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ON EULER–BOOLE FORMULAE

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Abstract. A number of inequalities, for functions whose derivatives are either functions of bounded variation or Lipschitzian functions or R-integrable functions, is proved by applying the Euler-Boole formulae. The results are applied to obtain some error estimates for the Boole quadrature rules.

1. Introduction

One of the quadrature rules of closed type is Boole's rule based on Boole's formula [3, p. 63]

$$\int_0^1 f(t)\mathrm{d}t = \frac{1}{90}\left[7f(0)+32f\left(\frac{1}{4}\right)+12f\left(\frac{1}{2}\right)+32f\left(\frac{3}{4}\right)+7f(1)\right] - \frac{1}{1935360}f^{(6)}(\xi), \tag{1.1}$$

where $0 \leqslant \xi \leqslant 1$. Formula (1.1) is valid for any function f with continuous sixth derivative $f^{(6)}$ on $[0,1]$. In the recent paper [4] the following two identities, named the extended Euler formulae, have been proved. For $n \geqslant 1$

$$f(x) = \int_0^1 f(t)\mathrm{d}t + T_n(x) + R_n^1(x) \tag{1.2}$$

and

$$f(x) = \int_0^1 f(t)\mathrm{d}t + T_{n-1}(x) + R_n^2(x), \tag{1.3}$$

where $T_0(x) = 0$ and

$$T_m(x) = \sum_{k=1}^{m} \frac{B_k(x)}{k!}\left[f^{(k-1)}(1) - f^{(k-1)}(0)\right], \tag{1.4}$$

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for $1 \leqslant m \leqslant n$, while

$$R_n^1(x) = -\frac{1}{n!} \int_0^1 B_n^*(x-t)\,\mathrm{d}f^{(n-1)}(t)$$

and

$$R_n^2(x) = -\frac{1}{n!} \int_0^1 [B_n^*(x-t) - B_n(x)]\,\mathrm{d}f^{(n-1)}(t).$$

Here, as in the rest of the paper, we write $\int_0^1 g(t)\,\mathrm{d}\varphi(t)$ to denote the Riemann-Stieltjes integral with respect to a function $\varphi : [0,1] \to \mathbf{R}$ of bounded variation, and $\int_0^1 g(t)\,\mathrm{d}t$ for the Riemann integral. The identities (1.2) and (1.3) extend the well known formula for the expansion of an arbitrary function in Bernoulli polynomials [6, p. 17]. They hold for every function $f : [0,1] \to \mathbf{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$, for some $n \geqslant 1$, and for every $x \in [0,1]$. The functions $B_k(t)$ are the Bernoulli polynomials, $B_k = B_k(0)$ are the Bernoulli numbers, and $B_k^*(t)$, $k \geqslant 0$, are periodic functions of period 1, related to the Bernoulli polynomials as

$$B_k^*(t) = B_k(t),\ 0 \leqslant t < 1, \text{ and } B_k^*(t+1) = B_k^*(t),\ t \in \mathbf{R}.$$

The Bernoulli polynomials $B_k(t)$, $k \geqslant 0$ are uniquely determined by the following identities

$$B_k'(t) = kB_{k-1}(t), \quad k \geqslant 1; \quad B_0(t) = 1 \tag{1.5}$$

and

$$B_k(t+1) - B_k(t) = kt^{k-1}, \quad k \geqslant 0. \tag{1.6}$$

For some further details on the Bernoulli polynomials and the Bernoulli numbers see for example [1] or [2]. We have

$$\begin{aligned} &B_0(t) = 1,\ B_1(t) = t - \frac{1}{2},\ B_2(t) = t^2 - t + \frac{1}{6},\ B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t \\ &B_4(t) = t^4 - 2t^3 + t^2 - \frac{1}{30},\ B_5(t) = t^5 - \frac{5}{2}t^4 + \frac{5}{3}t^3 - \frac{1}{6}t, \end{aligned} \tag{1.7}$$

so that $B_0^*(t) = 1$ and $B_1^*(t)$ is a discontinuous function with a jump of -1 at each integer. From (1.6) it follows that $B_k(1) = B_k(0) = B_k$ for $k \geqslant 2$, so that $B_k^*(t)$ are continuous functions for $k \geqslant 2$. Moreover, using (1.5) we get

$$B_k^{*\prime}(t) = kB_{k-1}^*(t), \tag{1.8}$$

for every $t \in \mathbf{R}$ when $k \geqslant 3$, and for every $t \in \mathbf{R} \setminus \mathbf{Z}$ when $k = 1, 2$.

The aim of this paper is to establish generalizations of the Boole formula (1.1) and give various error estimates for the quadrature rules based on such generalizations.

In Section 2 we use the extended Euler formulae (1.2) and (1.3) to obtain two new integral identities. We call them the Euler-Boole formulae, since they generalize the Boole formula (1.1).

In Section 3 we prove a number of inequalities related to the Euler-Boole formulae, for functions whose derivatives are either functions of bounded variation or Lipschitzian functions or R-integrable functions.

2. Euler-Boole formulae

For $k \geqslant 1$ define the functions $G_k(t)$ and $F_k(t)$ as

$$G_k(t) = 14B_k^*(1-t) + 32B_k^*\left(\frac{1}{4}-t\right) + 12B_k^*\left(\frac{1}{2}-t\right) + 32B_k^*\left(\frac{3}{4}-t\right),\ t \in \mathbf{R}$$

and

$$F_k(t) = G_k(t) - \widetilde{B}_k,\quad t \in \mathbf{R},\ k \geqslant 1,$$

where

$$\widetilde{B}_k = 7B_k(0) + 32B_k\left(\frac{1}{4}\right) + 12B_k\left(\frac{1}{2}\right) + 32B_k\left(\frac{3}{4}\right) + 7B_k(1),\quad k \geqslant 1.$$

Especially, using (1.7) we get

$$\widetilde{B}_1 = \widetilde{B}_2 = \widetilde{B}_3 = \widetilde{B}_4 = \widetilde{B}_5 = 0.$$

Also, for $k \geqslant 2$ we have $\widetilde{B}_k = G_k(0)$, that is

$$F_k(t) = G_k(t) - G_k(0),\ k \geqslant 2,\ \text{ and } F_1(t) = G_1(t),\quad t \in \mathbf{R}.$$

Obviously, $G_k(t)$ and $F_k(t)$ are periodic functions of period 1 and continuous for $k \geqslant 2$.

Let $f : [0,1] \to \mathbf{R}$ be such that $f^{(n-1)}$ exists on $[0,1]$ for some $n \geqslant 1$. We introduce the following notation

$$D(0,1) = \frac{1}{90}\left[7f(0) + 32f\left(\frac{1}{4}\right) + 12f\left(\frac{1}{2}\right) + 32f\left(\frac{3}{4}\right) + 7f(1)\right].$$

Further, we define $\widetilde{T}_0(0,1) = 0$ and, for $1 \leqslant m \leqslant n$,

$$\widetilde{T}_m(0,1) = \frac{1}{90}\left[7T_m(0) + 32T_m\left(\frac{1}{4}\right) + 12T_m\left(\frac{1}{2}\right) + 32T_m\left(\frac{3}{4}\right) + 7T_m(1)\right],$$

where $T_m(x)$ is given by (1.4). It is easy to see that $\widetilde{T}_1(0,1) = \widetilde{T}_2(0,1) = \widetilde{T}_3(0,1) = \widetilde{T}_4(0,1) = \widetilde{T}_5(0,1) = 0$ and for $m \geqslant 6$

$$\widetilde{T}_m(0,1) = \frac{1}{90}\sum_{k=6}^{m}\frac{\widetilde{B}_k}{k!}\left[f^{(k-1)}(1) - f^{(k-1)}(0)\right]. \tag{2.1}$$

In the next theorem we establish two formulae which play the key role in this paper. We call them the Euler-Boole formulae.

