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OPERATOR MONOTONE FUNCTIONS INDUCED FROM LÖWNER–HEINZ INEQUALITY AND STRICTLY CHAOTIC ORDER

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(*communicated by T. Furuta*)

Abstract. Furuta presented direct and simplified proofs of operator monotonicity of functions

$$\varphi(t) = \frac{t-1}{\log t} \quad \text{and} \quad \psi(t) = \frac{t\log t - t + 1}{(\log t)^2}$$

by using Löwner-Heinz inequality. Extending his method, we give a sequence of operator monotone functions $\{f_k(t)\}_{k=0}^{\infty}$ with $f_0(t) = \varphi(t)$ and $f_1(t) = \psi(t)$. We also study relations between $f_k(t)$ and strictly chaotic order defined among positive invertible operators and obtain some extensions of results due to Furuta.

1. Introduction

A (bounded linear) operator A on a Hilbert space H is positive, in symbol $A \geqslant 0$, if $(Ax, x) \geqslant 0$ for all $x \in H$. In particular, A is strictly positive, in symbol $A > 0$, if A is positive invertible. The well-known Löwner-Heinz inequality says that if $A \geqslant B \geqslant 0$ then $A^{\alpha} \geqslant B^{\alpha}$ for $0 \leqslant \alpha \leqslant 1$ ([7] – [12], etc.). This means that the function $t \mapsto t^{\alpha}$ $(0 \leqslant \alpha \leqslant 1)$ on $[0, \infty)$ is operator monotone. Another well-known example of operator monotone functions is $t \mapsto \log t$ on $(0, \infty)$ ([7], [9], [11], etc.). Based on this fact, the chaotic order $A \gg B$, which is weaker than the usual order $A \geqslant B$, is defined by $\log A \geqslant \log B$ among strictly positive operators. (Similarly, the strictly chaotic order $A \underset{s}{\gg} B$ is defined by $\log A > \log B$.)

Recently, Furuta [5] and [6], by using Löwner-Heinz inequality, presented direct and simplified proofs of operator monotonicity of the functions

$$\varphi(t) = \frac{t-1}{\log t}, \qquad \psi(t) = \frac{t\log t - t + 1}{(\log t)^2}$$

and their dual functions $t/\varphi(t)$ and $t/\psi(t)$. (The values of those functions at $t = 0$ and $t = 1$ are defined by their limits as $t \to +0$ and $t \to 1$, respectively.)

As an extension of results on chaotic order due to Fujii et al [2] – [4], Furuta showed in [5] (and [6]) the following fact related to the above functions:

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THEOREM F. *Let* $A \underset{s}{\gg} B$ *for* $A,\ B > 0$. *Then there exists* $\beta \in (0,1]$ *such that*

$$\varphi(A^\alpha) > \varphi(B^\alpha) \quad \big(\textit{and} \quad \psi(A^\alpha) > \psi(B^\alpha)\big) \quad \textit{for all } \alpha \in (0,\beta). \tag{1.1}$$

Furthermore, asking if the condition $A \underset{s}{\gg} B$ can be replaced by the weaker one $A \gg B$, Furuta, in [5] (and [6]), gave a decisive counterexample of a pair of positive 2×2 matrices such that $A \gg B$ but $\varphi(A^\alpha) \not\geqslant \varphi(B^\alpha)$ $\big($and $\psi(A^\alpha) \not\geqslant \psi(B^\alpha)\big)$ for all $\alpha > 0$.

In this paper, we induce from Löwner-Heinz inequality a sequence $\{f_k\}_{k=0}^\infty$ and some related sequences of operator monotone functions with $f_0 = \varphi$ and $f_1 = \psi$. We use a method of successive differentiation, the idea of which we owe to Furuta [5] and [6]. With respect to strictly chaotic order and functions $f_k(t)$ of the sequence, we show an extension of Theorem F, which allows us to replace φ or ψ in (1.1) by any f_k. We further, using the same matrices A and B chosen in [6] or somewhat general ones for a counterexample, show an example of positive matrices A and B such that $A \gg B$ but there is $\beta \in (0,1]$ with $f_k(A^\alpha) \not\geqslant f_k(B^\alpha)$ for all integers $k \geqslant 0$ and all $\alpha \in (0,\beta)$. This gives, for $k=1$, a weak counterexample for Furuta's question mentioned before.

