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A KALLMAN–ROTA INEQUALITY FOR EVOLUTION SEMIGROUPS

C. BUŞE AND S. S. DRAGOMIR

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Abstract. A Kallman-Rota type inequality for evolution semigroups and applications for real valued functions are given.

1. Introduction

Let X be a real or complex Banach space and $\mathcal{L}(X)$ the Banach algebra of all linear and bounded operators acting on X. The norms in X and in $\mathcal{L}(X)$ will be denoted by $\|\cdot\|$.

Let $\mathbb{R}_+$ the set of all non-negative real numbers and $\mathbf{J} \in \{\mathbb{R}_+, \mathbb{R}\}$. The set $\{(t,s) : t \geqslant s \in \mathbf{J}\}$ will be denoted by $\Delta_{\mathbf{J}}$. A family

$$\mathcal{U}_{\mathbf{J}} = \{U(t,s) : (t,s) \in \Delta_{\mathbf{J}}\} \subset \mathcal{L}(X)$$

is called an *evolution family* of bounded linear operators on X if $U(t,t) = I$ (the identity operator on X) and $U(t,s)U(s,r) = U(t,r)$ for all $t \geqslant s \geqslant r \in \mathbf{J}$. Such a family is said to be *strongly continuous* if for each $x \in X$, the maps

$$(t,s) \mapsto U(t,s)x : \Delta_{\mathbf{J}} \to X$$

are continuous. A strongly continuous evolution family is said to be *exponentially bounded* if there exist $\omega \in \mathbb{R}$ and $K_\omega \geqslant 1$ such that

$$||U(t,s)|| \leqslant K_\omega e^{\omega(t-s)} \text{ for all } (t,s) \in \Delta_{\mathbf{J}}$$

and *uniformly stable* if there exists $M \in \mathbb{R}_+$ such that

$$\sup_{(t,s)\in\Delta_{\mathbf{J}}} ||U(t,s)|| \leqslant M < \infty. \tag{1.1}$$

We remind that a family $\mathbf{T} = \{T(t) : t \geqslant 0\} \subset \mathcal{L}(X)$ is called *one-parameter semigroup* if $T(0) = I$ and $T(t+s) = T(t)T(s)$ for all $t \geqslant s \geqslant 0$. An one-parameter semigroup is called *strongly continuous* or C_0-semigroup if for each $x \in X$ the maps

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$t \mapsto T(t)x$ are continuous on $\mathbb{R}_+$. For a C_0-semigroup $\mathbf{T}$, its infinitesimal generator A with the domain $D(A)$ is defined by

$$D(A) := \left\{ x \in X : \text{ there exists in } X, \quad \lim_{t \to 0} \frac{T(t)x - x}{t} =: Ax \right\}.$$

It is easy to see that if $\mathbf{T} = \{T(t) : t \geqslant 0\}$ is a strongly continuous semigroup then the family $\mathcal{U}_{\mathbf{J}} = \{U(t,s) := T(t-s) : (t,s) \in \Delta_{\mathbf{J}}\}$ is a strongly continuous and exponentially bounded evolution family. Conversely, if $\mathcal{U}_{\mathbf{J}}$ is a strongly continuous evolution family and $U(t,s) = U(t-s,0)$ for all $(t,s) \in \Delta_{\mathbf{J}}$ then the family $\mathbf{T} := \{T(t) = U(t,0) : t \geqslant 0\}$ is a strongly continuous one-parameter semigroup. For more details about the strongly continuous semigroups and evolution families we refer to [3].

LEMMA 1. *Let* $\mathbf{T} := \{T(t) : t \geqslant 0\}$ *be a strongly continuous one-parameter semigroup and* $A : D(A) \subset X \to X$ *its infinitesimal generator. If* $\mathbf{T}$ *is uniformly stable, that is, there is a positive constant* M *such that* $\sup_{t \geqslant 0} ||T(t)|| \leqslant M$, *then*

$$\|Ax\|^2 \leqslant 4M^2 \left\|A^2x\right\| \|x\|, \quad \text{for all } x \in D\left(A^2\right). \tag{1.2}$$

Proof. See [4]. □

We are recalling the notion of evolution semigroup. For more details we refer to [1], [2] and references therein. We will consider the both cases, i.e., the evolution semigroups for evolution families on $\Delta_{\mathbb{R}_+}$ and on $\Delta_{\mathbb{R}}$.