THEOREM 1. *Let $f : [0,1] \to \mathbf{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$, for some $n \geqslant 1$. Then*

$$\int_0^1 f(t)\mathrm{d}t = D(0,1) - \widetilde{T}_n(0,1) + \widetilde{R}_n^1(f), \tag{2.2}$$

and

$$\int_0^1 f(t)\mathrm{d}t = D(0,1) - \widetilde{T}_{n-1}(0,1) + \widetilde{R}_n^2(f), \tag{2.3}$$

where

$$\widetilde{R}_n^1(f) = \frac{1}{90(n!)} \int_0^1 G_n(t)\,\mathrm{d}f^{(n-1)}(t),$$

and

$$\widetilde{R}_n^2(f) = \frac{1}{90(n!)} \int_0^1 F_n(t)\,\mathrm{d}f^{(n-1)}(t).$$

Proof. Put

$$x = 0,\ \frac{1}{4},\ \frac{1}{2},\ \frac{3}{4},\ 1$$

in formula (1.2) to get five new formulae. Then multiply these new formulae by

$$\frac{7}{90},\ \frac{32}{90},\ \frac{12}{90},\ \frac{32}{90},\ \frac{7}{90}$$

respectively, and add. The result is formula (2.2). Formula (2.3) is obtained from (1.3) by the same procedure. □

REMARK 1. The interval $[0,1]$ is used for simplicity and involves no loss in generality. In what follows this theorem and others will be applied, without comment, to any interval that is convenient.

So, it is simply to prove that if $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$, for some $n \geqslant 1$,

$$\int_a^b f(t)dt = D(a,b) - \widetilde{T}_n(a,b) + \frac{(b-a)^n}{90(n!)} \int_a^b G_n\left(\frac{t-a}{b-a}\right) df^{(n-1)}(t) \tag{2.4}$$

and

$$\int_a^b f(t)dt = D(a,b) - \widetilde{T}_{n-1}(a,b) + \frac{(b-a)^n}{90(n!)} \int_a^b F_n\left(\frac{t-a}{b-a}\right) df^{(n-1)}(t), \tag{2.5}$$

where

$$D(a,b) := \frac{b-a}{90}\left[7f(a) + 32\left(\frac{3a+b}{4}\right) + 12\left(\frac{2a+2b}{4}\right) + 32\left(\frac{a+3b}{4}\right) + 7f(b)\right],$$

and

$$\widetilde{T}_m(a,b) = \frac{1}{90}\sum_{k=1}^{m} \frac{(b-a)^k}{k!}\widetilde{B}_k\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right].$$

REMARK 2. Suppose that $f : [0,1] \to \mathbf{R}$ is such that $f^{(n)}$ exists and is integrable on $[0,1]$, for some $n \geqslant 1$. In this case (2.2) holds with

$$\widetilde{R}_n^1(f) = \frac{1}{90(n!)} \int_0^1 G_n(t) f^{(n)}(t)\mathrm{d}t,$$

while (2.3) holds with

$$\widetilde{R}_n^2(f) = \frac{1}{90(n!)} \int_0^1 F_n(t) f^{(n)}(t) \mathrm{d}t.$$

By direct calculations we get

$$F_1(t) = G_1(t) = \begin{cases} -7, & t = 0 \\ -90t + 7, & 0 < t \leqslant 1/4 \\ -90t + 39, & 1/4 < t \leqslant 1/2 \\ -90t + 51, & 1/2 < t \leqslant 3/4 \\ -90t + 83, & 3/4 < t \leqslant 1 \end{cases}, \tag{2.6}$$

$$F_2(t) = G_2(t) = \begin{cases} 90t^2 - 14t, & 0 \leqslant t \leqslant 1/4 \\ 90t^2 - 78t + 16, & 1/4 < t \leqslant 1/2 \\ 90t^2 - 102t + 28, & 1/2 < t \leqslant 3/4 \\ 90t^2 - 166t + 76, & 3/4 < t \leqslant 1 \end{cases}, \tag{2.7}$$

$$F_3(t) = G_3(t) = \begin{cases} -90t^3 + 21t^2, & 0 \leqslant t \leqslant 1/4 \\ -90t^3 + 117t^2 - 48t + 6, & 1/4 < t \leqslant 1/2 \\ -90t^3 + 153t^2 - 84t + 15, & 1/2 < t \leqslant 3/4 \\ -90t^3 + 249t^2 - 228t + 69, & 3/4 < t \leqslant 1 \end{cases}, \tag{2.8}$$

$$F_4(t) = G_4(t) = \begin{cases} 90t^4 - 28t^3, & 0 \leqslant t \leqslant 1/4 \\ 90t^4 - 156t^3 + 96t^2 - 24t + 2, & 1/4 < t \leqslant 1/2 \\ 90t^4 - 204t^3 + 168t^2 - 60t + 8, & 1/2 < t \leqslant 3/4 \\ 90t^4 - 332t^3 + 456t^2 - 276t + 62, & 3/4 < t \leqslant 1 \end{cases} \tag{2.9}$$

and

$$F_5(t) = G_5(t) = \begin{cases} -90t^5 + 35t^4, & 0 \leqslant t \leqslant 1/4 \\ -90t^5 + 195t^4 - 160t^3 + 60t^2 - 10t + 5/8, & 1/4 < t \leqslant 1/2 \\ -90t^5 + 255t^4 - 280t^3 + 150t^2 - 40t + 35/8, & 1/2 < t \leqslant 3/4 \\ -90t^5 + 415t^4 - 760t^3 + 690t^2 - 310t + 55, & 3/4 < t \leqslant 1 \end{cases}. \tag{2.10}$$

Applying (2.2) with $n = 1,\ 2,\ 3,\ 4,\ 5$ we get the identities

$$\begin{aligned} \int_0^1 f(t)\mathrm{d}t - D(0,1) &= \frac{1}{90} \int_0^1 G_1(t)\, \mathrm{d}f(t) \\ &= \frac{1}{180} \int_0^1 G_2(t)\, \mathrm{d}f^{(1)}(t) \\ &= \frac{1}{540} \int_0^1 G_3(t)\, \mathrm{d}f^{(2)}(t) \\ &= \frac{1}{2160} \int_0^1 G_4(t)\, \mathrm{d}f^{(3)}(t) \\ &= \frac{1}{10800} \int_0^1 G_5(t)\, \mathrm{d}f^{(4)}(t). \end{aligned}$$

The same identities are obtained from (2.3) with $n = 1, 2, 3, 4, 5$ since $\widetilde{T}_0(0,1) = \widetilde{T}_1(0,1) = \widetilde{T}_2(0,1) = \widetilde{T}_3(0,1) = \widetilde{T}_4(0,1) = 0$ and $F_k(t) = G_k(t)$ for $k = 1, 2, 3, 4, 5$ while (2.3) with $n = 6$ yields the identity

$$\int_0^1 f(t)\mathrm{d}t - D(0,1) = \frac{1}{64800}\int_0^1 F_5(t)\,\mathrm{d}f^{(5)}(t).$$

3. Some inequalities related to Euler-Boole formulae

In this section we use the Euler-Boole formulae established in Theorem 1 to prove a number of inequalities for various classes of functions. First, we need some properties of the functions $G_k(t)$ and $F_k(t)$ defined in the previous section.