2. Sequences of operator monotone functions induced from Löwner-Heinz inequality

In [5] and [6], Furuta presented direct and simplified proofs of operator monotonicity of $\varphi(t)$ and $\psi(t)$ stated before. The essence of his proofs is to make use of the function

$$T_n(x) = \frac{x^n - 1}{x-1} = x^{n-1} + \cdots + x + 1 \ \ (x \neq 1),\ T_n(1) = \lim_{x\to 1} T_n(x) = n$$

and its derivative $T_n'(x)$. In fact, if we define

$$U_{n,0}(t) = \frac{1}{n}\big[T_n(x)\big]_{x=t^{\frac{1}{n}}}$$

and

$$U_{n,1}(t) = \frac{1}{n^2}\big[T_n'(x)\big]_{x=t^{\frac{1}{n}}},$$

then $U_{n,0}(t)$ and $U_{n,1}(t)$ are operator monotone functions by Löwner-Heinz inequality, so that

$$\lim_{n\to\infty} U_{n,0}(t) = \frac{t-1}{\log t} = \varphi(t)$$

and

$$\lim_{n\to\infty} U_{n,1}(t) = \frac{t\log t - t + 1}{(\log t)^2} = \psi(t)$$

are also operator monotone. This consideration suggests constructing a sequence of operator monotone functions as follows.

THEOREM 2.1. *Let* $T_n^{(0)}(x) = T_n(x)$ *and let*

$$U_{n,k}(t) = \frac{1}{n^{k+1}}\left[T_n^{(k)}(x)\right]_{x=t^{\frac{1}{n}}} \quad \textit{for} \quad k = 0, 1, \dots, n-1.$$

Then for all integers $k \geqslant 0$

$$f_k(t) = \lim_{n\to\infty} U_{n,k}(t)$$

exist and they are operator monotone. Furthermore

$$f_k(t) = \frac{k!}{(\log\frac{1}{t})^{k+1}}\left\{1 - te_k\left(\log\frac{1}{t}\right)\right\}, \tag{2.1}$$

where

$$e_0(s) = 1 \quad \textit{and} \quad e_k(s) = 1 + \frac{s}{1!} + \cdots + \frac{s^k}{k!} \quad (k \geqslant 1).$$

Proof. It is not difficult to see that if $n > k \geqslant 1$ then

$$U_{n,k}(t) = \frac{1}{n^{k+1}}\left(c_{n-1,k}t^{\frac{n-1-k}{n}} + \cdots + c_{k+1,k}t^{\frac{1}{n}} + c_{k,k}\right), \tag{2.2}$$

where $c_{m,k} = m(m-1)\cdots(m-k+1)$. Hence $U_{n,k}(t)$ (and also its limit $f_k(t)$ if it exists) are operator monotone by Löwner-Heinz inequality. First for $k = 0$ we have, as stated before,

$$f_0(t) = \lim_{n\to\infty} U_{n,0}(t) = \frac{t-1}{\log t} = \frac{1 - te_0(\log\frac{1}{t})}{\log\frac{1}{t}}.$$

Next for $(n >)\ k \geqslant 1$, since $T_n(x)(x-1) = x^n - 1$, we have, by Leibniz' law on k-times differentiation,

$$T_n^{(k)}(x)(x-1) + kT_n^{(k-1)}(x) = n(n-1)\cdots(n-k+1)x^{n-k}.$$

Dividing both sides by n^k, we have, by the definition of $U_{n,k}(t)$,

$$U_{n,k}(t)\cdot n(t^{\frac{1}{n}} - 1) + kU_{n,k-1}(t) = \left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right)t^{1-\frac{k}{n}}.$$

Further, taking the limits of both sides as $n \to \infty$, we have

$$f_k(t)\log t + kf_{k-1}(t) = t \tag{2.3}$$

or

$$f_k(t) = \frac{1}{\log\frac{1}{t}}\{kf_{k-1}(t) - t\}. \tag{2.4}$$

(We note that this recurrence formula (2.3) (or (2.4)) ensures existence of limits $f_k(t)$ successively.) By an elementary computation we can obtain (2.1) from these formulae. □

