Mathematical
Inequalities
& Applications
Volume 7, Number 1 (2004), 87–93

NECESSARY AND SUFFICIENT TAUBERIAN CONDITIONS IN THE CASE OF WEIGHTED MEAN SUMMABLE INTEGRALS OVER $\mathbb{R}_+$

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(*communicated by Zs. Páles*)

Abstract. Let $0 \not\equiv p(x)$ be a nondecreasing function on $\mathbb{R}_+ := [0,\infty)$ such that $p(0) = 0$ and

$$\liminf_{t\to\infty} p(\lambda t)/p(t) > 1 \quad \text{for every} \quad \lambda > 1.$$

Given a real- or complex-valued function $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, we define

$$s(x) := \int_0^x f(u)du \quad \text{and} \quad \sigma(t) := \frac{1}{p(t)}\int_0^t s(x)dp(x), \qquad t > 0.$$

It is known that if the finite limit $\lim_{x\to\infty} s(x) = L$ exists, then the limit $\lim_{t\to\infty}\sigma(t) = L$ also exists. Our goal is to find necessary and sufficient conditions under which the converse implication holds. Most of these conditions are expressed in terms of inequalities.

In the case of real-valued functions we present one-sided Tauberian conditions, while in the case of complex-valued functions we present two-sided Tauberian conditions. As special cases, we obtain well-known Tauberian conditions such as slow decrease in the sense of R. Schmidt, slow oscillation in the sense of Hardy, and Landau type Tauberian c onditions.

1. Weighted mean summability of integrals over $\mathbb{R}_+$

Let $0 \not\equiv p : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function such that $p(0) = 0$ and

$$\liminf_{t\to\infty} p(\lambda t)/p(t) > 1 \quad \text{for every} \quad \lambda > 1. \tag{1.1}$$

In particular, it follows that $\lim_{t\to\infty} p(t) = \infty$.

Given a real- or complex-valued function f, integrable in Lebesgue's sense over any finite interval $(0,t)$ for $0 < t < \infty$, in symbol: $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, we set

$$s(x) := \int_0^x f(u)du \quad \text{and} \quad \sigma(t) := \frac{1}{p(t)}\int_0^t s(x)dp(x), \tag{1.2}$$

provided $p(t) > 0$ (by (1.1), this is the case if t is large enough). The second integral in (1.2) exists in the Riemann-Stieltjes sense.

Mathematics subject classification (2000): 40E05, 40G05; 39B72, 40A10.

Key words and phrases: improper integral over $\mathbb{R}_+$, weighted mean summability of integrals, Tauberian conditions, slow decrease, slow oscillation, Landau type Tauberian conditions..

This research was supported by the Hungarian National Foundation for Scientific Research under Grant T 046 192.

Paper MIA-07-10

The integral $\int_0^\infty f(x)dx$ is said to be summable (by the weighted mean method determined by the function p) if the finite limit

$$\lim_{t\to\infty} \sigma(t) = L \quad \text{exists.} \tag{1.3}$$

It is easy to check that if the finite limit

$$\lim_{x\to\infty} s(x) = L \quad \text{exists} \tag{1.4}$$

(that is, if the *improper integral* of f over $\mathbb{R}_+$ is convergent), then we also have (1.3). The converse implication is not true in general.

REMARK 1. However, if a real-valued function f is of constant sign on $\mathbb{R}_+$, then (1.3) and (1.4) are equivalent. In fact, this follows from (1.2) by applying Fubini's theorem:

$$\sigma(t) = \frac{1}{p(t)} \int_0^t dp(x) \int_0^x f(u)du = \int_0^t f(u)\Big\{1 - \frac{p(u)}{p(t)}\Big\}du.$$

The special case of summability $(C,1)$ corresponding to the choice $p(t) := t$, was treated by Hardy [2, p. 11] and Titchmarsh [6, p. 26].

2. Main results

First, we consider real-valued functions f and prove the following theorem under *one-sided Tauberian conditions*.

