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ON THE HYERS–ULAM–RASSIAS STABILITY OF AN n–DIMENSIONAL PEXIDERIZED QUADRATIC EQUATION

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(*communicated by Th. M. Rassias*)

Abstract. In this paper we prove the stability of an n-dimensional Pexiderized quadratic equation $f_1(\sum_{i=1}^{n} x_i) + \sum_{1\leqslant i<j\leqslant n} f_{\alpha(i,j)}(x_i - x_j) = n\sum_{i=1}^{n} f_{\beta(i)}(x_i)$ in the spirits of Hyers, Ulam and Rassias.

1. Introduction

In 1940, S. M. Ulam gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems (ref. [25]). Among those was the question concerning the stability of homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The case of approximately additive mappings was solved by D. H. Hyers [5] under the assumption that G_1 and G_2 are Banach spaces. In 1978, Th. M. Rassias [17] gave a significant generalization of the Hyers's result. P. Gavruta [4] also obtained a further generalization of the Hyers-Ulam-Rassias theorem.

The quadratic functional equation

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) = 0 \tag{1.1}$$

clearly has $f(x) = cx^2$ as a solution with c an arbitrary constant when f is a real function of a real variable. We define any solution of (1.1) to be a *quadratic function*. A Hyers-Ulam stability theorem for the equation (1.1) was proved by F. Skof for functions $f : V \to X$ where V is a normed space and X a Banach space (see [24]). In 1984, P. W. Cholewa [2] extended V by an Abelian group G in the Skof's result. In the paper [3], S. Czerwik proved the Hyers-Ulam-Rassias stability of the quadratic functional

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equation (1.1) and this result was generalized by a number of mathematicians (see [1,6,7,11–13,16,18–23]).

Throughout this paper, let V and X be a normed space and a Banach space, respectively. Lee and Jun [14,15] proved the Hyers-Ulam-Rassias stability of the Pexider equation of $f(x+y) = g(x) + h(y)$ (see also [10]) and also [8,9] proved the Hyers-Ulam-Rassias stability of the Pexiderized quadratic equation of $f(x+y) + g(x-y) - 2h(x) - 2k(y) = 0$.

In this paper, we prove the stability of the n-dimensional quadratic and the n-dimensional Pexiderized quadratic functional equations:

$$f\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\leqslant i<j\leqslant n} f(x_i - x_j) \; = \; n\sum_{i=1}^{n} f(x_i), \tag{1.2}$$

$$f_1\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\leqslant i<j\leqslant n} f_{\alpha(i,j)}(x_i - x_j) = n\sum_{i=1}^{n} f_{\beta(i)}(x_i). \tag{1.3}$$

2. Stability of the equation (1.2)

Let $n \geqslant 2$ be a given positive integer. We denote by $\varphi : (V\setminus\{0\})^n \to [0,\infty)$ a function such that

$$\tilde{\varphi}(x_1, x_2, \ldots, x_n) := \sum_{l=0}^{\infty} \frac{1}{n^{2(l+1)}} \varphi(n^l x_1, n^l x_2, \ldots, n^l x_n) < \infty \tag{a}$$

or

$$\tilde{\varphi}(x_1, x_2, \ldots, x_n) := \sum_{l=0}^{\infty} n^{2l} \varphi\left(\frac{x_1}{n^{l+1}}, \frac{x_2}{n^{l+1}}, \ldots, \frac{x_n}{n^{l+1}}\right) < \infty \tag{a'}$$

for all $x_1, x_2, \ldots, x_n \in V\setminus\{0\}$.

THEOREM 2.1 *Let φ be as above. Suppose that the function $f : V \to X$ satisfies*

$$\left\| f\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\leqslant i<j\leqslant n} f(x_i - x_j) - n\sum_{i=1}^{n} f(x_i) \right\| \leqslant \varphi(x_1, x_2, \ldots, x_n) \tag{2.1}$$

for all $x_1, x_2, \ldots, x_n \in V\setminus\{0\}$. Then there exists exactly one quadratic function $Q : V \to X$ such that

$$\left\| Q(x) - f(x) + \frac{n}{2(n+1)} f(0) \right\| \leqslant \tilde{\varphi}(x, x, \ldots, x)$$

for all $x \in V\setminus\{0\}$. The function Q is given by

$$Q(x) = \lim_{l\to\infty} \frac{f(n^l x)}{n^{2l}} \quad \textit{for all } x \in V \textit{ if } \varphi \textit{ satisfies (a)} \tag{2.2}$$

or

$$Q(x) = \begin{cases} \lim_{l\to\infty} n^{2l}\left[f\left(\frac{x}{n^l}\right) - \frac{n}{2(n+1)}f(0)\right] & \textit{for} \quad x \neq 0, \\ 0 & \textit{for} \quad x = 0 \end{cases}$$

if φ *satisfies* (a').

Proof. At first, we prove the result for the case that φ satisfies the condition (a). Replacing x_i by x for all $i = 1, 2, \ldots, n$ in (2.1), we obtain

$$\left\|f(nx) + \frac{n(n-1)}{2}f(0) - n^2 f(x)\right\| \leqslant \varphi(x, x, \ldots, x) \tag{2.3}$$

for $x \in V \setminus \{0\}$. Dividing by n^2 in (2.3), we easily obtain

$$\begin{aligned} &\left\|\frac{1}{n^2}\left(f(nx) - \frac{n}{2(n+1)}f(0)\right) - \left(f(x) - \frac{n}{2(n+1)}f(0)\right)\right\| \\ &\quad \leqslant \frac{1}{n^2}\varphi(x, x, \ldots, x) \end{aligned} \tag{2.4}$$

for $x \in V \setminus \{0\}$. Applying an induction argument to k in (2.4), we easily obtain

$$\begin{aligned} &\left\|\frac{1}{n^{2k}}\left(f(n^k x) - \frac{n}{2(n+1)}f(0)\right) - \left(f(x) - \frac{n}{2(n+1)}f(0)\right)\right\| \\ &\quad \leqslant \sum_{l=0}^{k-1} \frac{1}{n^{2(l+1)}}\varphi(n^l x, n^l x, \ldots, n^l x) \\ &\quad \leqslant \tilde{\varphi}(x, x, \ldots, x) \end{aligned} \tag{2.5}$$

for $x \in V \setminus \{0\}$ and for all $k \in \mathbb{N}$. Replacing x by $n^l x$ and dividing by n^{2l} in (2.5), we have

