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ON THE STABILITY OF THE FUNCTIONAL EQUATION $f(x+y-xy)+xf(y)+yf(x)=f(x)+f(y)$

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(*communicated by Zs. Páles*)

Abstract. In this paper we present a generalized version of the Hyers-Ulam stability and the superstability of the functional equation $f(x+y-xy)+xf(y)+yf(x)=f(x)+f(y)$, respectively.

1. Introduction

In 1940, S. M. Ulam [18] proposed the following problem for the stability of group homomorphisms:

Given a group G_1, *a metric group* G_2 *with metric* $d(\cdot,\cdot)$, *and a* $\varepsilon > 0$, *does there exist a* $\delta > 0$ *such that if a mapping* $f : G_1 \to G_2$ *satisfies* $d(f(xy), f(x)f(y)) \leqslant \delta$ *for all* $x, y \in G_1$, *then a homomorphism* $g : G_1 \to G_2$ *exists with* $d(f(x), g(x)) \leqslant \varepsilon$ *for all* $x \in G_1$ *?*

For Banach spaces the Ulam problem was first solved by D. H. Hyers [2] in 1941, which states that if $\delta > 0$ and $f : X \to Y$ is a mapping with X, Y Banach spaces, such that

$$||f(x+y)-f(x)-f(y)|| \leqslant \delta$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$||f(x)-T(x)|| \leqslant \delta$$

for all $x \in X$.

In 1978, Th. M. Rassias [12] gave a generalization of the Hyers' result in the following way:

Let X and Y be Banach spaces, let $\theta \geqslant 0$, and let $0 \leqslant p < 1$. If a function $f : X \to Y$ satisfies

$$\|f(x+y)-f(x)-f(y)\| \leqslant \theta(\|x\|^p+\|y\|^p)$$

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for all $x,\ y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$||f(x) - T(x)|| \leqslant \frac{2\theta}{2-2^p}||x||^p$$

for all $x \in X$.

Thereafter, P. Găvruta [1] generalized the stability of Rassias for the case of the bounded function, and various results concerning the stability of different functional equations have been obtained by numerous authors (see, for instance, [3-6,9,11,13,14,16]).

During the 34 th International Symposium on Functional Equations, Gy. Maksa [7] posed the problem concerning the Hyers-Ulam stability of the functional equation

$$f(xy) = xf(y) + yf(x) \tag{1}$$

on the interval $(0,1]$ and J. Tabor gave an answer to the question of Maksa in [17]. On the other hand, Zs. Páles [10] remarked that the functional equation (1) for real-valued functions has a superstability on the interval $[1,\infty)$.

Recently, Gy. Maksa and Zs. Páles [8] studied the stability problem of the generalized form of the equation (1) on the interval $(0,1]$:

$$f(xy) = x^{\alpha}f(y) + y^{\alpha}f(x) \quad (\alpha \in \mathbb{R}).$$

Here we introduce the following functional equation motivated by the functional equation (1):

$$f(x+y-xy) + xf(y) + yf(x) = f(x) + f(y). \tag{2}$$

In this paper, we will solve the functional equation (2) and then by following the ideas of J. Tabor [17] and Zs. Páles [10], a generalized version of the Hyers-Ulam stability and the superstability of the functional equation (2) will be investigated, respectively.

2. Solutions of eq. (2)

It is easy to see that the real-valued function $f(x) = (1-x)ln(1-x)$ is a solution of the functional equation (2) on the interval $(-\infty,\ 1)$. In the following theorem, we will find out the general solution of the functional equation (2) on the interval $(-\infty,\ 1)$.

THEOREM 2.1. *Let X be a vector space. A function $f : (-\infty,\ 1) \to X$ satisfies the functional equation (2) for all $x \in (-\infty,\ 1)$ if and only if there exists a solution $d : (0,\infty) \to X$ of the functional equation (1) such that*

$$f(x) = d(1-x)$$

for all $x \in (-\infty,\ 1)$.

Proof. $(\Rightarrow)$ Let us define the mapping $d : (0,\infty) \to X$ by $d(x) := f(1-x)$ for all $x \in (0,\infty)$.

We claim that d is a solution of the functional equation (1).

Indeed, for all $x,\ y \in (0,\infty)$, we have

$$\begin{aligned} d(xy) &= f(1-xy) \\ &= f((1-x)+(1-y)-(1-x)(1-y)) \\ &= f(1-x)+f(1-y)-(1-x)f(1-y)-(1-y)f(1-x) \\ &= xd(y)+yd(x). \end{aligned}$$

Therefore d is a solution of the functional equation (1), as claimed, and $f(x) = d(1-x)$ for all $x \in (-\infty,\ 1)$.