Let $\mathcal{U}_{\mathbb{R}_+}$ be a strongly continuous and exponentially bounded evolution family of bounded linear operators acting on X. Let us consider the following spaces:

- $C_{00}(\mathbb{R}_+, X)$ is the space consisting by all X-bounded, uniformly continuous functions on $\mathbb{R}_+$, such that
$$f(0) = \lim_{t \to \infty} f(t) = 0,$$
endowed with the sup-norm.
- $L_p(\mathbb{R}_+, X), 1 \leqslant p < \infty$ is the usual Lebesgue-Bochner space of all measurable functions $f : \mathbb{R}_+ \to X$, identifying functions which are equal almost everywhere, such that
$$||f||_p := \left(\int_0^\infty ||f(s)||^p ds \right)^{\frac{1}{p}} < \infty.$$
- $\chi_p(\mathbb{R}_+, X) \cap L_p(\mathbb{R}_+, X)$ endowed with the norm
$$||f||_{\chi_p} := ||f||_\infty + ||f||_p$$

Let $\mathcal{X}$ be either $C_{00}(\mathbb{R}_+, X)$ or $\chi_p(\mathbb{R}_+, X)$ and $f \in \mathcal{X}$.

It is easy to see that for each $t \geqslant 0$, the function $T(t)f$ given by

$$(T(t)f)(s) := \begin{cases} U(s, s-t)f(s-t), & s \geqslant t \\ 0, & 0 \leqslant s < t \end{cases} \tag{1.3}$$

belongs to $\mathcal{X}$, and the family $\mathbf{T} = \{T(t) : t \geqslant 0\}$ is an one-parameter semigroup of bounded linear operators acting on $\mathcal{X}$. Moreover, the following result, holds:

LEMMA 2. *The semigroup* $\mathbf{T}$ *defined in (1.3) is strongly continuous. If* $(A, D(A))$ *is the generator of* $\mathbf{T}$ *with its domain then for every* u, f *in* $\mathcal{X}$ *the following statements are equivalent:*

(i) $u \in D(A)$ *and* $Au = -f$ *;*

(ii) $u(t) = \int_0^t U(t,s)f(s)ds$ *;*

Proof. See [7]. □

The strongly continuous semigroup $\mathbf{T}$ defined in (1.3) is called *evolution semigroup* associated to $\mathcal{U}_{\mathbb{R}_+}$ on the space $\mathcal{X}$.

We will state here our first result.

THEOREM 1. *Let* $\mathcal{U}_{\mathbb{R}_+}$ *be a strongly continuous uniformly stable evolution family of bounded linear operators acting on* X*, and let* $g \in \mathcal{X}$*. Suppose that the following conditions are fulfilled:*

(i) $\int_0^{\cdot} U(\cdot,s)g(s)ds$ *belongs to* $\mathcal{X}$*;*

(ii) $\int_0^{\cdot} (\cdot - s)U(\cdot,s)g(s)ds$ *belongs to* $\mathcal{X}$*.*

Then the following inequality holds:

$$\left\| \int_0^{\cdot} U(\cdot,s)g(s)ds \right\|_{\mathcal{X}}^2 \leqslant 4M^2 \|g\|_{\mathcal{X}} \times \left\| \int_0^{\cdot} (\cdot - s)U(\cdot,s)g(s)ds \right\|_{\mathcal{X}}, \tag{1.4}$$

where M *is the constant from the estimation (1.1).*

$BUC(\mathbb{R},X)$ is the space of all X-valued, bounded and uniformly continuous functions on the real line endowed with the sup-norm. The following three spaces are closed subspaces of $BUC(\mathbb{R},X)$:

