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# A NEW SUBCLASS OF COMPLEX HARMONIC FUNCTIONS

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*Abstract.* Complex valued harmonic functions that are univalent and sense preserving in the unit disk $U$ can be written in the form $f = h + \overline{g}$, where $h$ and $g$ are analytic in $U$. In this paper, we introduce a class $HP(\alpha)$, $(\alpha \geqslant 0)$ of functions which are harmonic in $U$. We give sufficient coefficient conditions for normalized harmonic functions in $HP(\alpha)$. These conditions are also shown to be necessary when the coefficients are negative. This leads to distortion bounds and extreme points.

## 1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a domain $D \subset \mathbb{C}$ if both $u$ and $v$ are real harmonic in $D$. In any simply connected domain we can write $f = h + \bar{g}$, where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$ the co-analytic part of $f$. A necessary and sufficient condition for $f$ to be locally univalent and sense preserving in $D$ is that $|h'(z)| > |g'(z)|$ in $D$. See Clunie and Sheil-Small [1].

Denote by $S_H$ the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense preserving in the unit disk $U = \{ z : |z| < 1 \}$ for which $h(0) = f(0) - 1 = 0$. Then for $f = h + \bar{g} \in S_H$ we may express the analytic functions $h$ and $g$ as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{1}$$

Note that $S_H$ reduces to the class of normalized analytic univalent functions if the co-analytic part of its members is zero. In 1984 Clunie and Sheil-Small [1] investigated the class $S_H$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on $S_H$ and its subclasses.

The class of functions $f = h + \bar{g}$ with negative coefficients which are starlike and convex were investigated by Silverman [3].

We denote by $HP(\alpha)$ the class of all functions of the form (1) that satisfy the condition

$$\text{Re } \{\alpha z[h''(z) + g''(z)] + h'(z) + g'(z)\} > 0, \quad \alpha \geqslant 0. \tag{2}$$

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We further denote by $HF(\alpha)$ the subclass of $HP(\alpha)$ such that the functions $h$ and $g$ in $f=h+\bar{g}$ are of the form

$$h(z)=z-\sum_{n=2}^{\infty}|a_n|\,z^n, \qquad g(z)=-\sum_{n=1}^{\infty}|b_n|\,z^n. \tag{3}$$

## 2. Main results

THEOREM 2.1. *Let $f=h+\bar{g}$ be given by (1). Furthermore, let*

$$\sum_{n=1}^{\infty} n[\alpha(n-1)+1](|a_n|+|b_n|)\leqslant 2, \quad 0\leqslant |b_1|<1 \tag{4}$$

*where $a_1=1$ and $\alpha\geqslant 0$. Then $f$ is harmonic univalent sense preserving in $U$ and $f\in HP(\alpha)$.*

*Proof.* For $|z_1|\leqslant |z_2|<1$ we have by (4),

$$\begin{aligned}
|f(z_1)-f(z_2)| &\geqslant |h(z_1)-h(z_2)|-|g(z_1)-g(z_2)|\\
&= \left|(z_1-z_2)+\sum_{n=2}^{\infty}a_n(z_1^n-z_2^n)\right|-\left|\sum_{n=1}^{\infty}b_n(z_1^n-z_2^n)\right|\\
&\geqslant |z_1-z_2|\left(1-|b_1|-\sum_{n=2}^{\infty}n(|a_n|+|b_n|)|z_2|^{n-1}\right)\\
&\geqslant |z_1-z_2|\left(1-|b_1|-|z_2|\sum_{n=2}^{\infty}n[\alpha(n-1)+1](|a_n|+|b_n|)\right)\\
&\geqslant |z_1-z_2|(1-|b_1|)(1-|z_2|)>0.
\end{aligned}$$