We remark that if we define $\tilde{f}_0(t) = \left[\dfrac{e^x - 1}{x}\right]_{x=\log t} = \dfrac{t-1}{\log t}\big(= f_0(t)\big)$ and

$$\tilde{f}_k(t) = \left[\frac{d^k}{dx^k}\left(\frac{e^x - 1}{x}\right)\right]_{x=\log t}$$

for integers $k \geqslant 1$ then we have the same relation

$$\tilde{f}_k(t)\log t + k\tilde{f}_{k-1}(t) = t$$

as (2.3) (by Leibniz' law). Hence $\tilde{f}_k(t) = f_k(t)$ for all integers $k \geqslant 0$.

By the general theory on operator means [10] (cf. [9, p.169]), for a given operator monotone function $f(t) > 0$ on $(0,\infty)$, the following three functions

(i) ${}^t f(t) = tf\left(\frac{1}{t}\right)$ (transpose),
(ii) $f^*(t) = 1/f\left(\frac{1}{t}\right)$ (adjoint) and
(iii) $f^{\perp}(t) = t/f(t)$ (dual)

are defined, and they are all again operator monotone. Applying this general fact, we obtain the following result.

COROLLARY 2.2. *For each integer* $k \geqslant 0$

$$g_k(t)\left(= \frac{1}{k!}{}^t f_k(t)\right) = \frac{t - e_k(\log t)}{(\log t)^{k+1}},$$

$$h_k(t)\left(= k! f_k^*(t)\right) = \frac{t(\log t)^{k+1}}{t - e_k(\log t)}$$

and

$$j_k(t)\left(= k! f_k^{\perp}(t)\right) = \frac{t(\log \frac{1}{t})^{k+1}}{1 - te_k(\log \frac{1}{t})}$$

are operator monotone.

We here remark that by the similar argument taken as in Theorem 2.1 we can give (alternative) direct proofs of operator monotonicity of $g_k(t), h_k(t)$ and $j_k(t)$. In fact, if we put $V_{n,k}(t) = tU_{n,k}(\frac{1}{t}), W_{n,k}(t) = 1/U_{n,k}(\frac{1}{t})$ and $Z_{n,k}(t) = t/U_{n,k}(t)$, then we see them operator monotone from (2.2), so that as their limits $(n \to \infty)$ we obtain operator monotone functions ${}^t f_k(t), f_k^*(t)$ and $f_k^{\perp}(t)$, respectively.

If a function $f_\alpha(t) = f(t,\alpha)$ is an operator monotone function with a continuous parameter $\alpha \in [\alpha_1, \alpha_2]$, then its integral with respect to α is again operator monotone [1]. In particular, if we put $f_\alpha(t) = t^\alpha,\ \alpha \in [0,1]$, then by successive integration we have the sequence $\{g_k(t)\}_{k=0}^{\infty}$ as follows (cf. [9, p. 152]):

THEOREM 2.3. *Let* $q_0(t) = \displaystyle\int_0^1 t^\alpha\, d\alpha$ *and let*

$$q_k(t) = \int_0^1 \alpha^k q_{k-1}(t^\alpha)\, d\alpha \quad (k = 1, 2, \ldots).$$

Then $q_k(t) = g_k(t)$ *for all integers* $k \geqslant 0$.

Proof. Clearly $q_0(t) = g_0(t) = \dfrac{t-1}{\log t}$. Assume that $q_k(t) = g_k(t)$ for a $k \geqslant 0$. Then

$$\begin{aligned} q_{k+1}(t) &= \int_0^1 \alpha^{k+1} g_k(t^\alpha)\,d\alpha \\ &= \int_0^1 \alpha^{k+1} \frac{t^\alpha - e_k(\log t^\alpha)}{(\log t^\alpha)^{k+1}}\,d\alpha \\ &= \frac{1}{(\log t)^{k+1}} \int_0^1 \{t^\alpha - e_k(\alpha \log t)\}\,d\alpha \\ &= \frac{1}{(\log t)^{k+2}} \{t - e_{k+1}(\log t)\} = g_{k+1}(t). \end{aligned}$$

