Mathematical Inequalities & Applications
Volume 7, Number 1 (2004), 19–26

ON THE GENERALIZED HARDY–HILBERT INEQUALITY AND ITS APPLICATIONS

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(*communicated by J. Pečarić*)

Abstract. A generalized Hardy-Hilbert inequality with weight function of the form $B\left(\frac{p-2+\lambda}{p}, \frac{q-2+\lambda}{q}\right) - \theta_r(\lambda)/(2n+1)^{\lambda-\frac{2-\lambda}{r}}$ (with $\theta_r(\lambda) > 0$, $r = p, q$, $1-\frac{q}{p} < \lambda \leqslant 2$, $\frac{1}{p}+\frac{1}{q} = 1$ and $p \geqslant q > 1$) can be established by means of Euler-Maclaurin summation formula, where $B(m,n)$ is β function. In particular, when $\lambda = 1$, an improvement on Hardy-Hilbert's inequality is obtained. As its applications, Hardy-Littlewood's inequality is extended and refined.

1. Introduction

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers, $\frac{1}{p}+\frac{1}{q} = 1$ and $p \geqslant q > 1$. If $0 < \sum\limits_{n=0}^{\infty} a_n^p < +\infty$ and $0 < \sum\limits_{n=0}^{\infty} b_n^q < +\infty$, then

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{m+n+1} < \frac{\pi}{\sin\frac{\pi}{p}}\left(\sum_{n=0}^{\infty} a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=0}^{\infty} b_n^q\right)^{\frac{1}{q}} \tag{1}$$

where the constant factor $\frac{\pi}{\sin\frac{\pi}{p}}$ is best possible. This is famous Hardy-Hilbert's inequality (see [1]). Recently, Yang and Debnath [2] generalized this result. To be specific, they established the following inequality:

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}} &< B\left(\frac{p-2+\lambda}{p}, \frac{q-2+\lambda}{q}\right)\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda} a_n^p\right\}^{\frac{1}{p}} \\ &\times \left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda} b_n^q\right\}^{\frac{1}{q}}\end{aligned} \tag{2}$$

where the constant $B\left(\frac{p-2+\lambda}{p}, \frac{q-2+\lambda}{q}\right)$ is best possible.

The main purpose of this paper is to establish a strengthened result of (2) and to realize extensions and refinement of Hardy-Littlewood's inequality.

Mathematics subject classification (2000): 26D15.
Key words and phrases: Hardy-Hilbert's inequality, weight function, β function, Hardy-Littlewood's inequality.

2. Lemmas

In order to verify our main results, we need the following lemmas.

LEMMA 1. *Let* $f(x)$ *be a continuous and differentiable function in* $[0,+\infty)$. *If* $f(x)\downarrow 0$, *then*

$$\sum_{k=0}^{\infty} f(k) = \int_0^{\infty} f(x)\,dx + \frac{1}{2}f(0) + \int_0^{\infty} \rho(x)f'(x)\,dx, \tag{3}$$

where $\rho(x) = x - [x] - \frac{1}{2}$.

Proof. It follows from the paper [3] that

$$\sum_{k=n+1}^{m} f(k) = \int_n^{m} f(x)\,dx + \frac{1}{2}(f(m) - f(n)) + \int_n^{m} \rho(x)f'(x)\,dx, \tag{4}$$

where $\rho(x) = x - [x] - \frac{1}{2}$.

Let $m \to \infty$, $n = 0$ and notice that $f(x)\downarrow 0$. It follows from (4) that the equality (3) holds after simplifications.

LEMMA 2. *Let* $\varphi(x)\downarrow 0$. *Then*

$$-\int_0^{\infty} \rho(x)\varphi(x)\,dx < \frac{1}{8}\varphi(0), \tag{5}$$

where $\rho(x) = x - [x] - \frac{1}{2}$.