- $C_0(\mathbb{R},X)$ is the space of all X-valued, continuous functions on $\mathbb{R}$ such that $\lim_{t\to\infty} f(t) = 0$.
- $AP(\mathbb{R},X)$ is the space of all almost periodic functions, that is, the smallest closed subspace of $BUC(\mathbb{R},X)$ containing the functions of the form

$$t \mapsto e^{i\mu t}x, \quad \mu \in \mathbb{R} \text{ and } x \in X,$$

  see e.g. [6].
- $AAP(\mathbb{R},X)$ is the space of all X-valued asymptotically almost periodic functions on $\mathbb{R}$, i.e., the space consisting in all functions f for which there exist $g \in C_0(\mathbb{R},X)$ and $h \in AP(\mathbb{R},X)$ such that $f = g + h$.

Let $\mathcal{Y}$ one of the spaces described before and $f \in \mathcal{Y}$. If $\mathcal{U}_{\mathbb{R}}$ satisfies certain conditions, which will be outlined in Lemma 3 below, then for each $t \geqslant 0$ the function given by

$$s \mapsto (T(t)f)(s) := U(s,s-t)f(s-t) : \mathbb{R} \to X \tag{1.5}$$

belongs to $\mathcal{Y}$, and the family $\mathbf{T} := \{T(t) : t \geqslant 0\}$ is an one-parameter semigroup of bounded linear operators on $\mathcal{Y}$. The semigroup $\mathbf{T}$ can be not strongly continuous. However, in certain cases, this semigroup is strongly continuous, and is called *evolution semigroup* associated to $\mathcal{U}_{\mathbb{R}}$ on the space $\mathcal{Y}$.

LEMMA 3. *Let* $\mathcal{U}_{\mathbb{R}}$ *be a strongly continuous evolution family of bounded linear operators on* X*, and* q *be a fixed positive real number.*

(i) If $\mathcal{Y} = C_0(\mathbb{R}, X)$, and $\mathcal{U}_{\mathbb{R}}$ is exponentially bounded, then the semigroup associated to $\mathcal{U}_{\mathbb{R}}$, defined in (1.5), is a strongly continuous one-parameter semigroup of bounded linear operators on $\mathcal{Y}$;

(ii) If $\mathcal{Y}$ is either the spaces $AP(\mathbb{R}, X)$ or $AAP(\mathbb{R}, X)$ and $\mathcal{U}_{\mathbb{R}}$ is q-periodic, that is, $U(t+q, s+q) = U(t,s)$ for all $(t,s) \in \Delta_{\mathbb{R}}$, then the semigroup given in (1.5), is a strongly continuous semigroup on $\mathcal{Y}$.

Let $(B, D(B))$ the generator of the evolution semigroup given in (1.5). If u and g belongs to $\mathcal{Y}$ then the following statements are equivalent:

(iii) $u \in D(B)$ and $Bu = -g$;

(iv)

$$u(t) = U(t,s)u(s) + \int_s^t U(t,s)g(s)ds, \tag{1.6}$$

for all $t \geqslant s$.

Proof. See [5], [9] for evolution semigroups defined on $C_0(\mathbb{R}, X)$ and [8] for evolution semigroups on $AP(\mathbb{R}, X)$ or $AAP(\mathbb{R}, X)$. □

Let $\mathcal{Y}$ be one of the spaces $C_0(\mathbb{R}, X), AP(\mathbb{R}, X), AAP(\mathbb{R}, X)$ and let $\mathcal{Y}_0$ be the set of all functions $f \in \mathcal{Y}$ such that $\lim_{t \to (-\infty)} f(t) = 0$. It is clearly that $\mathcal{Y}_0$ is a closed subspace of $\mathcal{Y}$.

We may now state our second result.

THEOREM 2. *Let $\mathcal{U}_{\mathbb{R}}$ be a strongly continuous uniformly stable evolution family of bounded linear operators on X and $q > 0$, fixed. The following statements hold:*

(j) If $\mathcal{Y} = C_0(\mathbb{R}, X)$, then the evolution semigroup given in (1.5) is defined on $\mathcal{Y}_0$;

(jj) If $\mathcal{Y}$ is one of the both spaces $AP(\mathbb{R}, X)$ or $AAP(\mathbb{R}, X)$ and $\mathcal{U}_{\mathbb{R}}$ is q-periodic then the evolution semigroup given in (1.4) is defined on $\mathcal{Y}_0$.