$$\left\|\frac{1}{n^{2(k+l)}}\left(f(n^{(k+l)}x) - \frac{n}{2(n+1)}f(0)\right) - \frac{1}{n^{2l}}\left(f(n^l x) - \frac{n}{2(n+1)}f(0)\right)\right\|$$

$$\leqslant \frac{1}{n^{2l}}\tilde{\varphi}(n^l x, n^l x, \ldots, n^l x)$$

for all $l > 0$ and $x \in V \setminus \{0\}$. This shows that $\{\frac{1}{n^{2l}}(f(n^l x) - \frac{n}{2(n+1)}f(0))\}$ is a Cauchy sequence. Because X is a Banach space, the sequence $\{\frac{1}{n^{2l}}(f(n^l x) - \frac{n}{2(n+1)}f(0))\}$ converges. Define $Q : V \to X$ by

$$Q(x) = \lim_{l\to\infty} \frac{f(n^l x) - \frac{n}{2(n+1)}f(0)}{n^{2l}} = \lim_{l\to\infty} \frac{f(n^l x)}{n^{2l}} \tag{2.6}$$

for all $x \in V$. From (2.6), we easily get that

$$Q(0) = 0 \quad \text{and} \quad Q(nx) = n^2 Q(x)$$

for all $x \in V$. Replacing x_i by $n^l x_i$ for all i and dividing by n^{2l} in (2.1), we obtain

$$\left\| \frac{f(\sum_{i=1}^{n} n^l x_i)}{n^{2l}} + \sum_{1 \leqslant i < j \leqslant n} \frac{f(n^l x_i - n^l x_j)}{n^{2l}} - n \sum_{i=1}^{n} \frac{f(n^l x_i)}{n^{2l}} \right\| \leqslant \frac{\varphi(n^l x_1, n^l x_2, \ldots, n^l x_n)}{n^{2l}} \tag{2.7}$$

for all $x_1, x_2, \ldots, x_n \in V \setminus \{0\}$. Taking the limit in (2.7) as $l \to \infty$, we get

$$Q\left(\sum_{i=1}^{n} x_i\right) + \sum_{1 \leqslant i < j \leqslant n} Q(x_i - x_j) - n \sum_{i=1}^{n} Q(x_i) = 0 \tag{2.8}$$

for all $x_1, x_2, \ldots, x_n \in V \setminus \{0\}$. Replacing x_1 by y and x_i by x for all $i = 2, 3, \ldots, n$ in (2.8), we obtain

$$Q(y + (n-1)x) + (n-1)Q(y - x) - n(n-1)Q(x) - nQ(y) = 0 \tag{2.9}$$

for all $x, y \in V \setminus \{0\}$. Replacing x_2 by y and x_i by x for all $i = 1, 3, \ldots, n$ in (2.8), we obtain

$$Q(y + (n-1)x) + Q(x - y) + (n-2)Q(y - x) - n(n-1)Q(x) - nQ(y) = 0 \tag{2.10}$$

for all $x, y \in V \setminus \{0\}$. From (2.9) and (2.10), we have

$$Q(x) = Q(-x) \tag{2.11}$$

for all $x \in V$. When n is an even number, replacing x_i by x for $1 \leqslant i \leqslant \frac{n}{2} + 1$ and $-x$ for $\frac{n}{2} + 2 \leqslant i \leqslant n$ in (2.8), we obtain

$$Q(2x) + \left(\frac{n}{2} - 1\right)\left(n - \frac{n}{2} + 1\right)Q(2x) = n^2 Q(x) \tag{2.12}$$

for all $x \in V \setminus \{0\}$. When n is an odd number, replacing x_i by x for $1 \leqslant i \leqslant \frac{n+1}{2}$ and $-x$ for $\frac{n+1}{2} + 1 \leqslant i \leqslant n$ in (2.8), we obtain

$$Q(x) + \left(\frac{n+1}{2} - 1\right)\left(n - \frac{n+1}{2} + 1\right)Q(2x) = n^2 Q(x) \tag{2.13}$$

for all $x \in V \setminus \{0\}$. From (2.12) and (2.13), we obtain

$$Q(2x) = 4Q(x) \tag{2.14}$$

for all $x \in V$. Replacing x_1, x_2 by $x + y, x - y$ and x_i by x for $i = 3, 4, \ldots, n$ in (2.8), we obtain

$$\begin{aligned} &Q(nx) + Q(2y) + (n-2)Q(y) + (n-2)Q(-y) \\ &= n(n-2)Q(x) + nQ(x+y) + nQ(x-y) \end{aligned} \tag{2.15}$$

for $x, y \in V \setminus \{0\}$ with $x+y, x-y \in V \setminus \{0\}$. From (2.11), (2.14) and (2.15), we obtain

$$Q(x+y)+Q(x-y)-2Q(x)-2Q(y)=0$$

for $x, y \in V \setminus \{0\}$ with $x+y, x-y \in V \setminus \{0\}$. Hence we easily have that

$$Q(x+y)+Q(x-y)-2Q(x)-2Q(y)=0$$

for all $x, y \in V$. From (2.5), we have the inequality

$$\left\| Q(x)-f(x)+\frac{n}{2(n+1)}f(0) \right\| \leqslant \tilde{\varphi}(x,x,\ldots,x)$$

for all $x \in V \setminus \{0\}$. This completes the proof for the case that φ satisfies the condition (a).

Similarly, we can prove the result for the case that φ satisfies the condition (a'). □

COROLLARY 2.2. *Let $p \neq 2$, $\theta > 0$ be real numbers. Suppose that the function $f : V \to X$ satisfies*

$$\left\| f\left(\sum_{i=1}^{n} x_i\right) + \sum_{1 \leqslant i<j \leqslant n} f(x_i - x_j) - n\sum_{i=1}^{n} f(x_i) \right\| \leqslant \theta(\|x_1\|^p + \|x_2\|^p + \ldots + \|x_n\|^p)$$

for all $x_1, x_2, \ldots, x_n \in V \setminus \{0\}$. Then there exists exactly one quadratic function $Q : V \to X$ such that

$$\left\| Q(x)-f(x)+\frac{n}{2(n+1)}f(0) \right\| \leqslant \frac{n}{|n^2-n^p|}\theta\|x\|^p$$

for all $x \in V \setminus \{0\}$. The function Q is given by (2.2).