This completes the proof by induction. □

3. An extension of Theorem F

In the study of chaotic order Fujii et al [2] (and [3], [4]) pointed out that $A \underset{s}{\gg} B$ is equivalent to $A^\alpha > B^\alpha$ for an $\alpha > 0$. As stated before, Furuta [5] (and [6]) also presented Theorem F related to operator monotone functions and strictly chaotic order; if $A \underset{s}{\gg} B$ then there exists $\beta \in (0,1]$ such that $\varphi(A^\alpha) > \varphi(B^\alpha)$ $\big($*and* $\psi(A^\alpha) > \psi(B^\alpha)\big)$ for all $\alpha \in (0,\beta)$. We try to extend those results, giving the similar fact for all $f_k(t)$.

LEMMA 3.1. *Let A and B be selfadjoint operators and let $A > B$. Then there exisits $\beta \in (0,1]$ such that*

$$F_k(\alpha A) > F_k(\alpha B)$$

for all integers $k \geqslant 0$ and all $\alpha \in (0,\beta)$, where

$$F_k(t) = f_k(e^t) = \frac{k!}{(-t)^{k+1}}\{1 - e^t \cdot e_k(-t)\}.$$

Proof. We can obtain the expansion

$$F_k(t) = \frac{1}{k+1} + \frac{t}{(k+2)\cdot 1!} + \frac{t^2}{(k+3)\cdot 2!} + \cdots \tag{3.1}$$

for any integer $k \geqslant 0$ directly by Taylor's theorem, or by the induction method, using the relation (2.4) replaced t by e^t, i.e.,

$$F_k(t) = \frac{1}{t}\{e^t - kF_{k-1}(t)\} \quad (k = 1, 2, \ldots).$$

Since $A > B$, we may assume that $A - B \geqslant \varepsilon$ for some $\varepsilon > 0$. Hence for $0 < \alpha \leqslant 1$

$$\begin{aligned} F_k(\alpha A) - F_k(\alpha B) &= \frac{\alpha}{(k+2)\cdot 1!}(A - B) + \frac{\alpha^2}{(k+3)\cdot 2!}(A^2 - B^2) + \cdots \\ &\geqslant \frac{\alpha}{k+2}\Big\{\varepsilon - \Big(\frac{\|A\|^2 + \|B\|^2}{2!} + \frac{\|A\|^3 + \|B\|^3}{3!} + \cdots\Big)\alpha\Big\} \\ &\geqslant \frac{\alpha}{k+2}\{\varepsilon - (e^{\|A\|} + e^{\|B\|})\alpha\}. \end{aligned}$$

Putting $\beta = \min\left\{1, \dfrac{\varepsilon}{e^{\|A\|}+e^{\|B\|}}\right\}$, we have $F_k(\alpha A) - F_k(\alpha B) > 0$ for $\alpha \in (0,\beta)$. □

THEOREM 3.2. *Let A and B be strictly positive operators with $1 \notin \sigma(A), \sigma(B)$ and let $A \underset{s}{\gg} B$. Then there exists $\beta \in (0,1]$ such that*

$$f_k(A^\alpha) > f_k(B^\alpha) \tag{3.2}$$

for all integers $k \geqslant 0$ and all $\alpha \in (0,\beta)$.

Proof. We have only to replace A by $\log A$ and B by $\log B$, respectively in Lemma 3.1. □

4. Counterexamples

We want to show that in Theorem 3.2 the condition $A \underset{s}{\gg} B$ cannot be relaxed to $A \gg B$ in order to guarantee (3.2). We begin with the following example of 2×2 matrices with respect to Lemma 3.1.

