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SPECHT'S RATIO AND LOGARITHMIC MEAN IN THE YOUNG INEQUALITY

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Abstract. For a positive operator A with $0 < m \leqslant A \leqslant M$ $(m, M \in \mathbb{R})$, the Young operator inequality gives as follows: $\lambda A + (1-\lambda) \geqslant A^{\lambda}$ for $\lambda \in [0,1]$.

In this note, we prove that the estimation of the converse Young operator inequality is obtained by using Specht's ratio $S(t) = \frac{t^{\frac{1}{t-1}}}{e \log t^{\frac{1}{t-1}}}$ and the logarithmic mean $L(s,t) = \frac{t-s}{\log t - \log s}$ $(s, t > 0)$, that is, we have for a given p under some conditions

$$pA^{\lambda} + \max\left\{L(1,m)\log\frac{S(m)}{p}, L(1,M)\log\frac{S(M)}{p}\right\} \geqslant \lambda A + (1-\lambda)\ (\geqslant A^{\lambda}) \quad \text{for } \lambda \in [0,1].$$

Moreover by using operator means, we consider the converse Young operator inequality related to two operators A and B.

Furthermore we discuss reverse inequalities of the Hölder-McCarthy inequality and the inequality on the concavity of the logarithmic function.

1. Introduction

We cite the Young inequality which is considered as the λ-weighted arithmetic-geometric mean inequality as follows:

Let a and b be positive real numbers. Then the inequalities

$$\lambda a + (1-\lambda) \geqslant a^{\lambda} \tag{1.1}$$

and

$$\lambda a + (1-\lambda)b \geqslant a^{\lambda} b^{1-\lambda} \tag{1.2}$$

hold for every $\lambda \in [0,1]$.

In this note, an operator means a bounded linear operator acting on a complex Hilbert space H. The inequalities (1.1) and (1.2) are extended to an operator version. For it we use the following two means. Let A and B be positive invertible operators. For every $\lambda \in [0,1]$, we denote by ∇_{λ} the λ-weighted arithmetic mean as follows:

$$B \nabla_{\lambda} A := \lambda A + (1-\lambda)B,$$

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and by $\sharp_\lambda$ the λ-weighted geometric mean as follows:

$$B \sharp_\lambda A := B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^\lambda B^{\frac{1}{2}}.$$

The λ-weighted geometric mean is introduced by F. Kubo and T. Ando in [3]. The following Young operator inequality is regarded as an operator version of the λ-weighted arithmetic-geometric mean inequality:

THE YOUNG OPERATOR INEQUALITY. *Let A be a positive operator. Then the inequality*

$$\lambda A + (1-\lambda) \geqslant A^\lambda \tag{1.3}$$

holds for every $\lambda \in [0,1]$.

Furthermore, let A and B be positive invertible operators. Then the inequality

$$B \triangledown_\lambda A \geqslant B \sharp_\lambda A \tag{1.4}$$

holds for every $\lambda \in [0,1]$.

For the sake of convenience, we recall some constants as follows: Let m and M be real numbers with $0 < m < M$. Then the logarithmic mean $L(m,M)$ (cf. [2]) is defined by

$$L(m,M) = \frac{M-m}{\log M - \log m}.$$

Next the constant $S(h)$ defined by

$$S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}} \quad (h > 1)$$

is called Specht's ratio [1], [8], which is the best upper bound of the arithmetic mean by the geometric one for positive numbers: For $x_i \in [m,M]$ with $M > m > 0$ ($i = 1,2,\ldots,n$), the following inequality holds

$$S(h)\sqrt[n]{x_1x_2\ldots x_n} \geqslant \frac{x_1+x_2+\ldots+x_n}{n} \; (\geqslant \sqrt[n]{x_1\ldots x_n}),$$

where the constant $h = \frac{M}{m}$ is called a condition number in the sense of Turing [11].

In our previous note [10], we show converse ratio and difference inequalities of the Young operator inequality (1.4) independent of $\lambda \in [0,1]$ as follows: For positive invertible operators A and B with $0 < m \leqslant A, B \leqslant M$ and $h = \frac{M}{m} (> 1)$, the inequality

[Ratio inequality]	$S(h)(B \sharp_\lambda A) \geqslant B \triangledown_\lambda A \; (\geqslant B \sharp_\lambda A)$,
[Difference inequality]	$hL(m,M)\log S(h) \geqslant B \triangledown_\lambda A - B \sharp_\lambda A \; (\geqslant 0)$

hold for every $\lambda \in [0,1]$.