3. Stability of the equation (1.3)

DEFINITION 3.1. Let $\alpha : A \to \mathbb{N}$ and $\beta : \{1,2,\ldots,n\} \to \mathbb{N}$ be defined by

$$\alpha(i,j) = \frac{(i-1)(2n-i-2)}{2} + j,$$
$$\beta(i) = \frac{n(n-1)}{2} + i + 1,$$

where $A = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leqslant i < j \leqslant n\}$.

REMARK. Note that

$$\{\alpha(i,j) \mid (i,j) \in A\} = \left\{2, 3, \ldots, \frac{n(n-1)}{2} + 1\right\},$$
$$\{\beta(i) \mid i = 1,2,\ldots,n\} = \left\{\frac{n(n-1)}{2} + 2, \ldots, \frac{n(n+1)}{2} + 1\right\}.$$

The following lemma is seen in [7].

LEMMA 3.2. *Let* $\varphi : V \times V \to [0,\infty)$ *be a function such that*

$$\tilde{\varphi}(x,y) = \sum_{l=0}^{\infty} \frac{1}{4^{l+1}} \varphi(2^l x, 2^l y) < \infty \tag{b}$$

or

$$\tilde{\varphi}(x,y) = \sum_{l=0}^{\infty} 4^l \varphi\left(\frac{x}{2^{l+1}}, \frac{y}{2^{l+1}}\right) < \infty \tag{b'}$$

and

$$\check{\varphi}(x,y) = \sum_{l=0}^{\infty} \frac{1}{2^{l+1}} \varphi(2^l x, 2^l y) < \infty \tag{c}$$

or

$$\check{\varphi}(x,y) = \sum_{l=0}^{\infty} 2^l \varphi\left(\frac{x}{2^{l+1}}, \frac{y}{2^{l+1}}\right) < \infty \tag{c'}$$

for all $x,y \in V$. *Suppose that the functions* $f,g,h,k : V \to X$ *satisfy*

$$\|f(x+y) + g(x-y) - 2h(x) - 2k(y)\| \leqslant \varphi(x,y) \quad \textit{for all} \quad x,y \in V.$$

Then there exist exactly one quadratic function $Q : V \to X$ *and two unique additive functions* $T, T' : V \to X$ *such that*

$$\begin{aligned}
&\|f(x) - f(0) - Q(x) - T(x)\| \\
&\qquad \leqslant \frac{M_3(\varphi,x) + M_3(\varphi,-x) + M_4(\varphi,x,x) + M_4(\varphi,-x,-x)}{2}, \\
&\|g(x) - g(0) - Q(x) - T'(x)\| \\
&\qquad \leqslant \frac{M_3(\varphi,x) + M_3(\varphi,-x) + M_4(\varphi,x,-x) + M_4(\varphi,-x,x)}{2}, \\
&\left\|h(x) - h(0) - Q(x) - \frac{1}{2}(T(x) + T'(x))\right\| \\
&\qquad \leqslant \frac{M_3(\varphi,x) + M_3(\varphi,-x) + M_5(\varphi,x) + M_5(\varphi,-x)}{2} \\
&\qquad\quad + \frac{\varphi(x,0) + 2\varphi(0,0) + \varphi(-x,0)}{4}, \\
&\left\|k(x) - k(0) - Q(x) - \frac{1}{2}(T(x) - T'(x))\right\| \\
&\qquad \leqslant \frac{M_3(\varphi,x) + M_3(\varphi,-x) + M_5(\varphi,x) + M_5(\varphi,-x)}{2} \\
&\qquad\quad + \frac{\varphi(0,x) + 2\varphi(0,0) + \varphi(0,-x)}{4}
\end{aligned}$$

for all $x \in V$, *where*

$$\begin{aligned}
M_3(\varphi, x) &= \frac{1}{2}\Big[\tilde{\varphi}(x,x) + 2\tilde{\varphi}(0,x) + 2\tilde{\varphi}(x,0) \\
&\quad + \tilde{\varphi}(x,-x) + 2\tilde{\varphi}(0,0) + \varphi\Big(\frac{x}{2},\frac{x}{2}\Big) + \varphi\Big(\frac{x}{2},-\frac{x}{2}\Big)\Big], \\
M_4(\varphi, x, y) &= \check{\varphi}(x,0) + \check{\varphi}(0,y) + \check{\varphi}(x,y), \\
M_5(\varphi, x) &= \frac{1}{4}[\check{\varphi}(x,-x) + 2\check{\varphi}(x,0) + \check{\varphi}(2x,-x) + \check{\varphi}(-x,x) \\
&\quad + \check{\varphi}(0,x) + \check{\varphi}(-x,2x) + \check{\varphi}(x,x) + \check{\varphi}(2x,x) \\
&\quad + \check{\varphi}(-x,-x) + \check{\varphi}(0,-x) + \check{\varphi}(-x,-2x)]
\end{aligned}$$

for all $x, y \in V$. *The function* Q *is given by*

$$Q(x) = \begin{cases} \lim_{n\to\infty} \frac{f(2^n x)}{4^n} = \lim_{n\to\infty} \frac{g(2^n x)}{4^n} & \text{if } \varphi \text{ satisfies (b),} \\ \lim_{n\to\infty} \frac{4^n}{2}\big(f(\frac{x}{2^n}) + f(-\frac{x}{2^n}) - 2f(0)\big) & \text{if } \varphi \text{ satisfies (b')} \\ = \lim_{n\to\infty} \frac{4^n}{2}\big(g(\frac{x}{2^n}) + g(-\frac{x}{2^n}) - 2g(0)\big) & \end{cases}$$

and the functions T, T' *are given by*

$$T(x) = \begin{cases} \lim_{n\to\infty} \frac{f(2^n x) - f(-2^n x)}{2^{n+1}} & \text{if } \varphi \text{ satisfies (c),} \\ \lim_{n\to\infty} 2^n(f(2^{-n}x) - f(0)) & \text{if } \varphi \text{ satisfies (c'),} \end{cases}$$

$$T'(x) = \begin{cases} \lim_{n\to\infty} \frac{g(2^n x) - g(-2^n x)}{2^{n+1}} & \text{if } \varphi \text{ satisfies (c),} \\ \lim_{n\to\infty} 2^n(g(2^{-n}x) - g(0)) & \text{if } \varphi \text{ satisfies (c').} \end{cases}$$