THEOREM 1. *If a real-valued function $f \in L^1_{loc}(\mathbb{R}_+)$ is such that (1.3) holds, then (1.4) holds if and only if both*

$$\sup_{\lambda>1} \liminf_{t\to\infty} \frac{1}{p(\lambda t) - p(t)} \int_t^{\lambda t} [s(x) - s(t)]dp(x) \geqslant 0 \tag{2.1}$$

and

$$\sup_{0<\lambda<1} \liminf_{t\to\infty} \frac{1}{p(t) - p(\lambda t)} \int_{\lambda t}^{t} [s(t) - s(x)]dp(x) \geqslant 0. \tag{2.2}$$

A real-valued function $s(x)$ defined on $\mathbb{R}_+$ is said to be *slowly decreasing* if

$$\lim_{\lambda\to 1+} \liminf_{t\to\infty} \min_{t\leqslant x\leqslant \lambda t} [s(x) - s(t)] \geqslant 0. \tag{2.3}$$

In other words, (2.3) means that for every $\varepsilon > 0$ there exist $\lambda_1 = \lambda_1(\varepsilon) > 1$ and $t_1 = t_1(\varepsilon) > 0$ such that

$$s(x) - s(t) \geqslant -\varepsilon \quad \text{whenever} \quad t_1 \leqslant t \leqslant x \leqslant \lambda_1 t.$$

The term "slowly decreasing" was introduced by Schmidt [5] in the case of sequences of real numbers.

REMARK 2. It is easy to check that condition (2.3) can be equivalently reformulated as follows:

$$\lim_{\lambda\to 1-}\ \liminf_{t\to\infty}\ \min_{\lambda t\leqslant x\leqslant t}[s(t)-s(x)]\geqslant 0.$$

Thus, the following corollary of Theorem 1 is obvious.

COROLLARY 1. *If a real-valued function* $f\in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ *is such that (1.3) holds and its integral function* $s(x)$ *is slowly decreasing, then (1.4) holds.*

REMARK 3. Condition (2.3) is certainly satisfied if there exist constants $H>0$ and $t_1>0$ such that

$$tf(t)\geqslant -H\quad \text{for almost all}\quad t>t_1. \tag{2.4}$$

Indeed, for all $t_1<t<x<\infty$ we have

$$s(x)-s(t)=\int_t^x f(u)du\geqslant -H\int_t^x\frac{du}{u}=-H\ln\frac{x}{t},$$

whence it follows that

$$\min_{t\leqslant x\leqslant\lambda t}[s(x)-s(t)]\geqslant -H\ln\lambda,\quad t>t_1\quad\text{and}\quad\lambda>1.$$

Choosing λ sufficiently close to 1, inequality (2.3) is satisfied. We note that condition (2.4) resembles the one introduced by Landau [3] in the case of sequences of real numbers.

REMARK 4. There is an example in [4, pp. 56-57] which shows that conditions (2.1) and (2.2) in Theorem 1 are generally independent of one another.

REMARK 5. The proof of Theorem 1 in Section 3 below can be easily modified to prove the following assertion: Theorem 1 remains valid if conditions (2.1) and (2.2) are replaced by their symmetric counterparts:

$$\inf_{\lambda>1}\ \limsup_{t\to\infty}\frac{1}{p(\lambda t)-p(t)}\int_t^{\lambda t}[s(x)-s(t)]dp(x)\leqslant 0$$

and

$$\inf_{0<\lambda<1}\ \limsup_{t\to\infty}\frac{1}{p(t)-p(\lambda t)}\int_{\lambda t}^{t}[s(t)-s(x)]dp(x)\leqslant 0.$$

Second, we consider the general case where the function f may take on complex values. We shall prove the following theorem under *two-sided Tauberian condition*.

THEOREM 2. *If a complex-valued function* $f\in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ *is such that (1.3) holds, then (1.4) holds if and only if*

$$\inf_{\substack{0<\lambda<\infty\\ \lambda\neq 1}}\ \limsup_{t\to\infty}\left|\frac{1}{p(\lambda t)-p(t)}\int_t^{\lambda t}[s(x)-s(t)]dp(x)\right|=0. \tag{2.5}$$

We recall that a function $s(x)$ defined on $\mathbb{R}_+$ is said to be *slowly oscillating* if

$$\lim_{\lambda\to 1+}\ \limsup_{t\to\infty}\ \max_{t\leqslant x\leqslant \lambda t}|s(x)-s(t)|=0. \tag{2.6}$$

The term "slowly oscillating" was introduced by Hardy [1] in the case of sequences of real numbers. Again (cf. Remark 2), it is easy to check that condition (2.6) can be equivalently reformulated as follows:

$$\lim_{\lambda\to 1-}\ \limsup_{t\to\infty}\ \max_{\lambda t\leqslant x\leqslant t}|s(t)-s(x)|=0.$$

COROLLARY 2. *If a complex-valued function $f\in L^1_{\text{loc}}(\mathbb{R}_+)$ is such that (1.3) holds and its integral function $s(x)$ is slowly oscillating, then (1.4) holds.*