The purpose of this paper is to give complementary inequalities of the above converse ratio and difference inequalities, independent of a real number $\lambda \in [0,1]$: Let A and B be positive invertible operators with $0 < m \leqslant A, B \leqslant M$ and $h = \frac{M}{m} > 1$. For some real number $p > 0$, we show the following inequality:

$$p(B \sharp_\lambda A) + hL(m,M)\log\frac{S(h)}{p} \geqslant B \triangledown_\lambda A \; (\geqslant B \sharp_\lambda A).$$

To prove above complementary inequality, we show the following inequality which is the converse Young operator inequality of (1.3):

$$pA^{\lambda} + \max\left\{L(1,m)\log\frac{S(m)}{p}, L(1,M)\log\frac{S(M)}{p}\right\} \geqslant \lambda A + (1-\lambda)\ (\geqslant A^{\lambda}).$$

In the above two inequalities, we see that Specht's ratio and the logarithmic mean play the important role.

As applications, we consider the converse inequalities of well-known inequalities. Moreover, as the converse inequality of the Hölder-McCarthy inequality [4]:

$$\langle Ax, x\rangle^{\lambda} \geqslant \langle A^{\lambda}x, x\rangle \quad \text{for every } \lambda \in [0,1] \text{ and every unit vector } x \in H,$$

we show the following inequality:

$$p\langle A^{\lambda}x, x\rangle + \max\left\{L(1,m)\log\frac{S(m)}{p}, L(1,M)\log\frac{S(M)}{p}\right\} \geqslant \langle Ax, x\rangle^{\lambda}\ (\geqslant \langle A^{\lambda}x, x\rangle)$$

for every $\lambda \in [0,1]$ and every unit vector $x \in H$.

Furthermore the Young inequality (1.2) implies the concavity of the logarithmic function, that is, we have

$$\log(\lambda a + (1-\lambda)b) - \lambda\log a + (1-\lambda)\log b \geqslant 0.$$

We show the upper bound of the above inequality. Hence we give its operator version as follows:

$$2\log S(h) \geqslant \log(\lambda A + (1-\lambda)B) - (\lambda\log A + (1-\lambda)\log B) \geqslant 0.$$

2. Converse inequalities of the Young inequality

For the sake of convenience, we denote by I_a the following closed interval

$$I_a := \begin{cases} \left[L(1,\frac{1}{a}), L(1,a)\right] & \text{for } a \geqslant 1 \\ \left[L(1,a), L(1,\frac{1}{a})\right] & \text{for } 0 < a < 1\,. \end{cases} \tag{2.1}$$

We remark that a family of the interval I_a $(a > 0)$ has a monotone property in the sense that

$$I_a \subset I_b \quad \text{for } 1 \leqslant a < b \quad \text{or} \quad 0 < b < a < 1.$$

Moreover it is obvious that $1 \in I_a$ for $a > 0$ by the monotonicity of the logarithmic mean.

In this section, we give the converse inequalities of the Young inequalities (1.1), (1.2) and the Young operator inequalities (1.3), (1.4). For it we give the following lemmas. In our previous note [10], we obtain the following properties by considering Specht's ratio $S(t)$ as a function for $t > 0$:

LEMMA 2.1. *A function $S(t)$ is strictly decreasing for $0 < t < 1$ and strictly increasing for $t > 1$. Furthermore the following equations hold*

$$S(1) = 1 \quad \text{and} \quad S(t) = S\left(\frac{1}{t}\right) \quad \text{for all} \quad t > 0.$$

Moreover we show the following lemma related to the order of Specht's ratio and the logarithmic mean:

LEMMA 2.2. *The following inequality holds*

$$L\Big(1,\frac{1}{t}\Big) \leqslant 1 \leqslant S\Big(\frac{1}{t}\Big) = S(t) \leqslant L(1,t) \quad \textit{for} \quad t \geqslant 1$$

$$\Big(\textit{or}\ L(1,t) < 1 < S(t) = S\Big(\frac{1}{t}\Big) < L\Big(1,\frac{1}{t}\Big) \quad \textit{for} \quad 0 < t < 1\Big).$$

Proof. Let $t > 1$. From the property of the mean ($L(1,\frac{1}{t}) < 1 < L(1,t)$) and Lemma 2.1 ($S(t) = S(\frac{1}{t}) > 1$) we only prove an inequality $S(t) \leqslant L(1,t)$. Since it follows from the Klein inequality (i.e., $\log t \leqslant t-1$ for $t > 0$) that $\frac{\log t}{t-1} \leqslant 1$, we have

$$\log \frac{t^{\frac{1}{t-1}}}{e} \leqslant 0.$$

Taking an exponential and multiplying $L(1,t)$ in the both sides of the above inequality, we have

$$L(1,t)\frac{t^{\frac{1}{t-1}}}{e} \leqslant L(1,t)$$

and so the desired inequality is complete by $S(t) = \frac{t-1}{\log t} \cdot \frac{t^{\frac{1}{t-1}}}{e}$.