The Bernoulli polynomials are symmetric with respect to $\frac{1}{2}$, that is [1, 23.1.8]

$$B_k(1-t) = (-1)^k B_k(t), \quad \forall t \in \mathbf{R}, \quad k \geqslant 1. \tag{3.1}$$

Setting $t = \frac{1}{4}$ in (3.1) we get

$$B_k\left(\frac{3}{4}\right) = (-1)^k B_k\left(\frac{1}{4}\right), \quad k \geqslant 1.$$

Also, we have

$$B_k(1) = B_k(0) = B_k, \quad k \geqslant 2, \quad B_1(1) = -B_1(0) = \frac{1}{2}$$

and

$$B_{2j-1} = 0, \quad j \geqslant 2.$$

Therefore, using [1, 23.121] and [1, 23.122]

$$B_{2j}\left(\frac{1}{2}\right) = -\left(1 - 2^{1-2j}\right)B_{2j}, \quad j \geqslant 1,$$

and

$$B_{2j}\left(\frac{1}{4}\right) = B_{2j}\left(\frac{3}{4}\right) = -2^{-2j}\left(1 - 2^{1-2j}\right)B_{2j}, \quad j \geqslant 1,$$

we get

$$\widetilde{B}_{2j-1} = 0, \quad j \geqslant 1 \tag{3.2}$$

and

$$\widetilde{B}_{2j} = 14B_{2j} + 64B_{2j}\left(\frac{1}{4}\right) + 12B_{2j}\left(\frac{1}{2}\right) = \left(2 - 5\cdot 2^{3-2j} + 2^{7-4j}\right)B_{2j}, \quad j \geqslant 1.$$

Now, by (3.2) we have

$$F_{2j-1}(t) = G_{2j-1}(t), \quad j \geqslant 1, \tag{3.3}$$

and, by (3.3),

$$F_{2j}(t) = G_{2j}(t) - \widetilde{B}_{2j} = G_{2j}(t) - \left(2 - 5\cdot 2^{3-2j} + 2^{7-4j}\right)B_{2j}, \quad j \geqslant 1. \tag{3.4}$$

Further, the points 0 and 1 are the zeros of $F_k(t) = G_k(t) - G_k(0),\ k \geqslant 2,$ that is

$$F_k(0) = F_k(1) = 0,\ \ k \geqslant 2.$$

As we shall see below, 0 and 1 are the only zeros of $F_{2j}(t)$ for $j \geqslant 3$. Next, setting $t = \frac{1}{2}$ in (3.1) we get

$$B_k\left(\frac{1}{2}\right) = (-1)^k B_k\left(\frac{1}{2}\right),\ \ k \geqslant 1.$$

which implies that

$$B_{2j-1}\left(\frac{1}{2}\right) = 0,\ \ j \geqslant 1.$$

Using the above formulae, we get

$$F_{2j-1}\left(\frac{1}{2}\right) = G_{2j-1}\left(\frac{1}{2}\right) = 0,\ \ j \geqslant 1.$$

We shall see that 0, $\frac{1}{2}$ and 1 are the only zeros of $F_{2j-1}(t) = G_{2j-1}(t),$ for $j \geqslant 3$. Also, note that

$$G_{2j}\left(\frac{1}{2}\right) = 14B_{2j}\left(\frac{1}{2}\right) + 64B_{2j}\left(\frac{1}{4}\right) + 12B_{2j} = \left(-2 - 9 \cdot 2^{2-2j} + 2^{7-4j}\right) B_{2j},\ \ j \geqslant 1$$

and

$$F_{2j}\left(\frac{1}{2}\right) = G_{2j}\left(\frac{1}{2}\right) - \widetilde{B}_{2j} = -4\left(1 - 2^{-2j}\right) B_{2j},\ \ j \geqslant 1. \tag{3.5}$$

LEMMA 1. *For $k \geqslant 2$ we have*

$$G_k(1-t) = (-1)^k G_k(t),\ \ 0 \leqslant t \leqslant 1,$$

and

$$F_k(1-t) = (-1)^k F_k(t),\ \ 0 \leqslant t \leqslant 1.$$

Proof. As we noted in introduction, the functions $B_k^*(t)$ are periodic with period 1 and continuous for $k \geqslant 2$. Therefore, for $k \geqslant 2$ and $0 \leqslant t \leqslant 1$ we have

$$\begin{aligned}
G_k(1-t) &= 14B_k^*(t) + 32B_k^*\left(-\frac{3}{4} + t\right) + 12B_k^*\left(-\frac{1}{2} + t\right) + 32B_k^*\left(-\frac{1}{4} + t\right) \\
&= \begin{cases}
14B_k(t) + 32B_k\left(\frac{1}{4} + t\right) + 12B_k\left(\frac{1}{2} + t\right) + 32B_k\left(\frac{3}{4} + t\right), & 0 \leqslant t \leqslant \frac{1}{4}, \\
14B_k(t) + 32B_k\left(\frac{1}{4} + t\right) + 12B_k\left(\frac{1}{2} + t\right) + 32B_k\left(-\frac{1}{4} + t\right), & \frac{1}{4} < t \leqslant \frac{1}{2}, \\
14B_k(t) + 32B_k\left(\frac{1}{4} + t\right) + 12B_k\left(-\frac{1}{2} + t\right) + 32B_k\left(-\frac{1}{4} + t\right), & \frac{1}{2} < t \leqslant \frac{3}{4}, \\
14B_k(t) + 32B_k\left(-\frac{3}{4} + t\right) + 12B_k\left(-\frac{1}{2} + t\right) + 32B_k\left(-\frac{1}{4} + t\right), & \frac{3}{4} < t \leqslant 1,
\end{cases} \\
&= (-1)^k \times \\
&\quad \begin{cases}
14B_k(1-t) + 32B_k\left(\frac{3}{4} - t\right) + 12B_k\left(\frac{1}{2} - t\right) + 32B_k\left(\frac{1}{4} - t\right), & 0 \leqslant t \leqslant \frac{1}{4}, \\
14B_k(1-t) + 32B_k\left(\frac{3}{4} - t\right) + 12B_k\left(\frac{1}{2} - t\right) + 32B_k\left(\frac{5}{4} - t\right), & \frac{1}{4} < t \leqslant \frac{1}{2}, \\
14B_k(1-t) + 32B_k\left(\frac{3}{4} - t\right) + 12B_k\left(\frac{3}{2} - t\right) + 32B_k\left(\frac{5}{4} - t\right), & \frac{1}{2} < t \leqslant \frac{3}{4}, \\
14B_k(1-t) + 32B_k\left(\frac{7}{4} - t\right) + 12B_k\left(\frac{3}{2} - t\right) + 32B_k\left(\frac{5}{4} - t\right), & \frac{3}{4} < t \leqslant 1,
\end{cases} \\
&= (-1)^k G_k(t),
\end{aligned}$$

which proves the first identity. Further, we have $F_k(t) = G_k(t) - G_k(0)$ and $(-1)^k G_k(0) = G_k(0)$, since $G_{2j+1}(0) = 0$, so that we have

$$F_k(1-t) = G_k(1-t) - G_k(0) = (-1)^k [G_k(t) - G_k(0)] = (-1)^k F_k(t),$$

which proves the second identity. □

Note that the identities established in Lemma 1 are valid for $k = 1$, too, except at the points 0, $1/4$, $1/2$, $3/4$ and 1 of discontinuity of $F_1(t) = G_1(t)$.