REMARK 6. Condition (2.6) is certainly satisfied if there exist constants $H>0$ and $t_1>0$ such that

$$|tf(t)|\leqslant H \quad \text{for almost all} \quad t>t_1 \quad (\text{cf. } (2.4)).$$

REMARK 7. We emphasize that condition (1.1) is important in the proofs of Theorems 1 and 2. For example, in case $p(t):=t^\alpha(\log(1+t))^\beta$, condition (1.1) is satisfied if $\alpha>0$ and $\beta\in\mathbb{R}$, but it is not satisfied if $\alpha=0$ and $\beta>0$. In the particular case $p(t):=t$, one gets back the familiar Cesàro summability $(C,1)$. In this particular case, Theorems 1 and 2 were proved in [4].

3. Proofs

We begin with the following lemma, which is interesting in itself. It states that if we have (1.3), then the so-called *moving weighted avarages* of $s(x)$ also converge to the same limit.

LEMMA. *If the finite limit in (1.3) exists for a complex-valued function $f\in L^1_{\text{loc}}(R_+)$, then for every $0<\lambda<\infty$, $\lambda\neq 1$, we also have*

$$\lim_{t\to\infty}\frac{1}{p(\lambda t)-p(t)}\int_t^{\lambda t}s(x)dp(x)=L. \tag{3.1}$$

Proof. Case $\lambda>1$. By definition,

$$\frac{1}{p(\lambda t)-p(t)}\int_t^{\lambda t}s(x)dp(x)=\frac{1}{p(\lambda t)-p(t)}[p(\lambda t)\sigma(\lambda t)-p(t)\sigma(t)]$$

$$=\sigma(\lambda t)+\frac{p(t)}{p(\lambda t)-p(t)}[\sigma(\lambda t)-\sigma(t)]. \tag{3.2}$$

By (1.1), for each $\lambda > 1$ we have

$$0 < \limsup_{t\to\infty} \frac{p(t)}{p(\lambda t) - p(t)} = \left[\liminf_{t\to\infty} \frac{p(\lambda t)}{p(t)} - 1\right]^{-1} < \infty,$$

and (3.1) follows from (1.3) and (3.2).

Case $0 < \lambda < 1$. By definition,

$$\frac{1}{p(t) - p(\lambda t)} \int_{\lambda t}^{t} s(x)dp(x) = \frac{1}{p(t) - p(\lambda t)} [p(t)\sigma(t) - p(\lambda t)\sigma(\lambda t)]$$

$$= \sigma(t) + \frac{p(\lambda t)}{p(t) - p(\lambda t)} [\sigma(t) - \sigma(\lambda t)]. \tag{3.3}$$

By (1.1), for each $0 < \lambda < 1$ (then $1/\lambda > 1$) we have

$$0 < \limsup_{t\to\infty} \frac{p(\lambda t)}{p(t) - p(\lambda t)} = \left[\liminf_{t\to\infty} \frac{p(t)}{p(\lambda t)} - 1\right]^{-1} < \infty,$$

and (3.1) follows from (1.3) and (3.3).

Proof of Theorem 1. Necessity. Assume the fulfillment of (1.4) (then we also have (1.3)). Let $\lambda > 1$ be arbitrary. By the Lemma,

$$\lim_{t\to\infty} \frac{1}{p(\lambda t) - p(t)} \int_{t}^{\lambda t} [s(x) - s(t)]dp(x) \tag{3.4}$$

$$= \lim_{t\to\infty} \frac{1}{p(\lambda t) - p(t)} \int_{t}^{\lambda t} s(x)dp(x) - \lim_{t\to\infty} s(t) = L - L = 0.$$

This proves (2.1) even in a stronger form.

In case $0 < \lambda < 1$, we obtain in an analogous way that

$$\lim_{t\to\infty} \frac{1}{p(t) - p(\lambda t)} \int_{\lambda t}^{t} [s(t) - s(x)]dp(x) = 0; \tag{3.5}$$

which is stronger than (2.2).