Proof. Since $\int_k^{k+1} \rho(x)\,dx = 0$, $k = 0, 1, 2, \ldots$ we have

$$\begin{aligned}
-\int_0^{\infty} \rho(x)\varphi(x)\,dx &= \sum_{k=0}^{\infty} \int_k^{k+1} -\rho(x)\left(\varphi(x) - \varphi\left(k+\frac{1}{2}\right)\right) dx \\
&= \sum_{k=0}^{\infty} \left\{ \int_k^{k+\frac{1}{2}} -\rho(x)\left(\varphi(k) - \varphi\left(k+\frac{1}{2}\right)\right) dx \right. \\
&\qquad \left. + \int_{k+\frac{1}{2}}^{k+1} \rho(x)\left(\varphi\left(k+\frac{1}{2}\right) - \varphi(k+1)\right) dx \right\} + \sum_{k=0}^{\infty} \alpha_k \\
&= \frac{1}{8}\varphi(0) + \sum_{k=0}^{\infty} \alpha_k,
\end{aligned}$$

where

$$\alpha_k = \int_k^{k+\frac{1}{2}} -\rho(x)(\varphi(x) - \varphi(k))\,dx + \int_{k+\frac{1}{2}}^{k+1} \rho(x)(\varphi(k+1) - \varphi(x))\,dx.$$

Due to the fact that $\varphi(x)\downarrow 0$, hence $\alpha_k < 0$. Whence $\sum_{k=0}^{\infty} \alpha_k < 0$. It follows that the inequality (5) keeps valid.

LEMMA 3. *Let* $\frac{1}{p}+\frac{1}{q}=1$, $p \geqslant q > 1$, $n \in \mathbf{N}_0$ *and* $1-\frac{q}{p} < \lambda \leqslant 2$. *Define the function* F_r *by*

$$F_r(n,\lambda) = \Big(n+\frac{1}{2}\Big)^{1-\lambda} \int_0^{\frac{1}{2n+1}} \frac{1}{(1+x)^\lambda} \Big(\frac{1}{x}\Big)^{\frac{2-\lambda}{r}} dx, \tag{6}$$

where $r = p,\ q$. *Then*

$$F_r(n.\lambda) > \frac{(2n+1)^{\frac{2-\lambda}{r}}}{2(n+1)^\lambda} \left\{ \frac{1}{1-\frac{2-\lambda}{r}} + \frac{\lambda}{4(n+1)} \right\}. \tag{7}$$

Proof. Applying integration by parts to $F_r(n,\lambda)$ we have

$$\begin{aligned} F_r(n.\lambda) &= \frac{(2n+1)^{\frac{2-\lambda}{r}}}{2\big(1-\frac{2-\lambda}{r}\big)(n+1)^\lambda} + \Big(n+\frac{1}{2}\Big)^{1-\lambda} \frac{\lambda}{1-\frac{2-\lambda}{r}} \int_0^{\frac{1}{2n+1}} \frac{1}{(1+x)^{\lambda+1}} x^{1-\frac{2-\lambda}{r}} dx \\ &\geqslant \frac{(2n+1)^{\frac{2-\lambda}{r}}}{2\big(1-\frac{2-\lambda}{r}\big)(n+1)^\lambda} + \lambda \Big(n+\frac{1}{2}\Big)^{1-\lambda} \int_0^{\frac{1}{2n+1}} \frac{1}{(1+x)^{\lambda+1}} x^{1-\frac{2-\lambda}{r}} dx. \end{aligned} \tag{8}$$

Owing to the fact that

$$\begin{aligned} \int_0^{\frac{1}{2n+1}} \frac{x^{1-\frac{2-\lambda}{r}}}{(1+x)^{\lambda+1}} dx &> \int_0^{\frac{1}{2n+1}} \frac{x^{1-\frac{2-\lambda}{r}}}{\big(1+\frac{1}{2n+1}\big)^{\lambda+1}} dx \\ &= \frac{(2n+1)^{\lambda-1+\frac{2-\lambda}{r}}}{2^{\lambda+1}(n+1)^{\lambda+1}\big(2-\frac{2-\lambda}{r}\big)} \\ &\geqslant \frac{(2n+1)^{\lambda-1+\frac{2-\lambda}{r}}}{2^{\lambda+2}(n+1)^{\lambda+1}}. \end{aligned} \tag{9}$$

It shows from (8) and (9) that the inequality (7) is valid affter simplifications.

