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OPTIMAL INEQUALITIES BETWEEN SEIFFERT'S MEAN AND POWER MEANS

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Abstract. In this paper optimal inequalities between Seiffert's mean and power means are derived using a simple monotony property.

1. Introduction and main results

In this paper we will derive inequalities between Seiffert's mean and power means using elementary methods, namely monotonity properties. Our method yields optimal approximation results with quite little effort. Seiffert's mean was introduced by H.-J. Seiffert in [7] and is defined for distinct $x, y \in \mathbb{R}^{>} := (0, \infty)$ by

$$P(x, y) := \frac{x - y}{4 \arctan(\sqrt{x/y}) - \pi}$$

and by $P(x, x) := x$; an equivalent form is given in [8]. Inequalities of this mean have also been considered in [2, 3, 4, 9, 11, 5, 6] (more on these later), and G. Toader [10] has considered a generalization of it.

The classical power mean is defined by the formula

$$A_p(x, y) := \left(\frac{x^p + y^p}{2} \right)^{1/p}$$

for $x, y, p \in \mathbb{R}^{>}$ and $A_0(x, y) := \sqrt{xy}$ (for basic properties of the power means, see for instance [1]). The first inequalities between the Seiffert mean and power means were

$$A_{1/2} \leqslant P \leqslant A_{2/3},$$

proved by A. A. Jagers in [3]. In a recent paper, [2], the author complemented this result by showing that $1/2$ is the greatest value of p for which the ratio $P(x, 1)/A_p(x, 1)$ is increasing in $x > 1$ and $2/3$ is the least value of p for which the ratio $P(x, 1)/A_p(x, 1)$

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is decreasing in $x > 1$. These results led to the inequalities ([2, Corollary 1.7] with $\alpha = 1/2$)

$$A_{1/2} \leqslant P \leqslant (4/\pi)A_{1/2} \tag{1}$$

and [2, Corollary 6.3]

$$(2^{3/2}/\pi)A_{2/3} \leqslant P \leqslant A_{2/3}. \tag{2}$$

The largest ratio between the upper and lower bounds in these three double inequalities is $2^{1/2} \approx 1.414$, $4/\pi \approx 1.273$ and $\pi 2^{-3/2} \approx 1.111$, respectively. In this paper we will derive optimal inequalities of Seiffert's mean in the sense of minimizing this ratio. We start with an inequality similar to that derived by Jagers.

THEOREM 1.1. *Let* $p, q \in \mathbb{R}^{>}$. *Then*

$$A_p \leqslant P \leqslant A_q$$

if and only if $p \leqslant p_0 := \log 2/\log \pi$ *and* $q \geqslant 2/3$.

With the optimal choice $p = p_0$ and $q = 2/3$ in the previous theorem we get the maximal ratio $2^{\log \pi/\log 2 - 3/2} = \pi 2^{-3/2} \approx 1.111$ between the upper and lower bounds. If we allow multiplication of the power means by a constant, as in inequalities (1) and (2), we can obviously expect to get a better approximation:

THEOREM 1.2. *Let* $a, b, p, q \in \mathbb{R}^{>}$ *be such that*

$$aA_p \leqslant P \leqslant bA_q.$$

Then $\sup \frac{bA_q}{aA_p} \geqslant c_0$ *with equality if and only if* $p = q = p_0 := \log 2/\log \pi$, $a = 1$ *and* $b = c_0$, *where*

$$c_0 := \sup_{x \in [1,\infty)} \frac{P(x,1)}{A_{p_0}(x,1)} \approx 1.0298.$$

Before moving on to the proofs, let us note some other inequalities of Seiffert's mean. A lower bound was obtained by H.-J. Seiffert in [9] which involves power means and the logarithmic mean:

