in I$,

(ii) $c(t,s) = 0$, $h(t) = t$, then it reduces to the inequality given in Lemma, which in turn contains as a special case the inequality given in [2, p. 11].

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