($\Leftarrow$) This is obvious. $\square$

3. Generalized version of the Hyers-Ulam stability of eq. (2)

Throughout this section, we will assume that X is a sequentially complete topological vector space and V is a closed convex, bounded and symmetric with respect to zero subset of X.

DEFINITION. A function $g : [0,\infty) \to [0,\infty)$ is called exponentially increasing if it is increasing and there exists $\gamma > 1$ and $h \in [0,\infty)$ such that

$$g(x+h) \geqslant \gamma g(x)$$

for all $x \in [0,\infty)$.

First, we state a result of J. Tabor [17] concerning the stability of the additive functional equation $f(x+y) = f(x)+f(y)$ on the interval $[0,\infty)$:

PROPOSITION 3.1. *Suppose that $g : [0,\infty) \to [0,\infty)$ is exponentially increasing with constants γ and h as in Definition, and that $g(0) > 0$.*

Let $K := 2\frac{g(h)}{g(0)} + \frac{\gamma}{\gamma-1}$, and let $f : [0,\infty) \to X$ be an arbitrary function such that

$$f(x+y) - f(x) - f(y) \in g(x+y)V$$

for all $x \in [0,\infty)$. Then there exists a unique additive function $A : [0,\infty) \to X$ such that $A(h) = f(h)$ and that

$$f(x) - A(x) \in Kg(x)V$$

for all $x \in [0,\infty)$.

The main goal in this section is to examine a generalized version of the Hyers-Ulam stability of the functional equation (2) on the interval $[0,1)$ and the proof is very analogous to the one given in [17].

THEOREM 3.2. *Let* $f : [0,1) \to X$ *be a function such that*

$$f(x+y-xy) + xf(y) + yf(x) - f(x) - f(y) \in V \tag{3}$$

for all $x,\ y \in [0,1)$*, and let* $z \in (0,1)$ *be an arbitrary fixed. Then there exists a unique function* $F_z : [0,1) \to X$ *such that*

$$F_z(z) = f(z), \tag{4}$$

$$F_z(x+y-xy) + xF_z(y) + yF_z(x) = F_z(x) + F_z(y), \tag{5}$$

and that

$$f(x) - F_z(x) \in K_z V \tag{6}$$

for all $x,\ y \in [0,1)$*, where* $K_z := \frac{2}{1-z} + \frac{1}{z}$ *(the minimal value of* K_z *is equal to* $3+2\sqrt{2}$ *when* $z = \sqrt{2}-1$ *).*

Proof. Let K be a set of real numbers. By X^K we denote the vector space of all functions from K into X. We define the linear operator $\mathcal{A} : X^{[0,1)} \to X^{[0,\infty)}$ by the formula

$$\mathcal{A}(f)(x) := \exp(x)f(1-\exp(-x))$$

for all $x \in [0,\infty)$. The fact that f satisfies (3) is equivalent to

$$\mathcal{A}(f)(u+v) - \mathcal{A}(f)(u) - \mathcal{A}(f)(v) \in \exp(u+v)V$$

for all $u,\ v \in [0,\infty)$. Obviously exp is exponentially increasing with

$$h := -\exp^{-1}(1-z) = -ln(1-z),\ \gamma := \exp(h) = \frac{1}{1-z}.$$

Therefore by Proposition 3.1 there exists a unique $A_h : [0,\infty) \to X$ such that

$$A_h(h) = \mathcal{A}(f)(h), \tag{7}$$

$$A_h(u+v) = A_h(u) + A_h(v), \tag{8}$$

$$\mathcal{A}(f)(u) - A_h(u) \in K_z \exp(u)V \tag{9}$$

for all $x \in [0,\infty)$, where $K_z = 2\frac{\exp(h)}{\exp(0)} + \frac{\gamma}{\gamma-1} = \frac{2}{1-z} - \frac{1}{z}$.

Let $F_z := \mathcal{A}^{-1}(A_h)$. Then we can easily check that (7), (8) and (9) mean that F_z satisfies (4), (5) and (6), respectively.

We claim that F_z is unique. Suppose that there exists $F_z^{'}$ satisfying (7), (8) and (9). Then $\mathcal{A}(F_z^{'})$ satisfies (4), (5) and (6), hence $\mathcal{A}(F_z^{'}) = A_h = \mathcal{A}(F_z)$. Since $\mathcal{A}$ is a bijection, this implies that $F_z^{'} = F_z$. The proof of the theorem is complete. □

4. Superstability of eq. (2)

In this section, we will assume that X is a Banach space and we will investigate the superstability of the functional equation (2) on the interval $(-\infty,\ 0]$.