$$P(x,y) \geqslant A(x,y)G(x,y)\frac{\log\{x/y\}}{x-y},$$

for $x \neq y$. Here A denotes the arithemtic mean A_1 and G is the geometric mean A_0. Since this lower bound grows like $\sqrt{x}\log x$ as $x \to \infty$ (for fixed y) we see that it is a worse estimate than Theorem 1.1, which grows like x, for large x. On the other hand, numerical results suggest that it is better estimate for $x \approx y$. J. Sándor has recently derived several inequalities for Seiffert's mean in [4] using a sequential method. Some examples of his results are

$$A_{1/2} = \frac{A+G}{2} \leqslant P \leqslant \sqrt{\left(\frac{A+G}{2}\right)A} \quad ,$$

which appears in Theorem 1, and

$$\sqrt[3]{A^2G} \leqslant P \leqslant \frac{G+2A}{3} \quad ,$$

which is part of Theorem 2. (A and G are as in the previous paragraph.) As Sándor noted in Remark 4, the latter upper bound for P is better than the bound $A_{2/3}$ derived by Jagers (and also part of Theorem 1.1). The upper bound from Theorem 1.2 (that is $P \leqslant c_0 A_{p_0}$) grows like $c_0 2^{1/q_0} x$ whereas the upper bounds from the previous two inequalities grow like $2^{-3/2}x$ and $x/3$, respectively, for fixed y and $x \gg y$. We therefore see that Theorem 1.2 provides the best upper bound in this case, although it is obviously worse for $x \approx y$. Of the lower bounds the first is worse than that of Theorem 1.1, whereas the second lower bound seems to be better for $x \approx y$ but is worse for $x \gg y$. Some further inequalities between P, A, G and other meansare given in [11, 5, 6].

2. The proofs

Both of the theorems are based on the same fundamental lemma, the proof of which is postponed until the end of the paper, since it is somewhat technical. It should be noted that because of homogeneity in the inequality $aA_p(x,y) \leqslant P(x,y) \leqslant bA_q(x,y)$ we may assume without loss of generality that $y = 1 \leqslant x$, an assumption to which we will adhere for the rest of the paper.

LEMMA 2.1. *Let $1/2 < p < 2/3$. There exists a value $x_p \in (1,\infty)$ such that $P(x,1)/A_p(x,1)$ is increasing for $x \in [1,x_p]$ and decreasing for $x \in [x_p,\infty)$.*

Notice how this lemma complements the results in [2] which showed that the ratio is in the lemma is decreasing for $p \geqslant 2/3$ and decreasing for $p \leqslant 1/2$. Using this lemma we can easily prove Theorem 1.1.

Proof of Theorem 1.1. Let us first prove that $A_p \leqslant P$ if and only if $p \leqslant p_0$. Suppose first that $p \leqslant p_0$. Since $A_{p_0} \geqslant A_p$ by the well-known monotonicity property of the power mean, it suffices to show that $A_{p_0} \leqslant P$. Define $f(x) := P(x,1)/A_{p_0}(x,1)$. We have

$$\lim_{x\to\infty} f(x) = 2^{p_0}/\pi = 1$$

and also $f(1) = 1$, hence

$$f(x) \geqslant \min\{f(1), \lim_{x\to\infty} f(x)\} = 1$$

for all $x \in [1,\infty)$ by Lemma 2.1, from which it follows that $A_p \leqslant P$. Assuming conversely that $A_p \leqslant P$ we see that $2^p/\pi \geqslant 1$ by considering $x \to \infty$ and so $p \leqslant p_0$. Let us move on to the inequality $P \leqslant A_q$. The inequality $P \leqslant A_{2/3}$ was proved by A. A. Jagers in [3], as was already mentioned, and the inequality $P \leqslant A_q$ for $q \geqslant 2/3$ follows from the monotonicity of the power mean. Suppose then that $P \leqslant A_q$. It follows from the first paragraph in the proof of [2, Proposition 6.1] that $g(x) := P(x,1)/A_q(x,1)$ is strictly increasing for $x \in [1,x_0)$ for some $x_0 > 1$ if $q < 2/3$. Since $g(1) = 1$, it follows that $g(x) > 1$ with $x \in (1,x_0)$ for $q < 2/3$, which means that $P(x,1) > A_q(x,1)$ for the same x, contrary to the assumption $P \leqslant A_q$. $\square$

We will prove Theorem 1.2 in two steps, first assuming that $p = q$ and then considering what happens when we relax this constraint.