LEMMA 4. *With the same assumption as in Lemma 3, define the function* f *by*

$$f(x) = \frac{1}{(x+n+1)^\lambda} \Big(\frac{2n+1}{2x+1}\Big)^{\frac{2-\lambda}{r}}, \qquad x \in [0,+\infty). \tag{10}$$

Then

$$F_r(n,\lambda) - \frac{1}{2}f(0) - \int_0^\infty \rho(x) f'(x)\, dx > \frac{(2-\lambda)(r+2-\lambda)}{4r(r-2+\lambda)}, \tag{11}$$

where $\rho(x) = x - [x] - \frac{1}{2}$.

Proof. Let $\varphi(x) = -f'(x)$. Then $\varphi(x) \downarrow 0$. By Lemma 2 we have

$$\begin{aligned} \int_0^\infty \rho(x) f'(x)\, dx &= -\int_0^\infty \rho(x)\varphi(x)\, dx < \frac{1}{8}\varphi(0) \\ &= \frac{(2n+1)^{\frac{2-\lambda}{r}}}{8(n+1)^\lambda} \Big(\frac{\lambda}{n+1} + \frac{2(2-\lambda)}{r}\Big). \end{aligned}$$

According to the inequality (7) we can obtain

$$
\begin{aligned}
&F_r(n,\lambda)-\frac{1}{2}f(0)-\int_0^{\infty}\rho(x)f'(x)\,dx \\
&\qquad > \frac{(2n+1)^{\frac{2-\lambda}{r}}}{2(n+1)^{\lambda}}\left\{\frac{1}{1-\frac{2-\lambda}{r}}+\frac{\lambda}{4(n+1)}\right\}-\frac{(2n+1)^{\frac{2-\lambda}{r}}}{2(n+1)^{\lambda}} \\
&\qquad\quad -\frac{(2n+1)^{\frac{2-\lambda}{r}}}{8(n+1)^{\lambda}}\left\{\frac{\lambda}{n+1}+\frac{2(2-\lambda)}{r}\right\} \\
&\qquad = \frac{(2n+1)^{\frac{2-\lambda}{r}}}{4r(n+1)^{\lambda}}\left\{\frac{(2-\lambda)\left(1+\frac{2-\lambda}{r}\right)}{1-\frac{2-\lambda}{r}}\right\} \\
&\qquad > \frac{(2n+1)^{\frac{2-\lambda}{r}}}{4r(2n+1)^{\lambda}}\left\{\frac{(2-\lambda)\left(1+\frac{2-\lambda}{r}\right)}{1-\frac{2-\lambda}{r}}\right\} \\
&\qquad = \frac{\theta_r(\lambda)}{(2n+1)^{\lambda-\frac{2-\lambda}{r}}}
\end{aligned}
$$

where $\theta_r(\lambda)=\dfrac{(2-\lambda)(r+2-\lambda)}{4r(r-2+\lambda)}$.

Thus the lemma is proved.

3. Main results

In this section, we shall prove our main theorem and establish some important corollaries.

THEOREM 1. *Let* $a_n,\ b_n \geqslant 0\ \ (n=0,1,2,\ldots)$, $\frac{1}{p}+\frac{1}{q}=1$, $p\geqslant q>1$ *and* $1-\frac{q}{p}<\lambda\leqslant 2$. *If* $0<\sum\limits_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda}a_n^p<+\infty$ *and* $0<\sum\limits_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda}b_n^p<+\infty$, *then*

$$
\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}}<\left\{\sum_{n=0}^{\infty}\omega_q(\lambda,n)a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty}\omega_p(\lambda,n)b_n^q\right\}^{\frac{1}{q}} \tag{12}
$$

where $\omega_r(\lambda,n)=\left(n+\dfrac{1}{2}\right)^{1-\lambda}\left\{B\left(\dfrac{p-2+\lambda}{p},\dfrac{q-2+\lambda}{q}\right)-\dfrac{\theta_r(\lambda)}{(2n+1)^{\lambda-\frac{2-\lambda}{r}}}\right\}$ *and* $\theta_r(\lambda)=\dfrac{(2-\lambda)(r+2-\lambda)}{4r(r+\lambda-2)}$, $r=p,\ q$.