Now, we denote by $\varphi : V^n \to [0, \infty)$ a function such that

$$\tilde{\varphi}(x_1, x_2, \ldots, x_n) = \sum_{l=0}^{\infty} \frac{1}{4^{l+1}} \varphi(2^l x_1, 2^l x_2, \ldots, 2^l x_n) < \infty \tag{d}$$

or

$$\tilde{\varphi}(x_1, x_2, \ldots, x_n) = \sum_{l=0}^{\infty} 4^l \varphi\Big(\frac{x_1}{2^{l+1}}, \frac{x_2}{2^{l+1}}, \ldots, \frac{x_n}{2^{l+1}}\Big) < \infty \tag{d'}$$

and

$$\check{\varphi}(x_1, x_2, \ldots, x_n) = \sum_{l=0}^{\infty} \frac{1}{2^{l+1}} \varphi(2^l x_1, 2^l x_2, \ldots, 2^l x_n) < \infty \tag{e}$$

or

$$\check{\varphi}(x_1, x_2, \ldots, x_n) = \sum_{l=0}^{\infty} 2^l \varphi\Big(\frac{x_1}{2^{l+1}}, \frac{x_2}{2^{l+1}}, \ldots, \frac{x_n}{2^{l+1}}\Big) < \infty \tag{e'}$$

for all $x_1, x_2, \ldots, x_n \in V$. Define $\varphi_{i,j}, \varphi'_{i,j} : V \times V \to [0, \infty)$ by

$$\varphi_{i,j}(x,y) = \varphi(0, \ldots, 0, \overset{i-th}{x}, 0, \ldots, 0, \overset{j-th}{y}, 0, \ldots, 0),$$

$$\varphi'_{i,j}(x,y) = \varphi\left(\frac{x+y}{2},\ldots,\frac{x+y}{2},\overset{i-th}{x},\frac{x+y}{2},\ldots,\frac{x+y}{2},\overset{j-th}{y},\frac{x+y}{2},\ldots,\frac{x+y}{2}\right),$$

where $i<j$.

THEOREM 3.3. *Let* $\varphi : V^n \to [0,\infty)$ *be as above. Suppose that the functions* $f_k : V \to X$ *satisfy*

$$\left\| f_1\Big(\sum_{i=1}^{n} x_i\Big) + \sum_{1\leqslant i<j\leqslant n} f_{\alpha(i,j)}(x_i - x_j) - n\sum_{i=1}^{n} f_{\beta(i)}(x_i)\right\| \leqslant \varphi(x_1,x_2,\ldots,x_n) \tag{3.1}$$

for all $x_1,x_2,\ldots,x_n \in V$, *where* $k=1$, $\alpha(i,j)$, $\beta(i)$, $1\leqslant i,j\leqslant n$. *Then there exist exactly one quadratic function* $Q : V \to X$ *and unique additve functions* $T_k : V \to X$ *such that*

$$\|f_k(x) - f_k(0) - Q(x) - T_k(x)\| \leqslant M_k(x)$$

for all $x\in V$, *where*

$$M_1(x) = \inf_{1\leqslant i<j\leqslant n} \frac{M_3(\varphi_{i,j},x) + M_3(\varphi_{i,j},-x) + M_4(\varphi_{i,j},x,x) + M_4(\varphi_{i,j},-x,-x)}{2},$$

$$M_{\alpha(i,j)}(x) = \frac{M_3(\varphi_{i,j},x) + M_3(\varphi_{i,j},-x) + M_4(\varphi_{i,j},x,-x) + M_4(\varphi_{i,j},-x,x)}{2},$$

$$M_{\beta(1)}(x) = \inf_{1<j\leqslant n}\Big[\frac{M_3(\varphi'_{1,j},x) + M_3(\varphi'_{1,j},-x) + M_5(\varphi'_{1,j},x) + M_5(\varphi'_{1,j},-x)}{n} + \frac{\varphi'_{1,j}(x,0) + 2\varphi'_{1,j}(0,0) + \varphi'_{1,j}(-x,0)}{2n}\Big],$$

$$M_{\beta(j)}(x) = \frac{M_3(\varphi'_{1,j},x) + M_3(\varphi'_{1,j},-x) + M_5(\varphi'_{1,j},x) + M_5(\varphi'_{1,j},-x)}{n} + \frac{\varphi'_{1,j}(0,x) + 2\varphi'_{1,j}(0,0) + \varphi'_{1,j}(0,-x)}{2n}.$$

The functions Q *and* T_k *are given by*

$$Q(x) = \begin{cases} \lim_{n\to\infty} 4^{-n} f_k(2^n x) & \text{if } \varphi \text{ satisfies (d)},\\ \lim_{n\to\infty} 4^{n}\left(\frac{f_k(2^{-n}x)+f_k(-2^{-n}x)}{2} - f_k(0)\right) & \text{if } \varphi \text{ satisfies (d')}, \end{cases}$$

$$T_k(x) = \begin{cases} \lim_{n\to\infty} 2^{-n-1}(f_k(2^n x) - f_k(-2^n x)) & \text{if } \varphi \text{ satisfies (e)},\\ \lim_{n\to\infty} 2^{n-1}(f_k(2^{-n} x) - f_k(-2^{-n} x)) & \text{if } \varphi \text{ satisfies (e')} \end{cases}$$

for all $x\in V$. *The functions* T_k *'s satisfy the following equations;*

$$T_1 - \sum_{m=1}^{i-1} T_{\alpha(m,i)} + \sum_{m=i+1}^{n} T_{\alpha(i,m)} = nT_{\beta(i)},$$

$$T_1 = \sum_{i=1}^{n} T_{\beta(i)}.$$

Proof. If φ satisfies the condition (d') or (e'), then we easily have that $\varphi(0,0,\ldots,0)=0$. Hence

$$\left\| f_1(0) + \sum_{1\leqslant i<j\leqslant n} f_{\alpha(i,j)}(0) - n\sum_{i=1}^{n} f_{\beta(i)}(0) \right\| = 0 \tag{3.2}$$

if φ satisfies the condition (d') or (e'). Replacing x_i, x_j by x, y, respectively and replacing x_l by 0 in (3.1) for all $l \in \{1,2,\ldots,n\} - \{i,j\}$, where $1 \leqslant i < j \leqslant n$, we get