LEMMA 2.2. *Let* $a, b, p \in \mathbb{R}^{>}$. *If*

$$aA_p \leqslant P \leqslant bA_p$$

then b/a *can be minimized by the choise* $p = p_0 := \log 2/\log \pi$, *in which case the constants can be chosen so that* $b/a = c_0$, *where* c_0 *is as in Theorem 1.2.*

Before proving this lemma let us give an "intuitive proof" based on two graphs, which will hopefully also help in understanding the actual proof.

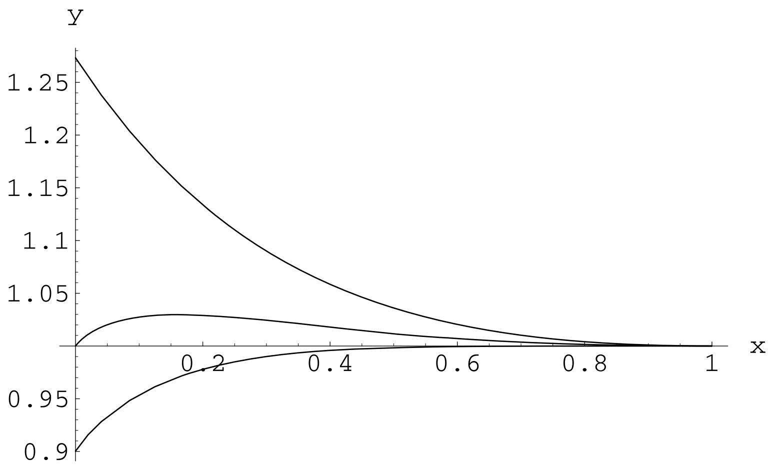

Figure 1: The function $y = P(x^{-2}, 1)/A_p(x^{-2}, 1)$ *for* $p = 1/2$ *(top),* $p = p_0$ *and* $p = 2/3$

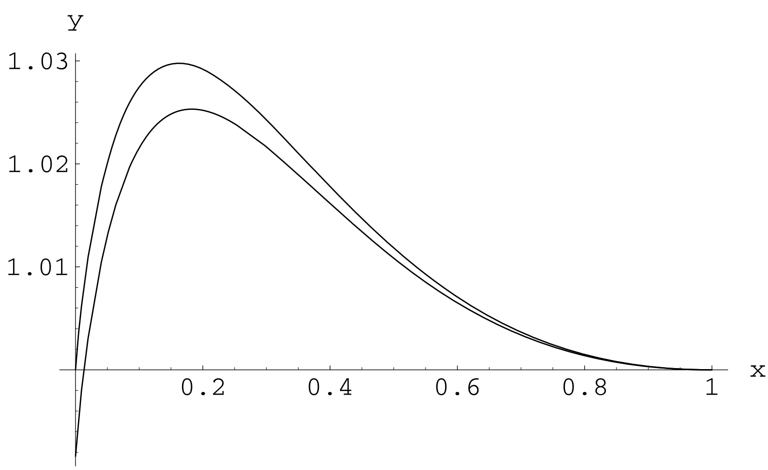

Figure 2: The function $y = P(x^{-2}, 1)/A_p(x^{-2}, 1)$ *for* $p = p_0$ *(top) and* $p = 0.61$

(Note the argument x^{-2} of the function – this was chosen to make the graph clearer, it obviously does not affect the argument below, which is based on differences in the y–coordinate.) The ratio that we seek to minimize is that between the maximum and the minimum value along a single curve. In Figure 2. we directly see that we should only consider curves close to the x-axis, in particular, which are not strictly positive at $x^{-2} = 0$. Moreover, as seen in the close-up in Figure 2., the curves are a lot steeper at the minimum at $x^{-2} = 0$ then at the maximum in the middle of the curve and hence it is probably not worth choosing a curve that goes below 0 as $x \to \infty$, since the gain in the upper bound is smaller than the loss in the lower bound. The next proof formalizes these thoughts.