Proof. We may apply Hölder's inequality to estimate the left-hand side of (12) as follows:

$$
\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m}{(m+n+1)^{\frac{\lambda}{p}}}\left(\frac{2m+1}{2n+1}\right)^{\frac{2-\lambda}{pq}}
$$

$$\cdot \frac{b_n}{(m+n+1)^{\frac{\lambda}{q}}}\left(\frac{2n+1}{2m+1}\right)^{\frac{2-\lambda}{pq}}$$

$$\leqslant \left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m^p}{(m+n+1)^{\lambda}}\left(\frac{2m+1}{2n+1}\right)^{\frac{2-\lambda}{q}}\right\}^{\frac{1}{p}}$$

$$\cdot \left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{b_n^q}{(m+n+1)^{\lambda}}\left(\frac{2n+1}{2m+1}\right)^{\frac{2-\lambda}{p}}\right\}^{\frac{1}{q}}$$

$$= \left\{\sum_{n=0}^{\infty}\omega_q(\lambda,n)a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty}\omega_p(\lambda,n)b_n^q\right\}^{\frac{1}{q}}$$

where $\omega_r(\lambda,n)=\sum_{m=0}^{\infty}\frac{1}{(m+n+1)^{\lambda}}\left(\frac{2n+1}{2m+1}\right)^{\frac{2-\lambda}{r}}$, $r=p,\ q$.

Let $f(x)$ be the function defined by (10). Clearly, $f(x)$ is continuous and differentiable in $[0,+\infty)$ and $f(x)\downarrow 0$. In view of Lemma 1 we have

$$\omega_r(\lambda,n)=\int_0^{\infty}f(x)\,dx+\frac{1}{2}f(0)+\int_0^{\infty}\rho(x)f'(x)\,dx, \tag{13}$$

where $\rho(x)=x-[x]-\frac{1}{2}$.

It is easy to deduce that

$$\begin{aligned}
\int_0^{\infty}f(x)\,dx &= \left(n+\frac{1}{2}\right)^{-\lambda}\int_0^{\infty}\frac{1}{\left(1+\frac{2x+1}{2n+1}\right)^{\lambda}}\left(\frac{2n+1}{2x+1}\right)^{\frac{2-\lambda}{r}}dx\\
&= \left(n+\frac{1}{2}\right)^{1-\lambda}\int_{\frac{1}{2n+1}}^{\infty}\frac{1}{(1+u)^{\lambda}}\left(\frac{1}{u}\right)^{\frac{2-\lambda}{r}}du\\
&= \left(n+\frac{1}{2}\right)^{1-\lambda}\left\{\int_0^{\infty}\frac{1}{(1+u)^{\lambda}}\left(\frac{1}{u}\right)^{\frac{2-\lambda}{r}}du-\int_0^{\frac{1}{2n+1}}\frac{1}{(1+u)^{\lambda}}\left(\frac{1}{u}\right)^{\frac{2-\lambda}{r}}du\right\}\\
&= \left(n+\frac{1}{2}\right)^{1-\lambda}B-F_r(n,\lambda),
\end{aligned} \tag{14}$$

where $F_r(n,\lambda)$ is defined by (6), and B is β function, here $B=B\left(\lambda-\left(1-\frac{2-\lambda}{r}\right),1-\frac{2-\lambda}{r}\right)$.

It follows from (13), (14) and (11) that

$$\begin{aligned}
\omega_r(\lambda,n) &= \left(n+\frac{1}{2}\right)^{1-\lambda}B-\left\{F_r(n,\lambda)-\frac{1}{2}f(0)-\int_0^{\infty}\rho(x)f'(x)\,dx\right\}\\
&< \left(n+\frac{1}{2}\right)^{1-\lambda}B-\frac{\theta_r(\lambda)}{(2n+1)^{\lambda-\frac{2-\lambda}{r}}},
\end{aligned}$$

where $\theta_r(\lambda)=\frac{(2-\lambda)(r+2-\lambda)}{4r(r-2+\lambda)}$.