LEMMA 2. *For $k \geqslant 3$ the function $G_{2k-1}(t)$ has no zeros in the interval $\left(0, \frac{1}{2}\right)$. The sign of this function is determined by*

$$(-1)^{k-1} G_{2k-1}(t) > 0, \quad 0 < t < \frac{1}{2}.$$

Proof. For $k = 3$, $G_5(t)$ is given by (2.10) and using the Sturm theorem it is easy to see that

$$G_5(t) > 0, \quad 0 < t < \frac{1}{2}.$$

Thus, our assertion is true for $k = 3$. Now, assume that $k \geqslant 4$. Then $2k-1 \geqslant 7$ and $G_{2k-1}(t)$ is continuous and at least twice differentiable function. Using (1.8) we get

$$G'_{2k-1}(t) = -(2k-1) G_{2k-2}(t)$$

and

$$G''_{2k-1}(t) = (2k-1)(2k-2) G_{2k-3}(t).$$

Let us suppose that G_{2k-3} has no zeros in the interval $\left(0, \frac{1}{2}\right)$. We know that 0 and $\frac{1}{2}$ are the zeros of $G_{2k-1}(t)$. Let us suppose that some α, $0 < \alpha < \frac{1}{2}$, is also a zero of $G_{2k-1}(t)$. Then inside each of the intervals $(0, \alpha)$ and $\left(\alpha, \frac{1}{2}\right)$ the derivative $G'_{2k-1}(t)$ must have at least one zero, say β_1, $0 < \beta_1 < \alpha$ and β_2, $\alpha < \beta_2 < \frac{1}{2}$. Therefore, the second derivative $G''_{2k-1}(t)$ must have at least one zero inside the interval (β_1, β_2). Thus, from the assumption that $G_{2k-1}(t)$ has a zero inside the interval $\left(0, \frac{1}{2}\right)$, it follows that $(2k-1)(2k-2) G_{2k-3}(t)$ also has a zero inside this interval. Thus, $G_{2k-1}(t)$ can not have a zero inside the interval $\left(0, \frac{1}{2}\right)$. To determine the sign of $G_{2k-1}(t)$, note that

$$G_{2k-1}\left(\frac{1}{4}\right) = -2 B_{2k-1}\left(\frac{1}{4}\right).$$

We have [1, 23.1.14]

$$(-1)^k B_{2k-1}(t) > 0, \quad 0 < t < \frac{1}{2},$$

which implies

$$(-1)^{k-1} G_{2k-1}\left(\frac{1}{4}\right) = 2 \cdot (-1)^k B_{2k-1}\left(\frac{1}{4}\right) > 0.$$

Consequently, we have

$$(-1)^{k-1} G_{2k-1}(t) > 0, \quad 0 < t < \frac{1}{2}. \quad \square$$

COROLLARY 1. *For $k \geqslant 3$ the functions $(-1)^k F_{2k}(t)$ and $(-1)^k G_{2k}(t)$ are strictly increasing on the interval $\left(0, \frac{1}{2}\right)$, and strictly decreasing on the interval $\left(\frac{1}{2}, 1\right)$. Further, for $k \geqslant 3$, we have*

$$\max_{t \in [0,1]} |F_{2k}(t)| = 4\left(1 - 2^{-2k}\right)|B_{2k}|,$$

and

$$\max_{t \in [0,1]} |G_{2k}(t)| = \left(2 + 9 \cdot 2^{2-2k} - 2^{7-4k}\right)|B_{2k}|.$$

Proof. Using (1.8) we get

$$\left[(-1)^k F_{2k}(t)\right]' = \left[(-1)^k G_{2k}(t)\right]' = 2k(-1)^{k-1} G_{2k-1}(t)$$

and $(-1)^{k-1} G_{2k-1}(t) > 0$ for $0 < t < \frac{1}{2}$, by Lemma 2. Thus, $(-1)^k F_{2k}(t)$ and $(-1)^k G_{2k}(t)$ are strictly increasing on the interval $\left(0, \frac{1}{2}\right)$. Also, by Lemma 1, we have $F_{2k}(1-t) = F_{2k}(t)$, $0 \leqslant t \leqslant 1$ and $G_{2k}(1-t) = G_{2k}(t)$, $0 \leqslant t \leqslant 1$, which implies that $(-1)^k F_{2k}(t)$ and $(-1)^k G_{2k}(t)$ are strictly decreasing on the interval $\left(\frac{1}{2}, 1\right)$. Further, $F_{2k}(0) = F_{2k}(1) = 0$, which implies that $|F_{2k}(t)|$ achieves its maximum at $t = \frac{1}{2}$, that is

$$\max_{t \in [0,1]} |F_{2k}(t)| = \left|F_{2k}\left(\frac{1}{2}\right)\right| = 4\left(1 - 2^{-2k}\right)|B_{2k}|.$$

Also

$$\begin{aligned} \max_{t \in [0,1]} |G_{2k}(t)| &= \max\left\{|G_{2k}(0)|, \left|G_{2k}\left(\frac{1}{2}\right)\right|\right\} \\ &= \left|G_{2k}\left(\frac{1}{2}\right)\right| = \left(2 + 9 \cdot 2^{2-2k} - 2^{7-4k}\right)|B_{2k}|, \end{aligned}$$

which completes the proof. □

COROLLARY 2. *For $k \geqslant 3$, we have*

$$\int_0^1 |F_{2k-1}(t)|\,\mathrm{d}t = \int_0^1 |G_{2k-1}(t)|\,\mathrm{d}t = \frac{4}{k}\left(1 - 2^{-2k}\right)|B_{2k}|.$$

Also, we have

$$\int_0^1 |F_{2k}(t)|\,\mathrm{d}t = \left|\widetilde{B}_{2k}\right| = \left(2 - 5 \cdot 2^{3-2k} + 2^{7-4k}\right)|B_{2k}|$$

and

$$\int_0^1 |G_{2k}(t)|\,\mathrm{d}t \leqslant 2\left|\widetilde{B}_{2k}\right| = \left(4 - 5 \cdot 2^{4-2k} + 2^{8-4k}\right)|B_{2k}|.$$