EXAMPLE 1. Let

$$A = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -5 \end{pmatrix}.$$

These are the same matrices adopted in [6] for a counterexample (showing that $A \geqslant B$, but $\psi(e^{\alpha A}) = F_1(\alpha A) \ngeqslant F_1(\alpha B) = \psi(e^{\alpha B})$ for all $\alpha > 0$.) We can then really show that for any integer $k \geqslant 0$ there exists some $\beta_k \in (0,1]$ such that

$$F_k(\alpha A) \ngeqslant F_k(\alpha B) \quad for \quad all \quad \alpha \in (0,\beta_k). \tag{4.1}$$

Here, however, let us consider somewhat general matrices instead: Let again

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} d & 0 \\ 0 & e \end{pmatrix}, \tag{4.2}$$

where all letters a, b, c, d and e are nonzero real numbers, and satisfy the conditions

$$a-d>0,\ c-e>0,\ d \neq e \quad \text{and} \quad (a-d)(c-e) = b^2. \tag{4.3}$$

Then it is easy to see $A \geqslant B$. Furthermore, (4.1), in the general setting above, is still true for each integer $k \geqslant 0$, which we shall show after Example 2.

EXAMPLE 2. Let

$$\log A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad \text{and} \quad \log B = \begin{pmatrix} d & 0 \\ 0 & e \end{pmatrix},$$

where a, b, c, d and e satisfy (4.3). Then from the fact stated above, $\log A = A_1 \geqslant B_1 = \log B$, i.e., $A \gg B$ but there exists some $\beta_k \in (0,1]$ for each integer $k \geqslant 0$ such that

$$f_k(A^\alpha) = F_k(\alpha A) \ngeqslant F_k(\alpha B) = f_k(B^\alpha)$$

for all $\alpha \in (0, \beta_k)$. Hence we have a counterexample with respect to Theorem 3.2.

Proof of (4.1) for A and B in the general setting (4.2). We may show that

$$D_k(\alpha) := F_k(\alpha A) - F_k(\alpha B) \not\geq 0 \qquad \text{or} \qquad \det D_k(\alpha) < 0 \tag{4.4}$$

for all sufficiently small $\alpha > 0$. By (3.1) we have

$$\begin{aligned} D_k(\alpha) &= \frac{\alpha}{(k+2)\cdot 1!}(A-B) + \frac{\alpha^2}{(k+3)\cdot 2!}(A^2-B^2) + \frac{\alpha^3}{(k+4)\cdot 3!}(A^3-B^3) + \cdots \\ &= \frac{\alpha}{2(k+2)(k+3)}\Big[\big\{2(k+3)(A-B) + \alpha(k+2)(A^2-B^2)\big\} \\ &\quad + \frac{\alpha^2(k+2)(k+3)}{3(k+4)}\big\{A^3 - B^3 + C(\alpha)\big\}\Big], \end{aligned}$$

where $C(\alpha)$ is the remainder such that $\lim_{\alpha\to+0} C(\alpha) = 0$. For convenience sake, write

$$L_k(\alpha) = 2(k+3)(A-B) + \alpha(k+2)(A^2-B^2) \tag{4.5}$$

and

$$M_k(\alpha) = A^3 - B^3 + C(\alpha). \tag{4.6}$$

Then, since $L_k(\alpha)$ is invertible as shown afterwards (in (i)), we see that

$$\begin{aligned} D_k(\alpha) &= \frac{\alpha}{2(k+2)(k+3)}\Big\{L_k(\alpha) + \frac{\alpha^2(k+2)(k+3)}{3(k+4)}M_k(\alpha)\Big\} \\ &= \frac{\alpha}{2(k+2)(k+3)}L_k(\alpha)N_k(\alpha), \end{aligned}$$

where

$$N_k(\alpha) = 1 + \frac{\alpha^2(k+2)(k+3)}{3(k+4)}L_k(\alpha)^{-1}M_k(\alpha). \tag{4.7}$$

Hence we see that

$$\det D_k(\alpha) = \left\{\frac{\alpha}{2(k+2)(k+3)}\right\}^2 \det L_k(\alpha)\det N_k(\alpha),$$

so that it suffices to show the following (i) and (ii) (in order to see (4.4)):
(i) $\det L_k(\alpha) < 0$ (hence $L_k(\alpha)$ is invertible), or precisely

$$\det L_k(\alpha) = -\alpha^2(k+2)^2 b^2(d-e)^2 < 0. \tag{4.8}$$

(ii) $\det N_k(\alpha) > 0$ for all sufficently small $\alpha > 0$, or the fact

$$\det N_k(0) =: \lim_{\alpha\to+0} \det N_k(\alpha) = \frac{k^2+6k+6}{3(k+2)(k+4)} > 0. \tag{4.9}$$