$$\begin{aligned}
&\Bigg\| f_1(x+y) + f_{\alpha(i,j)}(x-y) + \sum_{l=1}^{i-1} f_{\alpha(l,i)}(-x) + \sum_{\substack{i<l\leqslant n\\ l\neq j}} f_{\alpha(i,l)}(x) \\
&\qquad + \sum_{\substack{1\leqslant l<j\\ l\neq i}} f_{\alpha(l,j)}(-y) + \sum_{j<l\leqslant n} f_{\alpha(j,l)}(y) + \sum_{\substack{1\leqslant l<m\leqslant n\\ l\neq i,j\\ m\neq i,j}} f_{\alpha(l,m)}(0) \\
&\qquad - nf_{\beta(i)}(x) - nf_{\beta(j)}(y) - n\sum_{\substack{1\leqslant l\leqslant n\\ l\neq i,j}} f_{\beta(l)}(0) \Bigg\| \\
&\qquad \leqslant \varphi_{i,j}(x,y)
\end{aligned}$$

for all $x,y \in V$. Define $f,g,h,k : V \to X$ by

$$\begin{aligned}
f(x) &= f_1(x),\\
g(x) &= f_{\alpha(i,j)}(x),\\
2h(x) &= -\sum_{1\leqslant l<i\leqslant n} f_{\alpha(l,i)}(-x) - \sum_{\substack{1\leqslant i<l\leqslant n\\ l\neq j}} f_{\alpha(i,l)}(x) + nf_{\beta(i)}(x),\\
2k(x) &= -\sum_{\substack{1\leqslant l<j\leqslant n\\ l\neq i}} f_{\alpha(l,j)}(-y) - \sum_{1\leqslant j<l\leqslant n} f_{\alpha(j,l)}(y) - \sum_{\substack{1\leqslant l<m\leqslant n\\ l\neq i,j\\ m\neq i,j}} f_{\alpha(l,m)}(0)\\
&\qquad + nf_{\beta(j)}(y) + n\sum_{\substack{1\leqslant l\leqslant n\\ l\neq i,j}} f_{\beta(l)}(0)
\end{aligned}$$

for all $x \in V$. Then we get

$$\|f(x+y) + g(x-y) - 2h(x) - 2k(y)\| \leqslant \varphi_{i,j}(x,y)$$

for all $x,y \in V$. By Lemma 3.2, there exist exactly one quadratic function $Q : V \to X$ and two unique additive functions $T_1, T_{\alpha(i,j)} : V \to X$ satisfying

$$\begin{aligned}
&\|f_1(x) - f_1(0) - Q(x) - T_1(x)\| \\
&\qquad \leqslant \frac{M_3(\varphi_{i,j},x) + M_3(\varphi_{i,j},-x) + M_4(\varphi_{i,j},x,x) + M_4(\varphi_{i,j},-x,-x)}{2},
\end{aligned}$$

$$
\begin{aligned}
&\|f_{\alpha(i,j)}(x) - f_{\alpha(i,j)}(0) - Q(x) - T_{\alpha(i,j)}(x)\| \\
&\qquad \leqslant \frac{M_3(\varphi_{i,j},x) + M_3(\varphi_{i,j},-x) + M_4(\varphi_{i,j},x,-x) + M_4(\varphi_{i,j},-x,x)}{2}
\end{aligned}
$$

for all $x \in V$ and the functions $Q, T_1, T_{\alpha(i,j)}$ are given by

$$
Q(x) = \begin{cases}
\lim_{n\to\infty} \frac{f_1(2^n x)}{4^n} = \lim_{n\to\infty} \frac{f_{\alpha(i,j)}(2^n x)}{4^n} & \text{if } \varphi \text{ satisfies (d)}, \\
\lim_{n\to\infty} 4^n \left(\frac{f_1(2^{-n}x) + f_1(-2^{-n}x)}{2} - f_1(0) \right) & \text{if } \varphi \text{ satisfies (d')} \\
= \lim_{n\to\infty} 4^n \left(\frac{f_{\alpha(i,j)}(2^{-n}x) + f_{\alpha(i,j)}(-2^{-n}x)}{2} - f_{\alpha(i,j)}(0) \right), &
\end{cases}
$$

$$
T_1(x) = \begin{cases}
\lim_{n\to\infty} \frac{f_1(2^n x) - f_1(-2^n x)}{2^{n+1}} & \text{if } \varphi \text{ satisfies (e)}, \\
\lim_{n\to\infty} 2^{n-1}(f_1(2^{-n}x) - f_1(-2^{-n}x)) & \text{if } \varphi \text{ satisfies (e')},
\end{cases}
$$

$$
T_{\alpha(i,j)}(x) = \begin{cases}
\lim_{n\to\infty} \frac{f_{\alpha(i,j)}(2^n x) - f_{\alpha(i,j)}(-2^n x)}{2^{n+1}} & \text{if } \varphi \text{ satisfies (e)}, \\
\lim_{n\to\infty} 2^{n-1}(f_{\alpha(i,j)}(2^{-n}x) - f_{\alpha(i,j)}(-2^{-n}x)) & \text{if } \varphi \text{ satisfies (e')}
\end{cases}
$$

for all $x \in V$ where $i < j$. Replacing x_1, x_j by x, y, respectively and replacing x_l by $\frac{x+y}{2}$ in (3.1) for all $l \in \{2,3,4,\ldots,n\} - \{j\}$, where $1 < j \leqslant n$, we obtain

$$
\begin{aligned}
&\Bigg\| f_1\left(\frac{n(x+y)}{2}\right) + f_j(x-y) + \sum_{m=2}^{j-1} f_{\alpha(1,m)}\Big(\frac{x-y}{2}\Big) + \sum_{m=j+1}^{n} f_{\alpha(1,m)}\left(\frac{x-y}{2}\right) \\
&\qquad + \sum_{m=2}^{j-1} f_{\alpha(m,j)}\left(\frac{x-y}{2}\right) + \sum_{m=j+1}^{n} f_{\alpha(j,m)}\left(-\frac{x-y}{2}\right) + \sum_{\substack{1<l<m\leqslant n \\ l,m\neq j}} f_{\alpha(l,m)}(0) \\
&\qquad - nf_{\beta(1)}(x) - nf_{\beta(j)}(y) - n\sum_{\substack{1<l\leqslant n \\ l\neq j}} f_{\beta(l)}\left(\frac{x+y}{2}\right) \Bigg\| \\
&\qquad \leqslant \varphi'_{1,j}(x,y)
\end{aligned}
$$