Proof. Using (1.8) it is easy to see that

$$G_m'(t) = -mG_{m-1}(t), \quad m \geqslant 3. \tag{3.6}$$

Now, using Lemma 1, Lemma 2 and (3.6) we get

$$\begin{aligned}\int_0^1 |G_{2k-1}(t)|\,\mathrm{d}t &= 2\left|\int_0^{\frac{1}{2}} G_{2k-1}(t)\mathrm{d}t\right| = 2\left|-\frac{1}{2k}\, G_{2k}(t)|_0^{\frac{1}{2}}\right| \\ &= \frac{1}{k}\left|G_{2k}\left(\frac{1}{2}\right) - G_{2k}(0)\right| = \frac{4}{k}\left(1-2^{-2k}\right)|B_{2k}|,\end{aligned}$$

which proves the first assertion. By Corollary 1 and because $F_{2k}(0) = F_{2k}(1) = 0$, $F_{2k}(t)$ does not change its sign on the interval $(0,1)$. Therefore, using (3.4) and (3.6), we get

$$\begin{aligned}\int_0^1 |F_{2k}(t)|\,\mathrm{d}t = \left|\int_0^1 F_{2k}(t)\mathrm{d}t\right| &= \left|\int_0^1 \left[G_{2k}(t) - \widetilde{B}_{2k}\right]\mathrm{d}t\right| \\ &= \left|-\frac{1}{2k+1}\, G_{2k+1}(t)|_0^1 - \widetilde{B}_{2k}\right| = \left|\widetilde{B}_{2k}\right|,\end{aligned}$$

which proves the second assertion. Finally, we use (3.4) again and the triangle inequality to obtain

$$\int_0^1 |G_{2k}(t)|\,\mathrm{d}t = \int_0^1 \left|F_{2k}(t) + \widetilde{B}_{2k}\right|\mathrm{d}t \leqslant \int_0^1 |F_{2k}(t)|\,\mathrm{d}t + \left|\widetilde{B}_{2k}\right| = 2\left|\widetilde{B}_{2k}\right|,$$

which proves the third assertion. □

THEOREM 2. *Let $f : [0,1] \to \mathbf{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[0,1]$ for some $n \geqslant 1$. Then*

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1) + \widetilde{T}_{n-1}(0,1)\right| \leqslant \frac{1}{90(n!)}\int_0^1 |F_n(t)|\,\mathrm{d}t \cdot L \tag{3.7}$$

and

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1) + \widetilde{T}_n(0,1)\right| \leqslant \frac{1}{90(n!)}\int_0^1 |G_n(t)|\,\mathrm{d}t \cdot L. \tag{3.8}$$

Proof. For any integrable function $\Phi : [0,1] \to \mathbf{R}$ we have

$$\left|\int_0^1 \Phi(t)\mathrm{d}f^{(n-1)}(t)\right| \leqslant \int_0^1 |\Phi(t)|\,\mathrm{d}t \cdot L, \tag{3.9}$$

since $f^{(n-1)}$ is L-Lipschitzian function. Applying (3.9) with $\Phi(t) = F_n(t)$, we get

$$\left|\frac{1}{90(n!)}\int_0^1 F_n(t)\,\mathrm{d}f^{(n-1)}(t)\right| \leqslant \frac{1}{90(n!)}\int_0^1 |F_n(t)|\,\mathrm{d}t \cdot L.$$

Applying the above inequality, we get inequality (3.7) from identity (2.3). Similarly, we can apply inequality (3.9) with $\Phi(t) = G_n(t)$, and then use identity (2.2), to obtain inequality (3.8). □

COROLLARY 3. *Let $f : [0,1] \to \mathbf{R}$ be given function.*
If f is L-Lipschitzian on $[0,1]$, then

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) \right| \leqslant \frac{239}{3240} \cdot L.$$

If f' is L-Lipschitzian on $[0,1]$, then

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) \right| \leqslant \frac{1018}{273375} \cdot L.$$

Proof. Using (2.6) and (2.7) we get

$$\int_0^1 |F_1(t)|\,\mathrm{d}t = \frac{239}{36} \quad \text{and} \quad \int_0^1 |F_2(t)|\,\mathrm{d}t = \frac{4072}{6075}.$$

Therefore, applying (3.7) with $n = 1,2,$ we get the above inequalities. □

As we have already noted in Section 2, we have

$$\widetilde{T}_0(0,1) = \widetilde{T}_1(0,1) = \widetilde{T}_2(0,1) = \widetilde{T}_3(0,1) = \widetilde{T}_4(0,1) = \widetilde{T}_5(0,1) = 0. \tag{3.10}$$

Moreover, since $\widetilde{B}_{2k-1} = 0$ and $\widetilde{B}_{2k} = (2 - 5\cdot 2^{3-2k} + 2^{7-4k})B_{2k},\ k \geqslant 1,$ we have

$$\widetilde{T}_m(0,1) = \frac{1}{90}\sum_{k=3}^{\left[\frac{m}{2}\right]} \frac{1}{(2k)!}(2 - 5\cdot 2^{3-2k} + 2^{7-4k})B_{2k}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right],\ m \geqslant 6, \tag{3.11}$$

where $\left[\frac{m}{2}\right]$ is the greatest integer less than or equal to $\frac{m}{2}$.

COROLLARY 4. *Let $f : [0,1] \to \mathbf{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[0,1]$ for some $n \geqslant 5$. Set $D_1(f) = D_2(f) := 0$ and for any integer r such that $3 \leqslant r \leqslant \frac{n}{2}$ define*

$$D_r(f) := \frac{1}{90}\sum_{i=3}^{r} \frac{1}{(2i)!}(2 - 5\cdot 2^{3-2i} + 2^{7-4i})B_{2i}\left[f^{(2i-1)}(1) - f^{(2i-1)}(0)\right]. \tag{3.12}$$

If $n = 2k-1,\ k \geqslant 3,$ then

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) + D_{k-1}(f) \right| \leqslant \frac{4}{45\,[(2k)!]}\left(1 - 2^{-2k}\right)|B_{2k}| \cdot L.$$

If $n = 2k,\ k \geqslant 3,$ then

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) + D_{k-1}(f) \right| \leqslant \frac{1}{45\,[(2k)!]}(1 - 5\cdot 2^{2-2k} + 2^{6-4k})\,|B_{2k}| \cdot L$$

and

$$\left| \int_0^1 f(t)\mathrm{d}t - D(0,1) + D_k(f) \right| \leqslant \frac{1}{45\,[(2k)!]}(2 - 5\cdot 2^{3-2k} + 2^{7-4k})\,|B_{2k}| \cdot L.$$

Proof. For $n = 2k-1$, by (3.11) we have that $\widetilde{T}_{n-1}(0,1) = D_{k-1}(f)$. Thus, the first inequality follows from Corollary 2 and (3.7). For $n = 2k$, by (3.11) we have that $\widetilde{T}_{n-1}(0,1) = D_{k-1}(f)$ and $\widetilde{T}_n(0,1) = D_k(f)$. Now, the second inequality follows from Corollary 2 and (3.7), while the third one follows from Corollary 2 and (3.8). □

REMARK 3. Suppose that $f : [0,1] \to \mathbf{R}$ is such that $f^{(n)}$ exists and is bounded on $[0,1]$, for some $n \geqslant 1$. Therefore, the inequalities established in Theorem 2 hold with $L = \|f^{(n)}\|_\infty$.

THEOREM 3. *Let $f : [0,1] \to \mathbf{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$ for some $n \geqslant 1$. Then*

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1) + \widetilde{T}_{n-1}(0,1)\right| \leqslant \frac{1}{90(n!)} \max_{t\in[0,1]} |F_n(t)| \cdot V_0^1(f^{(n-1)}) \tag{3.13}$$

and

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1) + \widetilde{T}_n(0,1)\right| \leqslant \frac{1}{90(n!)} \max_{t\in[0,1]} |G_n(t)| \cdot V_0^1(f^{(n-1)}), \tag{3.14}$$

where $V_0^1(f^{(n-1)})$ is the total variation of $f^{(n-1)}$ on $[0,1]$.

