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ON GRÜSS TYPE DISCRETE INEQUALITIES

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Abstract. In this paper we establish two new discrete inequalities of the Grüss type by using a fairly elementary analysis.

1. Introduction

In 1935, G. Grüss [1] established the following integral inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx-\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right| \leqslant \frac{1}{4}(C-c)(D-d),$$

provided that f and g are two integrable functions on $[a,b]$ and satisfying the condition

$$c \leqslant f(x) \leqslant C, \qquad d \leqslant g(x) \leqslant D$$

for all $x\in[a,b]$, where c, C, d, D are given real constants.

In the years thereafter, numerious generalizations, variants, extensions and discretizations of the above inequality have appeared in the literature, see the chapter X of the book [2] by Mitrinović, Pečarić and Fink, where further references are given. In this paper we establish two new Grüss type discrete inequalities involving functions of two and three independent variables. The analysis used in the proofs is based on some discrete identities established in [3, 4] and our results provide new estimates on inequalities of this type.

2. Statement of results

In what follows $\mathbf{R}$ denotes the set of real numbers and $\mathbf{N}$ the set of natural numbers. Let $N_1=\{1,2,\ldots,k+1\}$, $N_2=\{1,2,\ldots,m+1\}$, $N_3=\{1,2,\ldots,n+1\}$ for k, m, $n\in\mathbf{N}$ and denote by $G=N_1\times N_2$, $H=N_1\times N_2\times N_3$. For functions $h:\mathbf{N}^2\to\mathbf{R}$ and $e:\mathbf{N}^3\to\mathbf{R}$ we define the operators $\Delta_1 h(x,y)=h(x+1,y)-h(x,y)$, $\Delta_2 h(x,y)=$

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$h(x,y+1)-h(x,y)$; $\Delta_2\Delta_1 h(x,y)=\Delta_2[\Delta_1 h(x,y)]$ and $\Delta_1 e(x,y,z)=e(x+1,y,z)-e(x,y,z)$, $\Delta_2 e(x,y,z)=e(x,y+1,z)-e(x,y,z)$, $\Delta_3 e(x,y,z)=e(x,y,z+1)-e(x,y,z)$; $\Delta_1\Delta_2 e(x,y,z)=\Delta_1[\Delta_2 e(x,y,z)]$, $\Delta_3\Delta_2\Delta_1 e(x,y,z)=\Delta_3[\Delta_2\Delta_1 e(x,y,z)]$.

First, we give the following notations used to simplify the details of presentation.

$$\begin{aligned}
A(\Delta_2\Delta_1 f(x,y)) &= A[1,1;x,y;k,m;\Delta_2\Delta_1 f] \\
&= \sum_{s=1}^{x-1}\sum_{t=1}^{y-1}\Delta_2\Delta_1 f(s,t) - \sum_{s=1}^{x-1}\sum_{t=y}^{m}\Delta_2\Delta_1 f(s,t) \\
&\quad - \sum_{s=x}^{k}\sum_{t=1}^{y-1}\Delta_2\Delta_1 f(s,t) + \sum_{s=x}^{k}\sum_{t=y}^{m}\Delta_2\Delta_1 f(s,t), \\
B(\Delta_3\Delta_2\Delta_1 f(r,s,t)) &= B[1,1,1;r,s,t;\Delta_3\Delta_2\Delta_1 f] \\
&= \sum_{u=1}^{r-1}\sum_{v=1}^{s-1}\sum_{w=1}^{t-1}\Delta_3\Delta_2\Delta_1 f(u,v,w) - \sum_{u=1}^{r-1}\sum_{v=1}^{s-1}\sum_{w=t}^{n}\Delta_3\Delta_2\Delta_1 f(u,v,w) \\
&\quad - \sum_{u=1}^{r-1}\sum_{v=s}^{m}\sum_{w=1}^{t-1}\Delta_3\Delta_2\Delta_1 f(u,v,w) - \sum_{u=r}^{k}\sum_{v=1}^{s-1}\sum_{w=1}^{t-1}\Delta_3\Delta_2\Delta_1 f(u,v,w) \\
&\quad + \sum_{u=1}^{r-1}\sum_{v=s}^{m}\sum_{w=t}^{n}\Delta_3\Delta_2\Delta_1 f(u,v,w) + \sum_{u=r}^{k}\sum_{v=s}^{m}\sum_{w=1}^{t-1}\Delta_3\Delta_2\Delta_1 f(u,v,w) \\
&\quad + \sum_{u=r}^{k}\sum_{v=1}^{s-1}\sum_{w=t}^{n}\Delta_3\Delta_2\Delta_1 f(u,v,w) - \sum_{u=r}^{k}\sum_{v=s}^{m}\sum_{w=t}^{n}\Delta_3\Delta_2\Delta_1 f(u,v,w), \\
E(f(x,y)) &= E[1,1;x,y;k+1,m+1;f] \\
&= \frac{1}{2}[f(x,1)+f(x,m+1)+f(1,y)+f(k+1,y)] \\
&\quad - \frac{1}{4}[f(1,1)+f(1,m+1)+f(k+1,1)+f(k+1,m+1)], \\
L(f(r,s,t)) &= L[1,1,1;r,s,t;k+1,m+1,n+1;f] \\
&= \frac{1}{8}[f(1,1,1)+f(k+1,m+1,n+1) \\
&\quad - \frac{1}{4}[f(r,1,1)+f(r,1,n+1)+f(r,m+1,1)+f(r,m+1,n+1)] \\
&\quad - \frac{1}{4}[f(k+1,s,n+1)+f(k+1,s,1)+f(1,s,n+1)+f(1,s,1)] \\
&\quad - \frac{1}{4}[f(k+1,m+1,t)+f(k+1,1,t)+f(1,m+1,t)+f(1,1,t)] \\
&\quad + \frac{1}{2}[f(1,s,t)+f(k+1,s,t)+\frac{1}{2}[f(r,1,t)+f(r,m+1,t) \\
&\quad + \frac{1}{2}[f(r,s,1)+f(r,s,n+1).
\end{aligned}$$