for all $x, y \in V$. Define $f, g, h, k : V \to X$ by

$$
\begin{aligned}
f(x) &= f_1\left(\frac{nx}{2}\right) - n\sum_{\substack{1<l\leqslant n \\ l\neq j}} f_{\beta(l)}\left(\frac{x}{2}\right) + \sum_{\substack{1<l<m\leqslant n \\ l,m\neq j}} f_{\alpha(l,m)}(0), \\
g(x) &= f_j(x) + \sum_{m=2}^{j-1} f_{\alpha(1,m)}\left(\frac{x}{2}\right) + \sum_{m=j+1}^{n} f_{\alpha(1,m)}\left(\frac{x}{2}\right) \\
&\quad + \sum_{m=2}^{j-1} f_{\alpha(m,j)}\left(\frac{x}{2}\right) + \sum_{m=j+1}^{n} f_{\alpha(j,m)}\left(-\frac{x}{2}\right), \\
2h(x) &= nf_{\beta(1)}(x), \\
2k(x) &= nf_{\beta(j)}(x)
\end{aligned}
$$

for all $x \in V$. Then we get

$$\|f(x+y)+g(x-y)-2h(x)-2k(y)\| \leqslant \varphi'_{1,j}(x,y)$$

for all $x,y \in V$. By Lemma 3.2, there exist exactly one quadratic function $Q' : V \to X$ and two unique additive functions $T_{\beta(1)}, T_{\beta(j)} : V \to X$ satisfying

$$\begin{aligned}
\|f_{\beta(1)}(x)-&f_{\beta(1)}(0) - Q'(x) - T_{\beta(1)}(x)\| \\
&\leqslant \frac{M_3(\varphi'_{1,j},x)+M_3(\varphi'_{1,j},-x)+M_5(\varphi'_{1,j},x)+M_5(\varphi'_{1,j},-x)}{n} \\
&\quad + \frac{\varphi'_{1,j}(x,0)+2\varphi'_{1,j}(0,0)+\varphi'_{1,j}(-x,0)}{2n}, \\
\|f_{\beta(j)}(x)-&f_{\beta(j)}(0) - Q'(x) - T_{\beta(j)}(x)\| \\
&\leqslant \frac{M_3(\varphi'_{1,j},x)+M_3(\varphi'_{1,j},-x)+M_5(\varphi'_{1,j},x)+M_5(\varphi'_{1,j},-x)}{n} \\
&\quad + \frac{\varphi'_{1,j}(0,x)+2\varphi'_{1,j}(0,0)+\varphi'_{1,j}(0,-x)}{2n}
\end{aligned}$$

for all $x \in V$. The functions $Q', T_{\beta(1)}, T_{\beta(j)}$ are given by

$$Q'(x) = \begin{cases} \lim\limits_{n\to\infty} \frac{f_{\beta(1)}(2^n x)}{4^n} = \lim\limits_{n\to\infty} \frac{f_{\beta(j)}(2^n x)}{4^n} & \text{if } \varphi \text{ satisfies (d)}, \\ \lim\limits_{n\to\infty} 4^n \left(\frac{f_{\beta(1)}(2^{-n}x)+f_{\beta(1)}(-2^{-n}x)}{2} - f_{\beta(1)}(0) \right) & \text{if } \varphi \text{ satisfies (d')} \\ \lim\limits_{n\to\infty} 4^n \left(\frac{f_{\beta(j)}(2^{-n}x)+f_{\beta(j)}(-2^{-n})}{2} - f_{\beta(j)}(0) \right), & \end{cases}$$

$$T_{\beta(1)}(x) = \begin{cases} \lim\limits_{n\to\infty} \frac{f_{\beta(1)}(2^n x)-f_{\beta(1)}(-2^n x)}{2^{n+1}} & \text{if } \varphi \text{ satisfies (e)}, \\ \lim\limits_{n\to\infty} 2^{n-1}(f_{\beta(1)}(2^{-n}x) - f_{\beta(1)}(-2^{-n}x)) & \text{if } \varphi \text{ satisfies (e')}, \end{cases}$$

$$T_{\beta(j)}(x) = \begin{cases} \lim\limits_{n\to\infty} \frac{f_{\beta(j)}(2^n x)-f_{\beta(j)}(-2^n x)}{2^{n+1}} & \text{if } \varphi \text{ satisfies (e)}, \\ \lim\limits_{n\to\infty} 2^{n-1}(f_{\beta(j)}(2^{-n}x) - f_{\beta(j)}(-2^{-n}x)) & \text{if } \varphi \text{ satisfies (e')} \end{cases}$$

for all $x \in V$ where $1 < j \leqslant n$. Replacing x_i by $2^m x$ for all $i = 1,2,\ldots,n$ and dividing by 4^m in (3,1), we obtain

$$\left\| \frac{f_1(2^m nx)}{4^m} + \sum_{1\leqslant i<j\leqslant n} \frac{f_{\alpha(i,j)}(0)}{4^m} - n\sum_{i=1}^{n} \frac{f_{\beta(i)}(2^m x)}{4^m} \right\| \leqslant \frac{\varphi(2^m x, 2^m x, \ldots, 2^m x)}{4^m} \tag{3.3}$$

for all x if φ satisfies (d). Replacing x_i by $\frac{x}{2^m}$ for all $i = 1,2,\ldots,n$ and multiplying by 4^m in (3,1), we obtain

$$\begin{aligned}
\left\| \frac{4^m}{2}\left[f_1\left(\frac{nx}{2^m}\right) + f_1\left(\frac{-nx}{2^m}\right)\right] + \sum_{1\leqslant i<j\leqslant n} 4^m f_{\alpha(i,j)}(0) - n\sum_{i=1}^{n} \frac{4^m}{2}\left[f_{\beta(i)}\left(\frac{x}{2^m}\right) + f_{\beta(i)}\left(\frac{-x}{2^m}\right)\right] \right\| \\
\leqslant \frac{4^m}{2}\left[\varphi\left(\frac{x}{2^m}, \frac{x}{2^m}, \ldots, \frac{x}{2^m}\right) + \varphi\left(\frac{-x}{2^m}, \frac{-x}{2^m}, \ldots, \frac{-x}{2^m}\right)\right]
\end{aligned} \tag{3.4}$$

for all x if φ satisfies (d'). From (3,2) and (3.4), we get

$$\begin{aligned}\Big\| 4^m\Big(\frac{1}{2}\Big(f_1\Big(\frac{nx}{2^m}\Big)+f_1\Big(\frac{-nx}{2^m}\Big)\Big)-f_1(0)\Big) \\ -n\sum_{i=1}^{n}4^m\Big(\frac{1}{2}\Big(f_{\beta(i)}\Big(\frac{x}{2^m}\Big)+f_{\beta(i)}\Big(\frac{-x}{2^m}\Big)\Big)-f_{\beta(i)}(0)\Big)\Big\| \\ \leqslant \frac{4^m}{2}\Big[\varphi\Big(\frac{x}{2^m},\frac{x}{2^m},\dots,\frac{x}{2^m}\Big)+\varphi\Big(\frac{-x}{2^m},\frac{-x}{2^m},\dots,\frac{-x}{2^m}\Big)\Big]\end{aligned} \tag{3.5}$$