If $(C, D(C))$ is the generator of the evolution semigroup on $\mathcal{Y}_0$, given in (1.5), and v, h belongs to $\mathcal{Y}_0$, then the following statements are equivalent:

(jjj) $v \in D(C)$ and $Cv = -h$;

(jv)

$$v(t) = \int_{-\infty}^t U(t,s)h(s)ds, \tag{1.7}$$

for every real number t. Moreover, the following inequality holds:

$$\left\| \int_{-\infty}^{\cdot} U(\cdot, s)h(s)ds \right\|_{\mathcal{Y}}^2 \leqslant 4M^2 \|h\|_{\mathcal{Y}} \times \left\| \int_{-\infty}^{\cdot} (\cdot - s)U(\cdot, s)h(s)ds \right\|_{\mathcal{Y}}. \tag{1.8}$$

2. Proofs

Proof of Theorem 1. Let $\mathbf{T}$ be the evolution semigroup associated to $\mathcal{U}_{\mathbb{R}_+}$ on the space $\mathcal{X}$ and $(A, D(A))$ its infinitesimal generator. From Lemma 2 it follows that the function $t \mapsto u(t) := \int_0^t U(t,s)g(s)ds$ belongs to $D(A)$ and $Au = -g$. The function

$t \mapsto v(t) := \int_0^t U(t,r)u(r)dr$ belongs to $\mathcal{X}$. Indeed, using the Fubini Theorem, we have:

$$\begin{aligned} v(t) &= \int_0^t \left[U(t,r) \int_0^r U(r,s)g(s)ds \right] dr \\ &= \int_0^t \left[\int_0^r U(t,s)g(s)ds \right] dr \\ &= \int_0^t \left[\int_0^t 1_{[0,r]}(s)U(t,s)g(s)ds \right] dr \\ &= \int_0^t \left[\int_s^t U(t,s)g(s)dr \right] ds \\ &= \int_0^t (t-s)U(t,s)g(s)ds, \end{aligned}$$

where $1_{[0,r]}$ is the characteristic function of the interval $[0,r]$. Using again Lemma 2 follows that $v \in D(A^2)$ and $A^2 v = A(Av) = -Au = g$.

Now the inequality (1.4) follows by Lemma 1, if we replace x with v in (1.2). $\square$

Proof of Theorem 2. Firstly we prove that $\mathcal{Y}_0$ is an invariant subspace for each operator $T(t), t \geqslant 0$, given in (1.5). By Lemma 3 it suffices to prove that $\lim_{s\to(-\infty)} (T(t)f)(s) = 0$ for each $t \geqslant 0$ and every $f \in \mathcal{Y}_0$, and this fact is an easy consequence of the following estimations:

$$\|(T(t)f)(s)\| \leqslant \|U(s,s-t)\| \, \|f(s-t)\| \leqslant M \|f(s-t)\| \to 0 \text{ as } s \to (-\infty),$$

where M is the positive constant from (1.1). Now, the implication *(jjj)* $\Rightarrow$*(jv)* follows from Lemma 3, passing to the limit for $s \to (-\infty)$. The converse implication *(jv)* $\Rightarrow$ *(jjj)* can be obtained on the following way.

Let v as in (1.7) and $t > 0$. Simple calculus gives

$$\frac{T(t)v - v}{t} = -\frac{\int_0^t T(r)h dr}{t} \to -h \text{ in } \mathcal{X}$$

when $t \to 0$, that is $v \in D(C)$ and $Cv = -h$. Now the inequality (1.8), can be established as in the proof of Theorem 1 and we omit the details. $\square$

3. Applications

In this section some scalar inequalities are presented.

COROLLARY 1. *Let $g : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function such that $g(0) = g(\infty) := \lim_{t\to\infty} g(t) = 0$. Suppose that the functions:*

$$t \mapsto h(t) := \int_0^t g(s)ds \text{ and } t \mapsto u(t) := \int_0^t (t-s)g(s)ds$$

verifies the condition $h(\infty) = u(\infty) = 0$.