Proof. If $\Phi : [0,1] \to \mathbf{R}$ is bounded on $[0,1]$ and the Riemann-Stieltjes integral $\int_0^1 \Phi(t)\, d f^{(n-1)}(t)$ exists, then

$$\left|\int_0^1 \Phi(t)\mathrm{d}f^{(n-1)}(t)\right| \leqslant \max_{t\in[0,1]} |\Phi(t)| \cdot V_0^1(f^{(n-1)}). \tag{3.15}$$

We apply estimate (3.15) to $\Phi(t) = F_n(t)$ to obtain

$$\left|\frac{1}{90(n!)}\int_0^1 F_n(t)\,\mathrm{d}f^{(n-1)}(t)\right| \leqslant \frac{1}{90(n!)} \max_{t\in[0,1]} |F_n(t)| \cdot V_0^1(f^{(n-1)}).$$

Now, we use the above inequality and identity (2.3) to obtain (3.13). In the same manner, we apply estimate (3.15) to $\Phi(t) = G_n(t)$, and then use identity (2.2), to obtain inequality (3.14). □

COROLLARY 5. *Let $f : [0,1] \to \mathbf{R}$ be given function.*
If f is a continuous function of bounded variation on $[0,1]$, then

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1)\right| \leqslant \frac{11}{60} \cdot V_0^1(f).$$

If f' is a continuous function of bounded variation on $[0,1]$, then

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1)\right| \leqslant \frac{17}{1440} \cdot V_0^1(f').$$

Proof. From explicit expressions (2.6) and (2.7), we get

$$\max_{t\in[0,1]} |F_1(t)| = -F_1\left(\frac{3}{4}\right) = \frac{33}{2} \quad \text{and}$$
$$\max_{t\in[0,1]} |F_2(t)| = F_2\left(\frac{1}{4}\right) = \frac{17}{8}.$$

Therefore, applying (3.13) with $n = 1, 2$ we get the above inequalities. □

COROLLARY 6. *Let $f : [0,1] \to \mathbf{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$ for some $n \geqslant 5$. Define $D_r(f)$, $r \geqslant 3$ as in Corollary 4. If $n = 2k-1$, $k \geqslant 3$, then*

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1) + D_{k-1}(f)\right| \leqslant \frac{1}{90\,[(2k-1)!]} \max_{t\in[0,1]} |F_{2k-1}(t)| \cdot V_0^1(f^{(2k-2)}).$$

If $n = 2k$, $k \geqslant 3$, then

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1) + D_{k-1}(f)\right| \leqslant \frac{2}{45\,[(2k)!]}\left(1 - 2^{-2k}\right)|B_{2k}| \cdot V_0^1(f^{(2k-1)})$$

and

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1) + D_k(f)\right| \leqslant \frac{1}{45\,[(2k)!]}\left(1 + 9\cdot 2^{1-2k} - 2^{6-4k}\right)|B_{2k}| \cdot V_0^1(f^{(2k-1)}).$$

Proof. The argument is similar to that used in the proof of Corollary 4. We apply Theorem 3 and use the formulae established in Corollary 1. □

REMARK 4. Suppose that $f^{(n)} : [0,1] \to \mathbf{R}$ is R-integrable function for some $n \geqslant 1$ In this case $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$ and we have

$$V_0^1(f^{(n-1)}) = \int_0^1 \left|f^{(n)}(t)\right| \mathrm{d}t = \|f^{(n)}\|_1.$$

Therefore, the inequalities established in Theorem 3 hold with $\|f^{(n)}\|_1$ in place of $V_0^1(f^{(n-1)})$. However, a similar observation can be made for the results of Corollaries 5 and 6.

THEOREM 4. *Assume (p,q) is a pair of conjugate exponents, that is $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ or $p = \infty$, $q = 1$. Let $\left|f^{(n)}\right|^p : [0,1] \to \mathbf{R}$ is R-integrable function for some $n \geqslant 1$. Then we have*

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1) + \widetilde{T}_{n-1}(0,1)\right| \leqslant K(n,p) \cdot \|f^{(n)}\|_p, \tag{3.16}$$

and

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1) + \widetilde{T}_n(0,1)\right| \leqslant K^*(n,p)\cdot \|f^{(n)}\|_p, \tag{3.17}$$

where

$$K(n,p) = \frac{1}{90(n!)}\left[\int_0^1 |F_n(t)|^q\,\mathrm{d}t\right]^{\frac{1}{q}},$$

and

$$K^*(n,p) = \frac{1}{90(n!)}\left[\int_0^1 |G_n(t)|^q\,\mathrm{d}t\right]^{\frac{1}{q}}.$$

Proof. Applying the Hölder inequality we have

$$\begin{aligned}\left|\frac{1}{90(n!)}\int_0^1 F_n(t) f^{(n)}(t)\mathrm{d}t\right| &\leqslant \frac{1}{90(n!)}\left[\int_0^1 |F_n(t)|^q\,\mathrm{d}t\right]^{\frac{1}{q}}\cdot \left\|f^{(n)}\right\|_p \\ &= K(n,p)\cdot \|f^{(n)}\|_p.\end{aligned}$$

Using the above inequality, by Remark 2, from (2.3) we get estimate (3.16). In the same manner, from (2.2) we get estimate (3.17). □

REMARK 5. For $p=\infty$ we have

$$K(n,\infty) = \frac{1}{90(n!)}\int_0^1 |F_n(t)|\,\mathrm{d}t \quad \text{and} \quad K^*(n,\infty) = \frac{1}{90(n!)}\int_0^1 |G_n(t)|\,\mathrm{d}t.$$

The results established in Theorem 4 for $p=\infty$ coincide with the results of Theorem 2 with $L=\|f^{(n)}\|_\infty$. Moreover, by Remark 3 and Corollary 3, we have

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1)\right| \leqslant K(n,\infty)\cdot \|f^{(n)}\|_\infty, \ \ n=1,2,$$

where

$$K(1,\infty) = \frac{239}{3240}, \qquad K(2,\infty) = \frac{1018}{273375}.$$

REMARK 6. Let us define for $p=1$

$$K(n,1) = \frac{1}{90(n!)}\max_{t\in[0,1]}|F_n(t)| \quad \text{and} \quad K^*(n,1) = \frac{1}{90(n!)}\max_{t\in[0,1]}|G_n(t)|.$$

Then, using Remark 4 and Theorem 3, we can extend the results established in Theorem 4 to the pair $p=1,\ q=\infty$. Also, by Remark 4 and Corollary 5, we have

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1)\right| \leqslant K(n,1)\cdot \|f^{(n)}\|_1, \ \ n=1,2,$$

where

$$K(1,1) = \frac{11}{60}, \qquad K(2,1) = \frac{17}{1440}.$$

REMARK 7. Note that $K^*(1,p) = K(1,p)$, for $1 < p \leqslant \infty$, since $G_1(t) = F_1(t)$. Also, for $1 < p \leqslant \infty$ we can easily calculate $K(1,p)$. We get

$$K(1,p) = \frac{1}{90}\left[\frac{12^{q+1}+14^{q+1}+31^{q+1}+33^{q+1}}{45(q+1)2^{q+1}}\right]^{\frac{1}{q}}, \quad 1 < p \leqslant \infty.$$