for all x if φ satisfies (d'). Taking the limit in (3.3) and (3.5) as $m\to\infty$, we get

$$Q(nx)=n^2Q'(x) \quad \text{i.e.} \quad Q(x)=Q'(x)$$

Let $f_k^-: V\to X$ be the odd functions defined by

$$f_k^-(x)=\frac{f_k(x)-f_k(-x)}{2} \quad \text{for all} \quad x\in V.$$

Then we can easily see that

$$\begin{aligned}\Big\| f_1^-(\sum_{i=1}^{n}x_i)+\sum_{1\leqslant i<j\leqslant n}f_{\alpha(i,j)}^-(x_i-x_j)-n\sum_{i=1}^{n}f_{\beta(i)}^-(x_i)\Big\| \\ \leqslant \frac{1}{2}[\varphi(x_1,x_2,\dots,x_n)+\varphi(-x_1,-x_2,\dots,-x_n)]\end{aligned} \tag{3.6}$$

for all $x_1,x_2,\dots,x_n\in V$. Replacing x_i by x and replacing x_l by 0 for all $l\in\{1,2,\dots,n\}-\{i\}$ in (3.6), we obtain

$$\begin{aligned}\Big\| f_1^-(x)+\sum_{l=1}^{i-1}f_{\alpha(l,i)}^-(-x)+\sum_{l=i+1}^{n}f_{\alpha(i,l)}^-(x)-nf_{\beta(i)}^-(x)\Big\| \\ \leqslant \frac{1}{2}[\varphi_{1,i}(0,x)+\varphi_{1,i}(0,-x)].\end{aligned} \tag{3.7}$$

Replacing x by 2^mx and dividing by 2^m in (3.7), we obtain

$$\begin{aligned}\Big\| \frac{f_1^-(2^mx)}{2^m}+\sum_{l=1}^{i-1}\frac{f_{\alpha(l,i)}^-(-2^mx)}{2^m}+\sum_{l=i+1}^{n}\frac{f_{\alpha(i,l)}^-(2^mx)}{2^m}-n\frac{f_{\beta(i)}^-(2^mx)}{2^m}\Big\| \\ \leqslant \frac{\varphi_{1,i}(0,2^mx)+\varphi_{1,i}(0,-2^mx)}{2^{m+1}}\end{aligned} \tag{3.8}$$

for all x if φ satisfies (e). Replacing x by $\frac{x}{2^m}$ and multiplying by 2^m in (3.7), we obtain

$$\begin{aligned}\Big\| 2^mf_1^-\Big(\frac{x}{2^m}\Big)+\sum_{l=1}^{i-1}2^mf_{\alpha(l,i)}^-\Big(-\frac{x}{2^m}\Big)+\sum_{l=i+1}^{n}2^mf_{\alpha(i,l)}^-\Big(\frac{x}{2^m}\Big)-n2^mf_{\beta(i)}^-\Big(\frac{x}{2^m}\Big)\Big\| \\ \leqslant 2^{m-1}\Big(\varphi_{1,i}\Big(0,\frac{x}{2^m}\Big)+\varphi_{1,i}\Big(0,-\frac{x}{2^m}\Big)\Big)\end{aligned} \tag{3.9}$$

for all x if φ satisfies (e'). Taking the limit in (3.8) and (3.9) as $m \to \infty$, we obtain

$$T_1(x) + \sum_{l=1}^{i-1} T_{\alpha(l,i)}(-x) + \sum_{l=i+1}^{n} T_{\alpha(i,l)}(x) - nT_{\beta(i)}(x) = 0.$$

Replacing x_i for all $i \in \{1, 2, \ldots, n\}$ by $2^m x$ dividing by 2^m in (3.6), we obtain

$$\left\| \frac{f_1^-(2^m nx)}{2^m} - n\sum_{i=1}^{n} \frac{f_{\beta(i)}^-(2^m x)}{2^m} \right\| \leqslant \frac{\varphi(2^m x, 2^m x, \ldots, 2^m x) + \varphi(-2^m x, -2^m x, \ldots, -2^m x)}{2^{m+1}} \tag{3.10}$$

for all x if φ satisfies (e). Replacing x_i by $\frac{x}{2^m}$ for all $i \in \{1, 2, \ldots, n\}$ and multiplying by 2^m in (3.6), we obtain

$$\left\| 2^m f_1^-\left(\frac{nx}{2^m}\right) - n\sum_{i=1}^{n} 2^m f_{\beta(i)}^-\left(\frac{x}{2^m}\right) \right\| \leqslant 2^{m-1}\left(\varphi\left(\frac{x}{2^m}, \frac{x}{2^m}, \ldots, \frac{x}{2^m}\right) + \varphi\left(-\frac{x}{2^m}, -\frac{x}{2^m}, \ldots, -\frac{x}{2^m}\right)\right) \tag{3.11}$$

for all x if φ satisfies (e'). Taking the limit in (3.10) and (3.11) as $m \to \infty$, we obtain

$$T_1(x) - \sum_{i=1}^{n} T_{\beta(i)}(x) = 0$$

for all $x \in V$. This completes the proof of the theorem. □

COROLLARY 3.4. *Let $p \neq 1, 2$, $\theta > 0$ be real numbers. Let $\psi : V \to [0, \infty)$ be a mapping such that $\psi(x) = \|x\|^p$ for $x \neq 0$ and $\psi(0) = 0$ if $p > 1$. Let $\alpha(i,j), \beta(i)$ be defined as in Theorem 3.3. Suppose that the functions $f_k : V \to X$ satisfy*

$$\left\| f_1\left(\sum_{i=1}^{n} x_i\right) + \sum_{1 \leqslant i < j \leqslant n} f_{\alpha(i,j)}(x_i - x_j) - n\sum_{i=1}^{n} f_{\beta(i)}(x_i) \right\| \leqslant \theta \sum_{i=1}^{n} \psi(x_i)$$

for all $x_1, x_2, \ldots, x_n \in V$, where $k = 1, \alpha(i,j), \beta(i), 1 \leqslant i,j \leqslant n$.