Our main results are established in the following theorems.

THEOREM 1. *Let* f, $g : G \to \mathbf{R}$ *be functions for which* $\Delta_2\Delta_1 f(x,y)$, $\Delta_2\Delta_1 g(x,y)$ *exist and*

$$|\Delta_2\Delta_1 f(x,y)| \leqslant K_1, \qquad |\Delta_2\Delta_1 g(x,y)| \leqslant K_2,$$

where K_1, K_2 *are given nonnegative constants. Then*

$$\begin{aligned}\Bigg|\sum_{x=1}^{k}\sum_{y=1}^{m} f(x,y)g(x,y) - \frac{1}{2}\sum_{x=1}^{k}\sum_{y=1}^{m}\big(E(f(x,y))g(x,y) + E(g(x,y))f(x,y)\big)\Bigg| \\ \leqslant \frac{1}{8}km\sum_{x=1}^{k}\sum_{y=1}^{m}(K_1|g(x,y)| + K_2|f(x,y)|).\end{aligned} \tag{2.1}$$

THEOREM 2. *Let* p, $q : H \to \mathbf{R}$ *be functions for which* $\Delta_3\Delta_2\Delta_1 p(r,s,t)$, $\Delta_3\Delta_2\Delta_1 q(r,s,t)$ *exist and*

$$|\Delta_3\Delta_2\Delta_1 p(r,s,t)| \leqslant M_1, \qquad |\Delta_3\Delta_2\Delta_1 q(r,s,t)| \leqslant M_2,$$

where M_1, M_2 *are given nonnegative constants. Then*

$$\begin{aligned}\Bigg|\sum_{r=1}^{k}\sum_{s=1}^{m}\sum_{t=1}^{n} p(r,s,t)q(r,s,t) \\ -\frac{1}{2}\sum_{r=1}^{k}\sum_{s=1}^{m}\sum_{t=1}^{n}\big(L(p(r,s,t))q(r,s,t) + L(q(r,s,t))p(r,s,t)\big)\Bigg| \\ \leqslant \frac{1}{16}kmn\sum_{r=1}^{k}\sum_{s=1}^{m}\sum_{t=1}^{n}\big(M_1|q(r,s,t)| + M_2|p(r,s,t)|\big).\end{aligned} \tag{2.2}$$