Consequently, $f$ is univalent in $U$. We note that $f$ is sense preserving in $U$. This is because

$$\begin{aligned}
|h'(z)| &\geqslant 1-\sum_{n=2}^{\infty}n|a_n||z|^{n-1}>1-\sum_{n=2}^{\infty}n|a_n|\geqslant 1-\sum_{n=2}^{\infty}n[\alpha(n-1)+1]|a_n|\\
&\geqslant \sum_{n=1}^{\infty}n[\alpha(n-1)+1]|b_n|>\sum_{n=1}^{\infty}n|b_n||z|^{n-1}\geqslant |g'(z)|.
\end{aligned}$$

Now we show that $f\in HP(\alpha)$. Using the fact that $\operatorname{Re} w>0$ if and only if $|1+w|>|1-w|$, it sufficies to show that

$$|1+\alpha z(h''(z)+g''(z))+h'(z)+g'(z)|-|1-\alpha z(h''(z)+g''(z))-h'(z)-g'(z)|>0. \tag{5}$$

Substituting for $h(z)$ and $g(z)$ in (5) yields by (4),

$$|1+\alpha z(h''(z)+g''(z))+h'(z)+g'(z)|-|1-\alpha z(h''(z)+g''(z))-h'(z)-g'(z)|$$

$$\begin{aligned}
&= \left|2+b_1+\sum_{n=2}^{\infty} n[\alpha(n-1)+1](a_n+b_n)z^{n-1}\right| \\
&\qquad - \left|-b_1-\sum_{n=2}^{\infty} n[\alpha(n-1)+1](a_n+b_n)z^{n-1}\right| \\
&\geqslant 2(1-|b_1|)-2\sum_{n=2}^{\infty} n[\alpha(n-1)+1](|a_n|+|b_n|)|z|^{n-1} \\
&> 2\left\{1-\left(\sum_{n=2}^{\infty} n[\alpha(n-1)+1]|a_n|+\sum_{n=1}^{\infty} n[\alpha(n-1)+1]|b_n|\right)\right\} \\
&\geqslant 0.
\end{aligned}$$

The harmonic mappings

$$f(z)=z+\sum_{n=2}^{\infty}\frac{x_n}{n[\alpha(n-1)+1]}z^n+\sum_{n=1}^{\infty}\frac{\bar{y}_n}{n[\alpha(n-1)+1]}\bar{z}^n, \tag{6}$$

where $\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=1$, show that the coefficient bound given by (4) is sharp.

The functions of the form (6) are in $HP(\alpha)$ because

$$\sum_{n=1}^{\infty} n[\alpha(n-1)+1](|a_n|+|b_n|)=1+\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=2.$$

The restriction placed in Theorem 2.1 on the moduli of the coefficients of $f=h+\bar{g}$ enables us to conclude, for an arbitrary rotation of the coefficients of $f$, that the resulting functions would still be harmonic univalent and $f\in HP(\alpha)$. Our next theorem establishes that such coefficient bounds cannot be improved.

THEOREM 2.2. *Let $f=h+\bar{g}$ be given by (3). Then $f\in HF(\alpha)$ if and only if*

$$\sum_{n=1}^{\infty} n[\alpha(n-1)+1](|a_n|+|b_n|)\leqslant 2, \tag{7}$$

*where $a_1=1$ and $\alpha\geqslant 0$.*

*Proof.* We first suppose that $f\in HF(\alpha)$. Then we find from (2) that

$$\operatorname{Re}\left\{1-|b_1|-\sum_{n=2}^{\infty} n[\alpha(n-1)+1](|a_n|+|b_n|)z^{n-1}\right\}>0,\ z\in U,\ \alpha\geqslant 0,\ |b_1|<1.$$

If we choose $z$ to be real and let $z\to 1^-$, we get

$$1-|b_1|-\sum_{n=2}^{\infty} n[\alpha(n-1)+1](|a_n|+|b_n|)\geqslant 0$$

which is precisely the assertion (7) of Theorem 2.2.