Sufficiency. This time we assume the fulfillments of (1.3), (2.1) and (2.2). We have to prove (1.4). To this end, let $\varepsilon > 0$ be given. By (2.1), there exists some $\lambda_1 > 1$ such that

$$\liminf_{t\to\infty} \frac{1}{p(\lambda_1 t) - p(t)} \int_{t}^{\lambda_1 t} [s(x) - s(t)]dp(x) \geqslant -\varepsilon; \tag{3.6}$$

and by (2.2), there exists some $0 < \lambda_2 < 1$ such that

$$\liminf_{t\to\infty} \frac{1}{p(t) - p(\lambda_2 t)} \int_{\lambda_2 t}^{t} [s(t) - s(x)]dp(x) \geqslant -\varepsilon. \tag{3.7}$$

By (1.3), (3.6) and the Lemma, we obtain

$$\begin{aligned} -\varepsilon &\leqslant \lim_{t\to\infty} \frac{1}{p(\lambda_1 t) - p(t)} \int_t^{\lambda_1 t} s(x)dp(x) - \limsup_{t\to\infty} s(t) \\ &= L - \limsup_{t\to\infty} s(t); \end{aligned}$$

while by (1.3), (3.7) and the Lemma, we obtain

$$\begin{aligned} -\varepsilon &\leqslant \liminf_{t\to\infty} s(t) - \lim_{t\to\infty} \frac{1}{p(t) - p(\lambda_2 t)} \int_{\lambda_2 t}^{t} s(x)dp(x) \\ &= \liminf_{t\to\infty} s(t) - L. \end{aligned}$$

Combining the last two inequalities yields

$$L - \varepsilon \leqslant \liminf_{t\to\infty} s(t) \quad \text{and} \quad \limsup_{t\to\infty} s(t) \leqslant L + \varepsilon.$$

Being $\varepsilon > 0$ arbitrary small, hence (1.4) follows. □

Proof of Theorem 2. Necessity. Assume the fulfillment of (1.4). In the same way as in the proof of the Necessity part of Theorem 1, we conclude (3.4) if $\lambda > 1$, and (3.5) if $0 < \lambda < 1$.

Sufficiency. Assume the fulfillments of (1.3) and (2.5). We shall prove (1.4). Given $\varepsilon > 0$, by (2.5) there exists some $0 < \lambda_1 < \infty$, $\lambda_1 \neq 1$, such that

$$L_1 := \limsup_{t\to\infty} \left| \frac{1}{p(\lambda_1 t) - p(t)} \int_t^{\lambda_1 t} [s(x) - s(t)]dp(x) \right| \leqslant \varepsilon. \tag{3.8}$$

For example, assume that $\lambda_1 > 1$. By (1.3), (3.8) and the Lemma, we can estimate as follows:

$$\begin{aligned} &\limsup_{t\to\infty} |L - s(t)| \\ &\leqslant \lim_{t\to\infty} \left| L - \frac{1}{p(\lambda_1 t) - p(t)} \int_t^{\lambda_1 t} s(x)dp(x) \right| + L_1 \leqslant \varepsilon. \end{aligned}$$

Being $\varepsilon > 0$ arbitrary, hence (1.4) follows. In the case where $0 < \lambda_1 < 1$ in (3.8), the proof of (1.4) is similar (cf. the proof of the Sufficency part of Theorem 1). □

REMARK 8. As a by-product of the proofs of Theorems 1 and 2, we can conclude the following

COROLARRY 3. *(i) If a real-valued function* $f \in L^1_{\text{loc}}(\mathbb{R}_+)$ *is such that (1.3), (2.1) and (2.2) hold, then (3.4) holds for every* $\lambda > 1$, *and (3.5) holds for every* $0 < \lambda < 1$. *(ii) If a complex-valued function* $f \in L^1_{\text{loc}}(\mathbb{R}_+)$ *is such that (1.3) and (2.5) hold, then (3.4) holds for every* $\lambda > 1$, *and (3.5) holds for every* $0 < \lambda < 1$.

REFERENCES

[1] G. H. HARDY, *Theorems relating to the summability and convergence of slowly oscillating series*, Proc. London Mat. Soc. (2), **8** (1910), 310–320.
[2] G. H. HARDY, *Divergent Series*, Clarendon Press, Oxford, 1949.
[3] E. LANDAU, *Über die Bedeutung einer neuerer Grenzwertsätze der Herren Hardy and Axel*, Prac. Mat.-Fiz. **21** (1910), 97–177.
[4] F. MÓRICZ AND Z. NÉMETH, *Tauberian conditions under which convergence of integrals follows from summability* $(C,1)$ *over* R_+, Analysis Math., **26** (2000), 53–61.
[5] R. SCHMIDT, *Über divergente Folgen und lineare Mittelbindungen*, Math. Z., **22** (1925), 89–152.
[6] E. C. TITCHMARSH, *Introduction to the Theory of Fourier Integrals*, Clarendon Press, Oxford, 1937.

(Received October 24, 2002)

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