3. Proof of Theorem 1

From the hypothes, we have the following identities (see [4]):

$$f(x,y) = E(f(x,y)) + \frac{1}{4}A(\Delta_2\Delta_1 f(x,y)), \tag{3.1}$$

$$g(x,y) = E(g(x,y)) + \frac{1}{4}A(\Delta_2\Delta_1 g(x,y)), \tag{3.2}$$

for $(x,y) \in G$. Multiplying (3.1) by $g(x,y)$, (3.2) by $f(x,y)$, adding the resulting identities, summing on G and rewriting we have

$$\begin{aligned}\sum_{x=1}^{k}\sum_{y=1}^{m} f(x,y)g(x,y) &= \frac{1}{2}\sum_{x=1}^{k}\sum_{y=1}^{m}\big(E(f(x,y))g(x,y) + E(g(x,y))f(x,y)\big) \\ &+ \frac{1}{8}\sum_{x=1}^{k}\sum_{y=1}^{m}\big(A(\Delta_2\Delta_1 f(x,y))g(x,y) + A(\Delta_2\Delta_1 g(x,y))f(x,y)\big).\end{aligned} \tag{3.3}$$

From the properties of modulus and sums, it is easy to see that

$$|A(\Delta_2\Delta_1 f(x,y))| \leqslant \sum_{x=1}^{k}\sum_{y=1}^{m} |\Delta_2\Delta_1 f(s,t)| \leqslant (km)K_1, \tag{3.4}$$

$$|A(\Delta_2\Delta_1 g(x,y))| \leqslant \sum_{x=1}^{k}\sum_{y=1}^{m} |\Delta_2\Delta_1 g(s,t)| \leqslant (km)K_2. \tag{3.5}$$

From (3.3)–(3.5) we get

$$\begin{aligned}
&\left|\sum_{x=1}^{k}\sum_{y=1}^{m} f(x,y)g(x,y) - \frac{1}{2}\sum_{x=1}^{k}\sum_{y=1}^{m}\big(E(f(x,y))g(x,y) + E(g(x,y))f(x,y)\big)\right| \\
&\quad \leqslant \frac{1}{8}\sum_{x=1}^{k}\sum_{y=1}^{m}\big(|A(\Delta_2\Delta_1 f(x,y))||g(x,y)| + |A(\Delta_2\Delta_1 g(x,y))||f(x,y)|\big) \\
&\quad \leqslant \frac{1}{8}km\sum_{x=1}^{k}\sum_{y=1}^{m}\big(K_1|g(x,y)| + K_2|f(x,y)|\big),
\end{aligned}$$

which is the required inequality in (2.1) and the proof is complete.

4. Proof of Theorem 2

From the hypotheses, we have the following identities (see [3]):

$$p(r,s,t) = L(p(r,s,t)) + \frac{1}{8}B(\Delta_3\Delta_2\Delta_1 p(r,s,t)), \tag{4.1}$$

$$q(r,s,t) = L(q(r,s,t)) + \frac{1}{8}B(\Delta_3\Delta_2\Delta_1 q(r,s,t)), \tag{4.2}$$

for $(r,s,t) \in H$. Multiplying (4.1) by $q(r,s,t)$, (4.2) by $p(r,s,t)$, adding the resulting identities, summing on H and rewriting we have

$$\begin{aligned}
&\sum_{r=1}^{k}\sum_{s=1}^{m}\sum_{t=1}^{n} p(r,s,t)q(r,s,t) \\
&\quad = \frac{1}{2}\sum_{r=1}^{k}\sum_{s=1}^{m}\sum_{t=1}^{n}\big(L(p(r,s,t))q(r,s,t) + L(q(r,s,t))p(r,s,t)\big) \\
&\quad + \frac{1}{16}\sum_{r=1}^{k}\sum_{s=1}^{m}\sum_{t=1}^{n}\big(B(\Delta_3\Delta_2\Delta_1 p(r,s,t))q(r,s,t) + B(\Delta_3\Delta_2\Delta_1 q(r,s,t))p(r,s,t)\big).
\end{aligned} \tag{4.3}$$

From the properties of modulus and sums, we observe that

$$|B(\Delta_3\Delta_2\Delta_1 p(r,s,t))| \leqslant \sum_{u=1}^{k}\sum_{v=1}^{m}\sum_{w=1}^{n} |\Delta_3\Delta_2\Delta_1 p(u,v,w)| \leqslant (kmn)M_1, \tag{4.4}$$

$$|B(\Delta_3\Delta_2\Delta_1 q(r,s,t))| \leqslant \sum_{u=1}^{k}\sum_{v=1}^{m}\sum_{w=1}^{n} |\Delta_3\Delta_2\Delta_1 q(u,v,w)| \leqslant (kmn)M_2. \tag{4.5}$$

Now, from (4.3)–(4.5) and following the same argument as in the proof of Theorem 1 with suitable changes we get the required inequality in (2.2). The proof is complete.

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