Let $0 < t < 1$. The desired inequality holds by replacing t with $\frac{1}{t}$ in the above case.

Let $t = 1$. Then we have the following equation (related to the property of the mean):

$$\lim_{t\to 1} L(1,t) = \lim_{t\to 1} \frac{t-1}{\log t} = \lim_{t\to 1} \frac{1}{\frac{1}{t}} = 1.$$

Hence we have the desired inequality from Lemma 2.1 ($S(1) = 1$). □

In the following theorem, we show converse inequalities of the Young inequalities (1.1) and (1.2):

THEOREM 2.3. *Let a be a positive number. Suppose that p be a positive number in I_a. Then the inequality*

$$pa^{\lambda} + L(1,a)\log\frac{S(a)}{p} \geqslant \lambda a + (1-\lambda)\ (\geqslant a^{\lambda}) \tag{2.2}$$

holds for every $\lambda \in [0,1]$.

Furthermore let a and b be positive numbers. Suppose that p be a positive number in $I_{\frac{a}{b}}$. Consequently, the inequality

$$pa^{\lambda}b^{1-\lambda} + L(a,b)\log\frac{S(\frac{a}{b})}{p} \geqslant \lambda a + (1-\lambda)b\ (\geqslant a^{\lambda}b^{1-\lambda}) \tag{2.3}$$

holds for every $\lambda \in [0,1]$.

Proof. Let $a \neq 1$. For every $p > 0$, we put a function $f_{p,a}(\lambda)$ derived from the Young inequality (1.1) as follows:

$$f_{p,a}(\lambda) := \lambda a + (1-\lambda) - pa^{\lambda} = (a-1)\lambda + 1 - pa^{\lambda}.$$

Then we want to determine the maximum of $f_{p,a}(\lambda)$. We have by a differential calculation

$$f'_{p,a}(\lambda) = (a-1) - pa^{\lambda}\log a$$

and so an equation $f'_{p,a}(\lambda) = 0$ has the following unique solution $\lambda = \lambda_{p,a}$:

$$\lambda_{p,a} = \frac{\log \frac{a-1}{p\log a}}{\log a} = \log_a \frac{a-1}{p\log a}.$$

The condition $p \in I_a$ for $a > 1$ is equivalent to the condition $\lambda_{p,a} \in [0,1]$. Indeed we have

$$p \in I_a \iff 1 \leqslant \frac{a-1}{p\log a} \leqslant a \iff \lambda_{p,a} \in [0,1]$$

by $L(1,\frac{1}{a}) = \frac{L(1,a)}{a}$. Similarly, the condition $p \in I_{\frac{1}{a}}$ for $0 < a < 1$ is equivalent to the condition $\lambda_{p,a} \in [0,1]$. It follows from $f''_{p,a}(\lambda) = -pa^{\lambda}(\log a)^2 < 0$ that $f_{p,a}(\lambda)$ is the strictly concave function. So a maximum of $f_{p,a}(\lambda)$ takes at $\lambda = \lambda_{p,a}$, and we have

$$\begin{aligned}\max_{0\leqslant\lambda\leqslant 1} f_{p,a}(\lambda) &= f_{p,a}(\lambda_{p,a})\\ &= \frac{a-1}{\log a}\log\frac{a-1}{p\log a} + 1 - \frac{a-1}{\log a} = \frac{a-1}{\log a}\left(\log\frac{a-1}{p\log a} + \frac{\log a}{a-1} - 1\right)\\ &= \frac{a-1}{\log a}\log\frac{a^{\frac{1}{a-1}}}{pe\log a^{\frac{1}{a-1}}} = L(1,a)\log\frac{S(a)}{p},\end{aligned}$$

and the desired inequality (2.2) is obtained.

Let $a = 1$. Then the inequality (2.2) is ensured by Lemma 2.1 ($S(1) = 1$), the property of the mean ($L(1,1) = 1$) and the Klein inequality ($\log p \leqslant p-1$ for $p > 0$).