Notice that $\frac{1}{p}+\frac{1}{q}=1$, hence $\lambda-\left(1-\frac{2-\lambda}{q}\right)=\frac{p-2+\lambda}{p}$. Whence

$$B\Big(\lambda-\Big(1-\frac{2-\lambda}{q}\Big),1-\frac{2-\lambda}{q}\Big)=B\Big(\frac{p-2+\lambda}{p},\frac{q-2+\lambda}{q}\Big).$$

Thus the proof of the theorem is completed.

When $p=q=2$, we obtain an improvement of the correspondent result of the paper [2].

COROLLARY 1. *If* $0<\sum\limits_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda}a_n^2<+\infty$ *and* $0<\sum\limits_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-\lambda}b_n^2<+\infty$*, then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_mb_n}{(m+n+1)^{\lambda}}<\left\{\sum_{n=0}^{\infty}\Big(n+\frac{1}{2}\Big)^{1-\lambda}\Big(B\Big(\frac{\lambda}{2},\frac{\lambda}{2}\Big)-\frac{\theta(\lambda)}{(2n+1)^{\frac{3}{2}\lambda-1}}\Big)a_n^2\right\}^{\frac{1}{2}}$$
$$\times\left\{\sum_{n=0}^{\infty}\Big(n+\frac{1}{2}\Big)^{1-\lambda}\Big(B\Big(\frac{\lambda}{2},\frac{\lambda}{2}\Big)-\frac{\theta(\lambda)}{(2n+1)^{\frac{3}{2}\lambda-1}}\Big)b_n^2\right\}^{\frac{1}{2}}$$

where $\theta(\lambda)=\dfrac{(2-\lambda)(4-\lambda)}{8\lambda}$*,* $B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$ *is* β *function.*

In particular, for $\lambda=2$, $\theta(2)=0$, the result of the paper [2] is obtained.
When $\lambda=1$, $B\left(\frac{1}{2},\frac{1}{2}\right)=\pi$, we can attain a refinement of Hilbert's inequality.

COROLLARY 2. *If* $0<\sum\limits_{n=0}^{\infty}a_n^2<+\infty$ *and* $0<\sum\limits_{n=0}^{\infty}b_n^2<+\infty$*, then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_mb_n}{m+n+1}<\left\{\sum_{n=0}^{\infty}\Big(\pi-\frac{3}{8\sqrt{2n+1}}\Big)a_n^2\right\}^{\frac{1}{2}}\left\{\sum_{n=0}^{\infty}\Big(\pi-\frac{3}{8\sqrt{2n+1}}\Big)b_n^2\right\}^{\frac{1}{2}}.$$

Also, for $\lambda=1$, it is observed from Theorem 1 that $B\left(\frac{p-1}{p},\frac{q-1}{q}\right)=B\left(1-\frac{1}{p},\frac{1}{q}\right)=\frac{\pi}{\sin\frac{\pi}{p}}$, we get consequently a sharp result on the inequality (1).

COROLLARY 3. *If* $0<\sum\limits_{n=0}^{\infty}a_n^p<+\infty$ *and* $0<\sum\limits_{n=0}^{\infty}b_n^q<+\infty$*, then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_mb_n}{m+n+1}<\left\{\sum_{n=0}^{\infty}\Big(\frac{\pi}{\sin\frac{\pi}{p}}-\frac{\theta_q}{(2n+1)^{\frac{1}{p}}}\Big)a_n^p\right\}^{\frac{1}{p}}$$
$$\times\left\{\sum_{n=0}^{\infty}\Big(\frac{\pi}{\sin\frac{\pi}{p}}-\frac{\theta_p}{(2n+1)^{\frac{1}{q}}}\Big)b_n^q\right\}^{\frac{1}{q}},$$

where $\theta_r=\dfrac{r+1}{4r(r-1)}$*,* $r=p,\ q$.