Conversely, suppose that the inequality (7) holds true. Then we find from the definition (2) that

$$\begin{aligned}\operatorname{Re}\left\{1-|b_1|-\sum_{n=2}^{\infty} n[\alpha(n-1)+1](|a_n|+|b_n|)z^{n-1}\right\}& \\ \geqslant 2-\sum_{n=1}^{\infty} n[\alpha(n-1)+1](|a_n|+|b_n|)|z|^{n-1}& \\ > 2-\sum_{n=1}^{\infty} n[\alpha(n-1)+1](|a_n|+|b_n|) \geqslant 0,&\end{aligned}$$

provided that the inequality (7) is satisfied.

THEOREM 2.3. *If $f\in HF(\alpha)$ then*

$$|f(z)| \leqslant (1+|b_1|)r+\frac{1-|b_1|}{2(1+\alpha)}r^2, \quad |z|=r<1,$$

*and*

$$|f(z)| \geqslant (1-|b_1|)r-\frac{1-|b_1|}{2(1+\alpha)}r^2, \quad |z|=r<1.$$

*Proof.* We only prove the left hand inequality. The right hand inequality can be proved using similar arguments. Let $f\in HF(\alpha)$, then by Theorem 2.2, we obtain

$$\begin{aligned}|f(z)| &\geqslant (1-|b_1|)r-\sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n \geqslant (1-|b_1|)r-\sum_{n=2}^{\infty}(|a_n|+|b_n|)r^2 \\ &= (1-|b_1|)r-\frac{1}{2(1+\alpha)}\sum_{n=2}^{\infty}2(1+\alpha)(|a_n|+|b_n|)r^2 \\ &\geqslant (1-|b_1|)r-\frac{1}{2(1+\alpha)}\sum_{n=2}^{\infty}n[\alpha(n-1)+1](|a_n|+|b_n|)r^2 \\ &\geqslant (1-|b_1|)r-\frac{1}{2(1+\alpha)}[1-|b_1|]r^2.\end{aligned}$$

The bounds given in Theorem 2.3 for the functions $f=h+\bar{g}$ of the form (3) also hold for functions of the form (1) if the coefficient condition (4) is satisfied. The functions

$$f(z)=z-|b_1|\,\bar{z}-\frac{1-|b_1|}{2(1+\alpha)}\,\bar{z}^2$$

and

$$f(z)=(1-|b_1|)\,z-\frac{1-|b_1|}{2(1+\alpha)}\,z^2$$

for $|b_1|<1$ show that the bounds given in Theorem 2.3 are sharp.

The following result follows from the second inequality in Theorem 2.3.

COROLLARY 2.4. *If* $f \in HF(\alpha)$ *then*

$$\left\{ w : |w| < \frac{1}{2(1+\alpha)}(1+2\alpha)(1-|b_1|) \right\} \subset f(U).$$

THEOREM 2.5. *If* $\alpha_1 \geqslant \alpha_2 \geqslant 0$, *then* $HF(\alpha_1) \subset HF(\alpha_2)$.

*Proof.* Let $f \in HF(\alpha_1)$ and let $\alpha_1 \geqslant \alpha_2 \geqslant 0$. Then by Theorem 2.2, we have

$$\sum_{n=1}^{\infty} n[\alpha_2(n-1)+1](|a_n|+|b_n|) \leqslant \sum_{n=1}^{\infty} n[\alpha_1(n-1)+1](|a_n|+|b_n|) \leqslant 2.$$

Hence $f$ is in $HF(\alpha_2)$.

Since $HF(\alpha)$ is a convex family, we will use the necessary and sufficient coefficient conditions of Theorem 2.2 to determine its extreme points.

THEOREM 2.6. *Set*

$$h_1(z) = z,\ h_n(z) = z - \frac{1}{n[\alpha(n-1)+1]} z^n, \quad (n = 2, 3, \ldots)$$

*and*

$$g_n(z) = z - \frac{1}{n[\alpha(n-1)+1]} \bar{z}^n, \quad (n = 1, 2, \ldots).$$

*Then* $f \in HF(\alpha)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=1}^{\infty} (\lambda_n h_n + \gamma_n g_n), \tag{8}$$

*where* $\lambda_n \geqslant 0,\ \gamma_n \geqslant 0,\ \sum_{n=1}^{\infty} (\lambda_n + \gamma_n) = 1.$

In particular, the extreme points of $HF(\alpha)$ are $\{h_n\}$ and $\{g_n\}$.