Proof of Lemma 2.2. Assume first that $p < p_0$. Then we may choose $a = 1$ (we always have $a \leqslant 1 \leqslant b$, since $A_p(1,1) = P(1,1) = 1$) but we have

$$b \geqslant \sup_{x\in[1,\infty)} \frac{P(x,1)}{A_p(x,1)} > \sup_{x\in[1,\infty)} \frac{P(x,1)}{A_{p_0}(x,1)} = c_0,$$

by the strict monotonicity of the power means (since the supremum does not occur at $x = 1$). If $p \geqslant 2/3$ then we may choose $b = 1$ but have $a \leqslant 2^{1/p}/\pi$ and so $b/a \geqslant \pi 2^{-1/p} > 1.110 > c_0 \approx 1.03$. We may assume then that $p_0 \leqslant p < 2/3$ and so Lemma 2.1 is applicable. Let us denote $f(x) := P(x,1)/A_p(x,1)$. Using the lemma we conclude that

$$2^{1/p}/\pi = \lim_{z\to\infty} f(z) = \min\{f(1), \lim_{z\to\infty} f(z)\} \leqslant f(x)$$

for every $x \in [1,\infty)$ and that there exists a $y \in (1,\infty)$ which maximizes f, hence $f(x) \leqslant f(y)$. It follows that

$$\frac{b}{a} \geqslant \frac{f(y)}{\lim_{z\to\infty} f(z)} = \frac{\pi(y-1)}{(4\arctan(\sqrt{y})-\pi)(y^p+1)^{1/p}}.$$

The right hand side grows as p grows, and so the minimal lower bound is obtained for the least p, hence $p = p_0$. For this value of p a numerical computation gives $y \approx 38.0586$ and so we find $c_0 = f(y) \approx 1.0298$. □

Proof of Theorem 1.2. If $aA_p \leqslant P \leqslant bA_q$, then the largest ratio between the upper and lower bounds is at $x = 1$ or at the limit $x \to \infty$ according as $p \geqslant q$ or $p \leqslant q$. Hence the ratio equals

$$\frac{b}{a}\max\{1, 2^{1/p-1/q}\}.$$

Consider fixed p and a. If $q \leqslant p$ then the ratio equals b/a and we should choose q so as to be able to minimize b. But this means choosing q as large as possible (since A_q grows in q), that is $q = p$, in which case we can use Lemma 2.2. For $q > p$ the ratio equals $(b2^{-1/q})/(a2^{-1/p})$. For fixed a we should choose p as large as possible, in particular, there is no need for considering $p < p_0$, since a can never be larger than 1. Similarly, we conclude that there is no need to consider $q > 2/3$. Since $p_0 \leqslant p < q \leqslant 2/3$ Lemma 2.1 is applicable to the lower bound. Hence we have $a = \min\{1, 2^{1/p}/\pi\}$ and so the ratio equals $\pi b 2^{-1/q}$. Since $b \geqslant P(x,1)/A_q(x,1)$ for every $x \in [1,\infty)$, we find that

$$\pi b 2^{-1/q} \geqslant \sup_{x\in[1,\infty)} \frac{\pi(x-1)}{(4\arctan(\sqrt{x})-\pi)(x^q+1)^{1/q}}.$$

As in the last part of the proof of Lemma 2.2, we find that the right hand side is increasing in q, hence q should be chosen as small as possible, i.e. $q = p_0$ (here we use continuity of the ratio, since actually we had $q > p$). $\square$

It remains only to prove Lemma 2.1. The proof is based on differentiating suitably, eliminating all difficult terms. This means that we start by defining a bunch of functions related to the previous functions, specifically, which are simultaneously positive. The actual argument begins only in the last paragraph of the proof, where we untwine this jungle of functions.