4. Applications

In this section, according to the above – mentioned result we will give extensions and a refinement on Hardy-Littlewood's inequality.

Let $f(x) \in L^2(0,1)$ and $f(x) \neq 0$. If

$$a_n = \int_0^1 x^n f(x)\,dx, \qquad n = 0,1,2,\ldots$$

then

$$\sum_{n=0}^{\infty} a_n^2 < \pi \int_0^1 f^2(x)\,dx, \tag{15}$$

where π is the best constant that keeps (15) valid. The inequality (15) is called Hardy-Littlewood's inequality (see $[1]$). With the use of Corollary 3 we establish firstly an extension of (15).

THEOREM 2. *With the above assumption, then*

$$\left\{\sum_{n=0}^{\infty} a_n^2\right\}^2 < \left\{\sum_{n=0}^{\infty}\left(\frac{\pi}{\sin\frac{\pi}{p}} - \alpha_q(n)\right)a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=0}^{\infty}\left(\frac{\pi}{\sin\frac{\pi}{p}} - \alpha_p(n)\right)a_n^q\right\}^{\frac{1}{q}} \int_0^1 f^2(x)\,dx \tag{16}$$

where $\alpha_r(n) = \dfrac{r+1}{4r(r-1)(2n+1)^{1-\frac{1}{r}}}$, $r = p,\ q$.

Proof. By our assumption, we have

$$a_n^2 = \int_0^1 a_n x^n f(x)\,dx.$$

Using the Cauchy-Schwarz inequality and Corollary 3 we obtain

$$\begin{aligned}
\left(\sum_{n=0}^{\infty} a_n^2\right)^2 &= \left\{\sum_{n=0}^{\infty}\int_0^1 a_n x^n f(x)\,dx\right\}^2 \\
&= \left\{\int_0^1 \left(\sum_{n=0}^{\infty} a_n x^n\right) f(x)\,dx\right\}^2 \\
&\leqslant \int_0^1 \left(\sum_{n=0}^{\infty} a_n x^n\right)^2 dx \int_0^1 f^2(x)\,dx \\
&= \int_0^1 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_m a_n x^{m+n}\,dx \int_0^1 f^2(x)\,dx \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m a_n}{m+n+1}\int_0^1 f^2(x)\,dx \\
&\leqslant \left\{\sum_{n=0}^{\infty}\omega_q(n) a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=0}^{\infty}\omega_p(n) a_n^q\right\}^{\frac{1}{q}} \int_0^1 f^2(x)\,dx,
\end{aligned} \tag{17}$$

where $\omega_r(n) = \frac{\pi}{\sin\frac{\pi}{p}} - \frac{r+1}{4r(r-1)(2n+1)^{1-\frac{1}{r}}}$, $r = p, q$.

Since $f(x) \neq 0$, $a_n^2 \neq 0$ for $n \in \mathbf{N}_0$. Therefore it is impossible to take equality in (17). It shows that the equality (16) is valid. Thus the theorem is proved.

If $\alpha_r(n)$ $(r = p, q)$ is replaced by zero, then we attain the following.

COROLLARY 4. *With the assumption as in Theorem 2, then*

$$\left(\sum_{n=0}^{\infty} a_n^2\right)^2 < \frac{\pi}{\sin\frac{\pi}{p}}\left(\sum_{n=0}^{\infty} a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=0}^{\infty} a_n^q\right)^{\frac{1}{q}}\int_0^1 f^2(x)\,dx. \tag{18}$$

It is evident that the inequality (18) is also an extension on (15).

At last, we establish an improvement on (15).

COROLLARY 5. *Assume that* $p = q = 2$, *then*

$$\left(\sum_{n=0}^{\infty} a_n^2\right)^2 < \sum_{n=0}^{\infty}\left(\pi - \frac{3}{8\sqrt{2n+1}}\right)a_n^2\int_0^1 f^2(x)\,dx.$$

This is obviously an immediate result of Theorem 2.

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(Received June 10, 2002)

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