In the limit case when $p \to 1$, that is when $q \to \infty$, we have

$$\lim_{q\to\infty} \frac{1}{90}\left[\frac{12^{q+1}+14^{q+1}+31^{q+1}+33^{q+1}}{45(q+1)2^{q+1}}\right]^{\frac{1}{q}} = \frac{11}{60} = K(1,1).$$

Now we use the formula (2.2) to obtain Grüss type inequality related to that Euler-Boole formula. To do this we need the following two technical lemmas. The first one is proved in [5, Lemma 2] and the second one is the key result from the recent paper [7]:

LEMMA 3. *Let $k \geqslant 1$ and $\gamma \in \mathbf{R}$. Then*

$$\int_0^1 B_k^*(\gamma - t)\mathrm{d}t = 0.$$

LEMMA 4. *Let $F, G : [0,1] \to \mathbf{R}$ be two integrable functions. If*

$$m \leqslant F(t) \leqslant M, \quad 0 \leqslant t \leqslant 1$$

and

$$\int_0^1 G(t)\mathrm{d}t = 0,$$

then

$$\left|\int_0^1 F(t)G(t)\mathrm{d}t\right| \leqslant \frac{M-m}{2} \cdot \int_0^1 |G(t)|\mathrm{d}t. \tag{3.18}$$

THEOREM 5. *Suppose that $f : [0,1] \to \mathbf{R}$ is such that $f^{(n)}$ exists and is integrable on $[0,1]$, for some $n \geqslant 1$. Assume that*

$$m_n \leqslant f^{(n)}(t) \leqslant M_n, \quad 0 \leqslant t \leqslant 1,$$

for some constants m_n and M_n. Then

$$\left|\int_0^1 f(t)\mathrm{d}t - D(0,1) + \widetilde{T}_n(0,1)\right| \tag{3.19}$$

$$\leqslant \begin{cases} \frac{239}{6480}(M_1 - m_1) & \text{for } n = 1, \\ \frac{509}{273375}(M_2 - m_2) & \text{for } n = 2, \\ \frac{2\left(1-2^{-2k}\right)}{45[(2k)!]}(M_{2k-1} - m_{2k-1})|B_{2k}| & \text{for } n = 2k-1,\ k \geqslant 3 \\ \frac{\left(1-5\cdot 2^{2-2k}+2^{6-4k}\right)}{45[(2k)!]}(M_{2k} - m_{2k})|B_{2k}| & \text{for } n = 2k,\ k \geqslant 3 \end{cases}.$$

Proof. By Remark 2 we can rewrite $\widetilde{R}_n^1(f)$ as

$$\widetilde{R}_n^1(f) = \frac{1}{90(n!)} \cdot \int_0^1 F(t)G(t)\mathrm{d}t,$$

where

$$F(t) = f^{(n)}(t), \quad G(t) = G_n(t), \quad 0 \leqslant t \leqslant 1.$$

Using Lemma 3 we get

$$\int_0^1 G(t)\mathrm{d}t = 0.$$

Also, using Corollary 2 for $n \geqslant 5$ we get

$$\int_0^1 |G(t)|\mathrm{d}t = \int_0^1 |G_n(s)|\mathrm{d}s \begin{cases} = \frac{4}{k}\left(1-2^{-2k}\right)|B_{2k}| & \text{for } n = 2k-1, \\ \leqslant \left(4 - 5\cdot 2^{4-2k} + 2^{8-4k}\right)|B_{2k}| & \text{for } n = 2k \end{cases}.$$

For $n = 1$ and $n = 2$ we have

$$\int_0^1 |G(t)|\mathrm{d}t = \frac{239}{36} \quad \text{and} \quad \int_0^1 |G(t)|\mathrm{d}t = \frac{4072}{6075}.$$

Now, we apply the inequality (3.18) to obtain the estimate

$$\left|\widetilde{R}_n^1(f)\right| \leqslant \frac{1}{90(n!)} \cdot \frac{M_n - m_n}{2} \cdot \int_0^1 |G(t)|\mathrm{d}t$$

$$\leqslant \begin{cases} \frac{239}{6480}(M_1 - m_1) & \text{for } n = 1, \\ \frac{509}{273375}(M_2 - m_2) & \text{for } n = 2, \\ \frac{2\left(1-2^{-2k}\right)}{45[(2k)!]}(M_{2k-1} - m_{2k-1})|B_{2k}| & \text{for } n = 2k-1, k \geqslant 3 \\ \frac{\left(1-5\cdot 2^{2-2k}+2^{6-4k}\right)}{45[(2k)!]}(M_{2k} - m_{2k})|B_{2k}| & \text{for } n = 2k, k \geqslant 3 \end{cases}.$$

which proves our assertion. □

In the following discussion we assume that $f : [0,1] \to \mathbf{R}$ has a continuous derivative of order n, for some $n \geqslant 1$. In this case we can use (2.3) and the second formula from Remark 2 to obtain, for $n = 2k$.

$$\widetilde{R}_{2k}^2(f) = \frac{1}{90(2k)!} \int_0^1 F_{2k}(s) f^{(2k)}(s)\mathrm{d}s. \tag{3.20}$$

THEOREM 6. *If $f : [0,1] \to \mathbf{R}$ is such that $f^{(2k)}$ is a continuous function on $[0,1]$, for some $k \geqslant 3$, then there exists a point $\eta \in [0,1]$ such that*

$$\widetilde{R}_{2k}^2(f) = -\frac{1}{90[(2k)!]}(2 - 5\cdot 2^{3-2k} + 2^{7-4k})B_{2k} f^{(2k)}(\eta). \tag{3.21}$$

Proof. Using (3.20) we can rewrite $\widetilde{R}^2_{2k}(f)$ as

$$\widetilde{R}^2_{2k}(f) = (-1)^k \frac{1}{90[(2k)!]} J_k, \tag{3.22}$$

where

$$J_k = \int_0^1 (-1)^k F_{2k}(s) f^{(2k)}(s) \mathrm{d}s. \tag{3.23}$$

From Corollary 1 follows that

$$(-1)^k F_{2k}(s) \geqslant 0, \quad 0 \leqslant s \leqslant 1,$$

and the claim follows from the mean value theorem for integrals and Corollary 2. □

REMARK 8. For $k = 3$ formula (3.21) reduces to

$$\widetilde{R}^2_6(f) = -\frac{1}{1935360} f^{(6)}(\eta)$$

i.e. to (1.1).