Then the following inequality holds:

$$\sup_{t\geqslant 0}\left|\int_0^t g(s)ds\right|^2 \leqslant 4\cdot\sup_{t\geqslant 0}|g(t)| \times \sup_{t\geqslant 0}\left|\int_0^t (t-s)g(s)ds\right|.$$

Proof. We apply Theorem 1 for $\mathcal{X} = C_{00}(\mathbb{R}_+,\mathbb{R})$ and for $U(t,s)x = x$, where $t \geqslant s \geqslant 0$ and $x \in \mathbb{R}$. $\square$

COROLLARY 2. *Let* g,h,u *as in Corollary 1 and* f *be a continuous, positive and nondecreasing function on* $\mathbb{R}_+$. *The following inequality holds:*

$$\sup_{t\geqslant 0}\left[\frac{\left|\int_0^t f(s)g(s)ds\right|^2}{f(t)^2}\right] \leqslant 4\sup_{t\geqslant 0}|g(t)|\sup_{t\geqslant 0}\left[\frac{\left|\int_0^t (t-s)f(s)g(s)ds\right|}{f(t)}\right].$$

Proof. Follows by Theorem 1 for $\mathcal{X} = C_{00}(\mathbb{R}_+,\mathbb{R})$ and $U(t,s) = \frac{f(s)}{f(t)}$. $\square$

COROLLARY 3. *Let* $1 \leqslant p < \infty$ *and* $f \in \chi_p(\mathbb{R}_+,\mathbb{R})$. *If the functions*

$$t \mapsto g(t) := \int_0^t f(s)ds \text{ and } t \mapsto h(t) := \int_0^t (t-s)f(s)ds$$

belongs to $\chi_p(\mathbb{R}_+,\mathbb{R})$, *then the following inequality, holds:*

$$\|g\|_{\chi_p}^2 \leqslant 4\|f\|_{\chi_p} \times \|h\|_{\chi_p}.$$

Proof. Follows by Theorem 1 for $\mathcal{X} = \chi_p(\mathbb{R}_+,\mathbb{R})$ and for $U(t,s)x = x$ where $t \geqslant s \geqslant 0$ and $x \in \mathbb{R}$. $\square$

COROLLARY 4. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be an almost periodic or asymptotically almost periodic function such that* $g(-\infty) = 0$. *Then*

$$\sup_{t\in\mathbb{R}}\left|\int_{-\infty}^t \frac{1+\sin^2 s}{1+\sin^2 t}g(s)ds\right|^2 \leqslant 16\sup_{t\in\mathbb{R}}|g(t)| \times \sup_{t\in\mathbb{R}}\left|\int_{-\infty}^t (t-s)\frac{1+\sin^2 s}{1+\sin^2 t}g(s)ds\right|.$$

Proof. Follows by Theorem 2 for $\mathcal{Y} = AP(\mathbb{R},\mathbb{R})$ or $\mathcal{Y} = AAP(\mathbb{R},\mathbb{R})$ and $U(t,s)x = \frac{1+\sin^2 s}{1+\sin^2 t}x$ where $t \geqslant s$ and $x \in \mathbb{R}$. It is clear that $\mathcal{U} = \{U(t,s); t \geqslant s\}$ is a π-periodic family consisting in operators acting on $\mathbb{R}$, and $\sup_{t\geqslant s} U(t,s) \leqslant 2$. $\square$

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C. Buşe
Department of Mathematics
West University of Timisoara
Timisoara, 1900, Bd. V. Parvan. Nr. 4
Romania
e-mail: buse@tim1.uvt.ro
URL: http://rgmia.vu.edu.au/BuseCVhtml

S. S. Dragomir
School of Communications and Informatics
Victoria University of Technology
P. O. Box 14428
Melbourne City MC, Victoria 8001
Australia
e-mail: sever.dragomir@vu.edu.au
URL: http://rgmia.vu.edu.au/SSDragomirWeb.html