Proof of Lemma 2.1. Let us denote $f(x) := P(x,1)/A_p(x,1)$. We have to prove that f is initially increasing and then decreasing. We can do this by showing that the logarithmic derivative has characteristic $+|-$. Since f is continuous, we may assume that $x > 1$. We then have

$$\frac{d\log f(x)}{dx} = \frac{1}{x-1} - \frac{2}{(x+1)\sqrt{x}}\frac{1}{4\arctan(\sqrt{x})-\pi} - \frac{x^{p-1}}{x^p+1}.$$

Let us multiply the inequality $d\log f/dx \geqslant 0$ by $x(x-1)(x^p+1)(4\arctan(\sqrt{x})-\pi)$ and divide by $x^p + x$. We get the inequality

$$g(x) := 4\arctan(\sqrt{x}) - \pi - 2\sqrt{x}\frac{(x-1)(x^p+1)}{(x+1)(x^p+x)} \geqslant 0,$$

which holds if and only if $d\log f/dx \geqslant 0$. We will show that $g'(x)$ has characteristic $+|-$. We find that

$$g'(x) = \frac{(x-1)\sqrt{x}}{(x+1)^2(x^p+x)^2}[-(x+3)x^{2p-1} - (2p-1)(x^2-1)x^{p-1} + 3x + 1]$$

hence g' is positive if and only if $h(x) := -(x+3)x^{2p-1} - (2p-1)(x^2-1)x^{p-1} + 3x + 1$ is. Now we find that

$$m(x) := h''(x)x^{3-p} = 2(1-2p)px^{p+1} - 6(2p^2-3p+1)x^p \\ - (2p^2+p-1)px^2 + 2p^3 - 7p^2 + 7p - 2.$$

Clearly $h''(x)$ is positive if and only if $m(x)$ is. We find that

$$m'(x) = -2(2p^2+p-1)p(x^p+x) - 6(2p^2-3p+1)px^{p-1} \\ \leqslant -2(5p-1)(2p-1)px^{p-1},$$

where the inequality follows since $x^p + x \geqslant 2x^{p-1}$. Therefore $m'(x) < 0$ for $p > 1/2$, and hence m is decreasing. Since m is decreasing and $m(1) = -24p^2 + 28p - 8 = -4(2p-1)(3p-2) > 0$ and $m(x) \to -\infty$ as $x \to \infty$ for $1/2 < p < 2/3$ it follows that m and hence h'' have characteristic $+|-$. Hence h' is first increasing and then decreasing. Since $h'(1) = 4(2-3p) > 0$ it follows that h' is first increasing and positive then decreasing and eventually negative, since $\lim_{x\to\infty} h'(x) = -\infty$, hence h' has characteristic $+|-$. In the same way we also conclude that h has characteristic $+|-$, which means that g' has characteristic $+|-$. Going through the same steps once more, we find that g too has characteristic $+|-$, which concludes the proof since g was constructed so as to be positive if and only if $d\log f/dx \geqslant 0$. $\square$

In conclusion, some ways in which the results of this paper could be extended are stated. One possibility would be to replace the Seiffert mean by Seiffert-Toader type means, defined in [10], for instance with $M = A_q$.

Another interesting extension would be to consider approximation of Seiffert's mean with extended mean values, which are generalizations of the power means. Indeed some results in this direction were already obtained in [2] (Theorem 1.6 and Corollary 6.2), although these are obviously far from optimal. Of course it might also be possible to combine both of the proposed extensions in a more general result.

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