COROLLARY 7. *Let $f \in C^\infty[0,1]$ and $\lambda \in \mathbf{R}$ be such that $0 < \lambda < 2\pi$ and $|f^{(2k)}(t)| \leqslant \lambda^{2k}$ for $t \in [0,1]$ and $k \geqslant k_0$ for some $k_0 \geqslant 3$. Then*

$$\int_0^1 f(t)dt = D(0,1) - \frac{1}{90} \sum_{j=3}^{\infty} \frac{1}{(2j)!} (2 - 5 \cdot 2^{3-2j} + 2^{7-4j}) B_{2j} \left[f^{(2j-1)}(1) - f^{(2j-1)}(0) \right]. \tag{3.24}$$

Proof. From Theorem 6 when $k \geqslant k_0$ we have that

$$|\widetilde{R}^2_{2k}(f)| \leqslant \frac{1}{90(2k)!} 2|B_{2k}| \lambda^{2k} \approx \frac{1}{90(2k)!} \cdot 2 \cdot 2 \frac{(2k)!}{(2\pi)^{2k}} \lambda^{2k} = \frac{2}{45} \left(\frac{\lambda}{2\pi} \right)^{2k},$$

so, (3.24) follows. □

THEOREM 7. *If $f : [0,1] \to \mathbf{R}$ is such that $f^{(2k)}$ is a continuous function on $[0,1]$, for some $k \geqslant 3$, and does not change its sign on $[0,1]$, then there exists a point $\theta \in [0,1]$ such that*

$$\widetilde{R}^2_{2k}(f) = -4\theta \frac{1}{90[(2k)!]} \left(1 - 2^{-2k}\right) B_{2k} \left[f^{(2k-1)}(1) - f^{(2k-1)}(0) \right]. \tag{3.25}$$

Proof. Suppose that $f^{(2k)}(t) \geqslant 0$, $0 \leqslant t \leqslant 1$. From Corollary 1 it follows that

$$0 \leqslant (-1)^k F_{2k}(s) \leqslant (-1)^k F_{2k}\left(\frac{1}{2}\right), \quad 0 \leqslant s \leqslant 1.$$

Therefore, if J_k is given by (3.23), then

$$0 \leqslant J_k \leqslant (-1)^k F_{2k}\left(\frac{1}{2}\right) \int_0^1 f^{(2k)}(s) \mathrm{d}s.$$

Using (3.5), we get

$$0 \leqslant J_k \leqslant (-1)^{k-1} 4\left(1-2^{-2k}\right) B_{2k}\left[f^{(2k-1)}(1)-f^{(2k-1)}(0)\right],$$

which means that there must exist a point $\theta \in [0,1]$ such that

$$J_k = \theta(-1)^{k-1} 4\left(1-2^{-2k}\right) B_{2k}\left[f^{(2k-1)}(1)-f^{(2k-1)}(0)\right].$$

Combining this with (3.22) we get (3.25). The argument is the same when $f^{(2k)}(t) \leqslant 0$, $0 \leqslant t \leqslant 1$, since in that case we get

$$(-1)^{k-1} 4\left(1-2^{-2k}\right) B_{2k}\left[f^{(2k-1)}(1)-f^{(2k-1)}(0)\right] \leqslant J_k \leqslant 0. \quad \square$$

REMARK 9. The same series expansion of $\int_0^1 f(t)dt$ as in Corollary 7 can be obtain from previous theorem under assumption $\left|f^{(2k-1)}(1)-f^{(2k-1)}(0)\right| \leqslant \lambda^{2k}$ for every $k \geqslant k_0$ for some $k_0 \geqslant 3$ where $0<\lambda<2\pi$.

REMARK 10. If we approximate $\int_0^1 f(t)dt$ by

$$I_{2k}(f) = D(0,1) - \frac{1}{90}\sum_{j=3}^{k-1}\frac{1}{(2j)!}(2-5\cdot 2^{3-2j}+2^{7-4j})B_{2j}\left[f^{(2j-1)}(1)-f^{(2j-1)}(0)\right],$$

then the next approximation will be $I_{2k+2}(f)$. The difference

$$\Delta_{2k}(f) = I_{2k+2}(f) - I_{2k}(f)$$

is equal to the last term in $I_{2k+2}(f)$, that is

$$\Delta_{2k}(f) = -\frac{1}{90[(2k)!]}(2-5\cdot 2^{3-2k}+2^{7-4k})B_{2k}\left[f^{(2k-1)}(1)-f^{(2k-1)}(0)\right].$$

We see that, under the assumptions of Theorem 7, $\widetilde{R}^2_{2k}(f)$ and $\Delta_{2k}(f)$ are of the same sign. Moreover, we have

$$\widetilde{R}^2_{2k}(f) = 2\theta\frac{1-2^{-2k}}{1-5\cdot 2^{2-2k}+2^{6-4k}}\Delta_{2k}(f).$$

Thus, we have the following estimate for the remainder $\widetilde{R}^2_{2k}(f)$:

$$\left|\widetilde{R}^2_{2k}(f)\right| \leqslant 2\left|\Delta_{2k}(f)\right|.$$

THEOREM 8. *Suppose that* $f:[0,1]\to\mathbf{R}$ *is such that* $f^{(2k+2)}$ *is a continuous function on* $[0,1]$ *for some* $k \geqslant 3$. *If*

$$f^{(2k)}(x) \geqslant 0 \quad \textit{and} \quad f^{(2k+2)}(x) \geqslant 0, \quad x \in [0,1],$$

or

$$f^{(2k)}(x) \leqslant 0 \quad \textit{and} \quad f^{(2k+2)}(x) \leqslant 0, \quad x \in [0,1],$$

then the remainder $\widetilde{R}^2_{2k}(f)$ *has the same sign as the first neglected term* $\Delta_{2k}(f)$ *and*

$$\left|\widetilde{R}^2_{2k}(f)\right| \leqslant \left|\Delta_{2k}(f)\right|.$$

Proof. We have

$$\Delta_{2k}(f) + \widetilde{R}^2_{2k+2}(f) = \widetilde{R}^2_{2k}(f),$$

that is

$$\Delta_{2k}(f) = \widetilde{R}^2_{2k}(f) - \widetilde{R}^2_{2k+2}(f). \tag{3.26}$$

By (3.20) we have

$$\widetilde{R}^2_{2k}(f) = \frac{1}{90[(2k)!]}\int_0^1 F_{2k}(s)f^{(2k)}(s)\mathrm{d}s$$

and

$$-\widetilde{R}^2_{2k+2}(f) = \frac{1}{90[(2k+2)!]}\int_0^1 [-F_{2k+2}(s)]f^{(2k+2)}(s)\mathrm{d}s.$$

Under the assumptions made on f, we see that for all $s \in [0,1]$ either

$$f^{(2k)}(s) \geqslant 0 \quad \text{and} \quad f^{(2k+2)}(s) \geqslant 0$$

or

$$f^{(2k)}(s) \leqslant 0 \quad \text{and} \quad f^{(2k+2)}(s) \leqslant 0.$$

Also, from Corollary 1 it follows that for all $s \in [0,1]$

$$(-1)^k F_{2k}(s) \geqslant 0 \quad \text{and} \quad (-1)^k[-F_{2k+2}(s)] \geqslant 0.$$

We conclude that $\widetilde{R}^2_{2k}(f)$ has the same sign as $-\widetilde{R}^2_{2k+2}(f)$. Therefore, because of (3.26), $\Delta_{2k}(f)$ must have the same sign as $\widetilde{R}^2_{2k}(f)$ and $-\widetilde{R}^2_{2k+2}(f)$. Moreover, it follows that

$$\left|\widetilde{R}^2_{2k}(f)\right| \leqslant |\Delta_{2k}(f)| \quad \text{and} \quad \left|-\widetilde{R}^2_{2k+2}(f)(f)\right| \leqslant |\Delta_{2k}(f)|. \quad \square$$

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