*Proof.* For functions $f$ of the form $(8)$ we have

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} (\lambda_n h_n + \gamma_n g_n) \\ &= \sum_{n=1}^{\infty} (\lambda_n + \gamma_n) z - \sum_{n=2}^{\infty} \frac{1}{n[\alpha(n-1)+1]} \lambda_n z^n - \sum_{n=1}^{\infty} \frac{1}{n[\alpha(n-1)+1]} \gamma_n \bar{z}^n. \end{aligned}$$

Then

$$\begin{aligned} \sum_{n=2}^{\infty} n[\alpha_2(n-1)+1] \frac{\lambda_n}{n[\alpha(n-1)+1]} &+ \sum_{n=1}^{\infty} n[\alpha_2(n-1)+1] \frac{\gamma_n}{n[\alpha(n-1)+1]} \\ &= \sum_{n=1}^{\infty} (\lambda_n + \gamma_n) - \lambda_1 = 1 - \lambda_1 \leqslant 1 \end{aligned}$$

and so $f \in HF(\alpha)$.

Conversely, suppose that $f \in HF(\alpha)$. $\lambda_n = n[\alpha(n-1)+1]|a_n|$ $(n = 2, 3, \ldots)$ and $\gamma_n = n[\alpha(n-1)+1]|b_n|$ $(n = 1, 2, 3, \ldots)$. Then note that by Theorem 2.2., $0 \leqslant \lambda_n \leqslant 1$ $(n = 2, 3, \ldots)$ and $0 \leqslant \gamma_n \leqslant 1$ $(n = 1, 2, 3, \ldots)$. We define

$$\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n - \sum_{n=1}^{\infty} \gamma_n$$

and note that, by Theorem 2.2., $\lambda_1 \geqslant 0$. Consequently, we obtain

$$f(z) = \sum_{n=1}^{\infty} (\lambda_n h_n + \gamma_n g_n)$$

as required.

Following Ruscheweyh [2] we call the $\delta$-neighborhood of $f$ the set

$$\begin{aligned} N_\delta(f) = \Bigg\{ & F : F(z) = z - \sum_{n=2}^{\infty} |A_n|\, z^n - \sum_{n=1}^{\infty} |B_n|\, \bar{z}^n \text{ and} \\ & \sum_{n=2}^{\infty} n(|a_n - A_n| + |b_n - B_n|) + |b_1 - B_1| \leqslant \delta \Bigg\}. \end{aligned}$$

In particular, for the identity function $I(z) = z$, we immediately have

$$\begin{aligned} N_\delta(I) = \Bigg\{ & f : f(z) = z - \sum_{n=2}^{\infty} |a_n|\, z^n - \sum_{n=1}^{\infty} |b_n|\, \bar{z}^n \text{ and} \\ & \sum_{n=2}^{\infty} n(|a_n| + |b_n|) + |b_1| \leqslant \delta \Bigg\}. \end{aligned}$$

THEOREM 2.7. *Let*

$$\delta = \frac{1 + \alpha|b_1|}{\alpha + 1}$$

*then* $HF(\alpha) \subset N_\delta(I)$.

*Proof.* Let $f$ belong to $HF(\alpha)$. We have

$$\begin{aligned} |b_1| + \sum_{n=2}^{\infty} n(|a_n| + |b_n|) &\leqslant |b_1| + \frac{1}{\alpha+1} \sum_{n=2}^{\infty} n[\alpha_2(n-1)+1](|a_n| + |b_n|) \\ &\leqslant |b_1| + \frac{1}{\alpha+1}(1 - |b_1|) = \delta. \end{aligned}$$

Hence $f(z) \in N_\delta(I)$.

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