First for (i), put $L_k(\alpha) = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix}$. Then from (4.5), we have

$$\begin{aligned} l_{11}(\alpha) &= 2(k+3)(a-d) + \alpha(k+2)(a^2+b^2-d^2), \\ l_{12}(\alpha) &= l_{21}(\alpha) = \{2(k+3)+\alpha(k+2)(a+c)\}b, \\ l_{22}(\alpha) &= 2(k+3)(c-e) + \alpha(k+2)(b^2+c^2-e^2). \end{aligned} \tag{4.10}$$

Using the identity $b^2 = (a-d)(c-e)$, we have

$$a^2+b^2-d^2 = (a^2-d^2)+b^2 = (a-d)(a+d+c-e),$$

and similarly

$$b^2+c^2-e^2 = (c-e)(a-d+c+e),$$

so that we can rewrite $l_{11}(\alpha)$ and $l_{22}(\alpha)$ as follows;

$$l_{11}(\alpha) = \{2(k+3)+\alpha(k+2)(a+d+c-e)\}(a-d), \tag{4.11}$$

$$l_{22}(\alpha) = \{2(k+3)+\alpha(k+2)(a-d+c+e)\}(c-e). \tag{4.12}$$

Further, then from (4.10), (4.11), (4.12) and the identity $b \times \frac{b}{a-d} = c-e$, we have

$$\begin{aligned} l_{21}(\alpha)' &:= l_{21}(\alpha) - l_{11}(\alpha) \times \frac{b}{a-d} = -\alpha(k+2)b(d-e), \\ l_{22}(\alpha)' &:= l_{22}(\alpha) - l_{12}(\alpha) \times \frac{b}{a-d} = -\alpha(k+2)(c-e)(d-e). \end{aligned}$$

Hence we see that

$$\begin{aligned} \det L_k(\alpha) &= \det \begin{pmatrix} l_{11}(\alpha) & l_{12}(\alpha) \\ l_{21}(\alpha) & l_{22}(\alpha) \end{pmatrix} = \det \begin{pmatrix} l_{11}(\alpha) & l_{12}(\alpha) \\ l_{21}(\alpha)' & l_{22}(\alpha)' \end{pmatrix} \\ &= -\alpha^2(k+2)^2 b^2 (d-e)^2. \qquad (\text{Use } \ (a-d)(c-e) = b^2 \ \text{ again.}) \end{aligned}$$

This is the desired result. Next for (ii), we begin with computation of the product $L_k(\alpha)^{-1}M_k(\alpha)$. For $L_k(\alpha)^{-1}$, we note that

$$L_k(\alpha)^{-1} = \frac{1}{\det L_k(\alpha)} \begin{pmatrix} l_{22}(\alpha) & -l_{12}(\alpha) \\ -l_{21}(\alpha) & l_{11}(\alpha) \end{pmatrix}, \tag{4.13}$$

and for $M_k(\alpha)$, putting $M_k(\alpha) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$, we have, from (4.6), that

$$\begin{aligned} m_{11}(\alpha) &= a^3 + 2ab^2 + b^2c - d^3 + f_{11}(\alpha), \\ m_{12}(\alpha) &= m_{21}(\alpha) = b(a^2+ac+b^2+c^2) + f_{12}(\alpha) \quad (f_{12} = f_{21}), \\ m_{22}(\alpha) &= ab^2 + 2b^2c + c^3 - e^3 + f_{22}(\alpha), \end{aligned}$$

where $f_{ij}(\alpha)(i,j=1,2)$ are components of $C(\alpha)$ with $\lim_{\alpha \to +0} f_{ij}(\alpha) = 0$. Hence we have, from (4.8) and (4.13),

$$\alpha^2 L_k(\alpha)^{-1} M_k(\alpha) = -\frac{1}{(k+2)^2 b^2 (d-e)^2} \begin{pmatrix} l_{22}(\alpha) & -l_{12}(\alpha) \\ -l_{21}(\alpha) & l_{11}(\alpha) \end{pmatrix} \begin{pmatrix} m_{11}(\alpha) & m_{12}(\alpha) \\ m_{21}(\alpha) & m_{22}(\alpha) \end{pmatrix}.$$