Then there exists exactly one quadratic function $Q : V \to X$ and unique additve functions $T_k : V \to X$ such that

$$\|f_k(x) - f_k(0) - Q(x) - T_k(x)\| \leqslant M_k(x)$$

for all $x \in V$, where

$$M_1(x) = \left(\frac{4}{|4 - 2^p|} + \frac{2}{2^p} + \frac{4}{|2 - 2^p|}\right)\|x\|^p + \frac{10 + 16(n-2)}{3}\psi(0),$$

$$M_{\alpha(i,j)}(x) = \left(\frac{4}{|4 - 2^p|} + \frac{2}{2^p} + \frac{4}{|2 - 2^p|}\right)\|x\|^p + \frac{10 + 16(n-2)}{3}\psi(0),$$

$$M_{\beta(i)}(x) = \frac{1}{n}\left[\left(\frac{8}{|4-2^p|} + \frac{4}{2^p} + \frac{8+2\cdot 2^p}{|2-2^p|} + 1\right)\|x\|^p + \frac{23}{3}\psi(0)\right] + \frac{(n-2)}{n}\left[\left(\frac{2}{2^p} + \frac{2^p+4}{2^p|4-2^p|} + \frac{3+2^p+3^p}{2^p|2-2^p|}\right)\|x\|^p + 4\psi(0)\right].$$

The functions Q and T_k are given by

$$Q(x) = \begin{cases} \lim_{n\to\infty} 4^{-n} f_k(2^n x) & \text{if } p<2, \\ \lim_{n\to\infty} 4^n \left(\frac{f_k(2^{-n}x)+f_k(-2^{-n}x)}{2} - f(0)\right) & \text{if } p>2, \end{cases}$$

$$T_k(x) = \begin{cases} \lim_{n\to\infty} 2^{-n-1}(f_k(2^n x) - f_k(-2^n x)) & \text{if } p<1, \\ \lim_{n\to\infty} 2^{n-1}(f_k(2^{-n}x) - f_k(-2^{-n}x)) & \text{if } p>1 \end{cases}$$

for all $x \in V$.

COROLLARY 3.5. *Let $\alpha(i,j), \beta(i)$ be defined as in Theorem 3.3. Suppose that the functions $f_k : V \to X$ satisfy*

$$f_1(\sum_{i=1}^n x_i) + \sum_{1\leqslant i<j\leqslant n} f_{\alpha(i,j)}(x_i - x_j) = n\sum_{i=1}^n f_{\beta(i)}(x_i)$$

for all $x_1, x_2, \ldots, x_n \in V$, where $k = 1,\ \alpha(i,j),\ \beta(i),\ 1 \leqslant i,j \leqslant n$.

Then there exists exactly one quadratic function $Q : V \to X$ and additve functions $T_k : V \to X$ such that

$$f_k(x) = Q(x) + T_k(x) + f_k(0)$$

for all $x \in V$.

REFERENCES

[1] C. BORELLI AND G. L. FORTI, *On a general Hyers-Ulam-stability result*, Internat. J. Math. Math. Sci. **18** (1995), 229–236.

[2] P. W. CHOLEWA, *Remarks on the stability of functional equations*, Aequationes Math. **27** (1984), 76–86.

[3] S. CZERWIK, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg. **62** (1992), 59–64.

[4] P. GĂVRUTA, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. and Appl. **184** (1994), 431–436.

[5] D. H. HYERS, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.

[6] D. H. HYERS, G. ISAC AND TH. M. RASSIAS, *"Stability of Functional Equations in Several Variables"*, Birkhäuser (1998).

[7] D. H. HYERS AND TH. M. RASSIAS, *Approximate homomorphisms*, Aeq. Math. **44** (1992), 125–153.

[8] K. -W. JUN AND Y. -H. LEE, *On the Hyers-Ulam-Rassias stability of a generalized quadratic equation*, Bull. Korean Math. Soc. **38** (2001), 261–272.

[9] K. -W. JUN AND Y. -H. LEE, *On the Hyers-Ulam-Rassias stability of a Pexiderized quadratic inequality*, Math. Ineq. Appl. **4** (2001), 93–118.

[10] K. -W. JUN, D.-S. SHIN AND B. -D. KIM, *On Hyers-Ulam-Rassias stability of the pexider equation*, J. Math. Anal. Appl. **239** (1999), 20–29.

[11] S.-M. JUNG, *On the Hyers-Ulam stability of the functional equations that have the quadratic property*, J. Math. Anal. Appl. **222** (1998), 126–137.

[12] S.-M. JUNG, *On the Hyers-Ulam-Rassias stability of a quadratic functional equation*, J. Math. Anal. Appl. **232** (1999), 384–393.
[13] PL. KANNAPPAN, *Quadratic functional equation and inner product spaces*, Results Math. **27** (1995), 368–372.
[14] Y. -H. LEE AND K. -W. JUN, *A generalization of the Hyers-Ulam-Rassias stability of Pexider equation*, J. Math. Anal. Appl. **246** (2000), 627–638.
[15] Y. -H. LEE AND K. -W. JUN, *A note on the Hyers-Ulam-Rassias stability of Pexider equation*, J. Korean Math. Soc. **37** (2000), 111–124.
[16] J. M. RASSIAS, *On the stability of the Euler-Lagrange functional equation*, Chinese J. Math. **20** (1992), 185–190.
[17] TH. M. RASSIAS, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[18] TH. M. RASSIAS, *Stability and set-valued functions,* in: Analysis and Topology, World Scientific Publ. Co., 1988, pp. 585–614.
[19] TH. M. RASSIAS, *On the stability of the quadratic functional equation and its applications*, Studia Univ. "Babes-Bolyai" **XLIII (3)** (1998), 89–124.
[20] TH. M. RASSIAS, *Functional equations and inequalities,* Kluwer Academic Publishers, Dordrecht, 2000.
[21] TH. M. RASSIAS, *On the stability of functional equations and a problem of Ulam*, Acta Math. **62** (2000), 23–130.
[22] TH. M. RASSIAS, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. **251** (2000), 264–284.
[23] TH. M. RASSIAS, *On the stability of the quadratic functional equation*, Mathematica, XLV(2) (2000), 77–114.
[24] F. SKOF, *Proprietà locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.
[25] S. M. ULAM, *"Problems in Modern Mathematics"*, Chap. VI, Wiley, New York, 1960.

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