THEOREM 4.1. *Let X be a Banach space, and let $f:(-\infty,\ 0]\to X$ be a mapping satisfying the inequality*

$$\|f(x+y-xy)+xf(y)+f(x)y-f(x)-f(y)\|\leqslant\delta \tag{10}$$

for some $\delta>0$ and for all $x,\ y\in(-\infty,\ 0]$. Then f satisfies the functional equation (2) *for all $x,\ y\in(-\infty,\ 0]$.*

Proof. Let $G:(-\infty,\ 0]\to X$ be the mapping defined by

$$G(x)=\frac{f(x)}{1-x}$$

for all $x\in(-\infty,\ 0]$. Then, by (10), we see that g satisfies the inequality

$$\|G(x+y-xy)-G(y)-G(x)\|\leqslant\frac{\delta}{(1-x)(1-y)}$$

for all $x,\ y\in(-\infty,\ 0]$. Define the mapping $F:[0,\infty)\to X$ by

$$F(u)=G(1-\exp(-u))$$

for all $u\in[0,\infty)$. The inequality (10) guarantees that

$$\|F(u+v)-F(u)-F(v)\|\leqslant\delta\exp(-(u+v)) \tag{11}$$

for all $u,\ v\in[0,\infty)$. By applying Skof's extension procedure [17], we will show that F is additive on the interval $[0,\infty)$.

Let $\varepsilon>0$ be given. Then the inequality (11) implies that there exists a $c>0$ satisfying

$$\|F(u+v)-F(u)-F(v)\|<\frac{\varepsilon}{9} \tag{12}$$

for all $u,\ v\in[0,\infty)$ with $u+v>c$.

Suppose that positive real numbers u and $v\neq 0$ are given. Let m and n be integers great than 1. Then it is easy to see that

$$\begin{aligned}F(nu)-F((n-1)u)-F(u)=\ &F(u+mv)-F(u)-F(mv)\\&+F((n-1)u+(u+mv))-F((n-1)u)-F(u+mv)\\&-(F(nu+mv)-F(nu)-F(mv))\end{aligned}$$

If m is so large that $m>(c+u)/v$, then the last equality implies

$$\|F(nu)-F((n-1)u)-F(u)\|<\frac{\varepsilon}{3} \tag{13}$$

for all $u\in[0,\infty)$ and all integers $n>1$. The relation

$$F(nu)-nF(u)=\sum_{k=2}^{n}(F(ku)-F((k-1)u)-F(u))$$

together with (13), yields

$$\|F(nu) - nF(u)\| < \frac{(n-1)\varepsilon}{3} \tag{14}$$

for all $u \in [0,\infty)$ and all integers $n > 1$. Obviously, it follows from (14) that

$$\begin{aligned} &\|(F(nu+nv) - F(nu) - F(nv)) - n(F(u+v) - F(u) - F(v))\| \\ &\quad \leqslant \|F(nu+nv) - nF(u+v)\| + \|F(nu) - nF(u)\| + \|F(nv) - nF(v)\| \\ &\quad < (n-1)\varepsilon \end{aligned}$$

for all $u,\ v \in [0,\infty)$ with $(u,\ v) \neq (0,\ 0)$. Dividing by n both sides of the last inequality and then letting $n \to \infty$ and considering the fact that $(1/n)(F(nu+nv) - F(nu) - F(nv)) \to 0$ as $n \to \infty$, we get

$$\|F(u+v) - F(u) - F(v)\| \leqslant \varepsilon \tag{15}$$

for all $u,\ v \in [0,\infty)$ with $(u,\ v) \neq (0,\ 0)$.

Let $u \in [0,\infty)$ with $u > c$. Then the inequality (12) with such an u and $v = 0$ yields $\|F(0)\| < \frac{\varepsilon}{9}$. Hence we obtain $\|F(0+0) - F(0) - F(0)\| = \|F(0)\| < \frac{\varepsilon}{9}$. Therefore the inequality (15) holds for all $u,\ v \in [0,\infty)$. Since $\varepsilon > 0$ was arbitrary, we conclude that F is additive on $[0,\infty)$.

Now, according to the definitions of F and G, we have

$$\frac{f(x)}{1-x} = F(ln(1-x))$$

for all $x \in (-\infty,\ 0]$, i.e.,

$$f(x) = (1-x)F(ln(1-x))$$

for all $x \in (-\infty,\ 0]$, and so we see that f satisfies the functional equation (2) for all $x,\ y \in (-\infty,\ 0]$ since F is additive. This completes the proof of the theorem. □

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