Now taking the limit as $\alpha \to +0$, we have

$$\lim_{\alpha\to 0} \alpha^2 L_k(\alpha)^{-1} M_k(\alpha) = -\frac{2(k+3)}{(k+2)^2 b^2 (d-e)^2}$$
$$\times \begin{pmatrix} c-e & -b \\ -b & a-d \end{pmatrix} \begin{pmatrix} a^3+2ab^2+b^2c-d^3 & b(a^2+ac+b^2+c^2) \\ b(a^2+ac+b^2+c^2) & ab^2+2b^2c+c^3-e^3 \end{pmatrix}.$$

Put $\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$ the product of the two matrices right above. Then for p_{11}, using the identity $b^2 = (a-d)(c-e)$ (or $(a-d)(c-e) = b^2$) repeatedly, we have

$$\begin{aligned} p_{11} &= (c-e)(a^3+2ab^2+b^2c-d^3) - b^2(a^2+ac+b^2+c^2) \\ &= (c-e)\{(a^3-d^3)+(2a+c)b^2\} - (a-d)(c-e)(a^2+ac+b^2+c^2) \\ &= (c-e)(a-d)\{a^2+ad+d^2+(2a+c)(c-e)-(a^2+ac+b^2+c^2)\} \\ &= (c-e)(a-d)\{d^2+(a+c-e)d-(a+c)e\} \\ &= b^2(d-e)(d+a+c). \end{aligned}$$

Similarly we have

$$\begin{aligned} p_{12} &= b(c-e)(d-e)(e+a+c), \\ p_{21} &= -b(a-d)(d-e)(d+a+c), \\ p_{22} &= -b^2(d-e)(e+a+c). \end{aligned}$$

Hence for $N_k(0)$, we have (from (4.7))

$$\begin{aligned} N_k(0) &:= \lim_{\alpha\to+0} N_k(\alpha) = \lim_{\alpha\to+0}\left\{ I + \frac{(k+2)(k+3)}{3(k+4)} \alpha^2 L_k(\alpha)^{-1} M_k(\alpha) \right\} \\ &= I + \frac{(k+2)(k+3)}{3(k+4)} \times \left\{ -\frac{2(k+3)}{(k+2)^2 b^2 (d-e)^2} \right\} \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \\ &= I - \frac{2(k+3)^2}{3(k+2)(k+4)b(d-e)} \begin{pmatrix} b(d+a+c) & (c-e)(e+a+c) \\ -(a-d)(d+a+c) & -b(e+a+c) \end{pmatrix}. \end{aligned}$$

Now if we rewrite $N_k(0)$ as

$$N_k(0) = \frac{1}{3(k+2)(k+4)b(d-e)} \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix}, \tag{4.14}$$

then we see that

$$\begin{aligned} n_{11} &= \{3(k+2)(k+4)(d-e) - 2(k+3)^2(d+a+c)\}b, \\ n_{12} &= -2(k+3)^2(c-e)(e+a+c), \\ n_{21} &= 2(k+3)^2(a-d)(d+a+c), \\ n_{22} &= \{3(k+2)(k+4)(d-e) + 2(k+3)^2(e+a+c)\}b. \end{aligned}$$

Further, we have, using $b^2 = (a-d)(c-e)$ (or $b \times \frac{b}{a-d} = c-e$) again,

$$n'_{11} := n_{11} + n_{21} \times \frac{b}{a-d} = 3(k+2)(k+4)b(d-e),$$
$$n'_{12} := n_{12} + n_{22} \times \frac{b}{a-d} = 3(k+2)(k+4)(d-e)(c-e).$$

Hence

$$\det\begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} = \det\begin{pmatrix} n'_{11} & n'_{12} \\ n_{21} & n_{22} \end{pmatrix} = 3(k+2)(k+4)b^2(d-e)^2(k^2+6k+6). \tag{4.15}$$

Consequently, we have, from (4.14) and (4.15),

$$\det N_k(0) = \left\{\frac{1}{3(k+2)(k+4)b(d-e)}\right\}^2 \det\begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} = \frac{k^2+6k+6}{3(k+2)(k+4)}.$$

This is the desired (4.9).

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