The desired inequality (2.3) is obtained by replacing a with $\frac{a}{b}$ in (2.2). $\square$

From the above theorem, we see that the estimation of the converse Young inequality is obtained by using Specht's ratio and the logarithmic mean. Moreover we see that the estimation is the best upper bound from the proof of Theorem 2.3.

In the following remark we give the upper bound of $\lambda a + (1-\lambda)a - pa^{\lambda}$ in $p \notin I_a$ for $a > 0$:

REMARK 2.4. Let $a \geqslant 1$ and $p \notin I_a$ in Theorem 2.3. Then the value of $f'_{p,a}(\lambda)$ for $\lambda \in [0,1]$ is positive if $0 < p < L(1,\frac{1}{a})$ and negative if $L(1,a) < p$. So the maximum of $f_{p,a}(\lambda)$ takes at $\lambda = 1$ if $0 < p < L(1,\frac{1}{a})$ and $\lambda = 0$ if $L(1,a) < p$, i.e.,

$$\begin{aligned}\lambda a + (1-\lambda) - pa^{\lambda} &\leqslant \begin{cases}(1-p)a\ (= f(1)) & \text{if} \quad 0 < p < L(1,\frac{1}{a})\\ 1-p\ (= f(0)) & \text{if} \quad L(1,a) < p\end{cases}\\ &< L(1,a)\log\frac{S(a)}{p}.\end{aligned}$$

On the other hand, let $0 < a < 1$ and $p \notin I_a$ in Theorem 2.3. Then the maximum of $f_{p,a}(\lambda)$ is given as follows:

$$\lambda a + (1-\lambda) - pa^{\lambda} \leqslant \begin{cases} 1-p \ (=f(0)) & \text{if} \quad 0<p<L(1,a) \\ (1-p)a \ (=f(1)) & \text{if} \quad L(1,\frac{1}{a})<p \end{cases}$$
$$< L(1,a)\log\frac{S(a)}{p}.$$

As the special case of Theorem 2.3, we have the following converse ratio and difference inequalities of the Young inequalities (1.1) and (1.2) (cf. [10]):

COROLLARY 2.5. *Let a and b be positive numbers. Then the following converse ratio and difference inequalities hold for every $\lambda \in [0,1]$*
[*Ratio inequalities*]

$$S(a)a^{\lambda} \geqslant \lambda a + (1-\lambda)\ (\geqslant a^{\lambda}), \tag{2.4}$$

$$S\left(\frac{a}{b}\right)a^{\lambda}b^{1-\lambda} \geqslant \lambda a + (1-\lambda)b\ (\geqslant a^{\lambda}b^{1-\lambda}), \tag{2.5}$$

[*Difference inequalities*]

$$L(1,a)\log S(a) \geqslant \lambda a + (1-\lambda) - a^{\lambda}\ (\geqslant 0), \tag{2.6}$$

$$L(a,b)\log S\left(\frac{a}{b}\right) \geqslant \lambda a + (1-\lambda)b - a^{\lambda}b^{1-\lambda}\ (\geqslant 0). \tag{2.7}$$

Proof. Since we have $S(a), 1 \in I_a$ for $a > 0$ from Lemma 2.2, the inequality (2.2) suffices for the inequalities (2.4) and (2.6) by putting $p = S(a)$ and 1, respectively. Moreover the inequality (2.3) suffices for the inequalities (2.5) and (2.7) by putting $p = S(\frac{a}{b})$ and $1 \in I_{\frac{a}{b}}$, respectively. $\square$

In the following theorem, we show the converse inequality of the Young operator inequality (1.3), that is, the operator version of (2.2):

THEOREM 2.6. *Let A be a positive operator with $0 < m \leqslant A \leqslant M$ and $m < M$. Suppose that p be a positive number satisfying one of the following conditions*

$$\begin{array}{llll} \text{(i)} & p \in I_m & \textit{if} & (mM \geqslant M >)\ m \geqslant 1, \\ \text{(ii)} & p \in I_m & \textit{if} & (M >)\ mM \geqslant 1 > m > 0, \\ \text{(iii)} & p \in I_M & \textit{if} & M \geqslant 1 > mM\ (\geqslant m > 0), \\ \text{(iv)} & p \in I_M & \textit{if} & 1 > M\ (> m > mM > 0). \end{array}$$

Then the inequality

$$pA^{\lambda} + \max\left\{L(1,m)\log\frac{S(m)}{p}, L(1,M)\log\frac{S(M)}{p}\right\} \geqslant \lambda A + (1-\lambda)\ (\geqslant A^{\lambda}) \tag{2.8}$$

holds for every $\lambda \in [0,1]$.

Proof. Let t be a positive number $m \leqslant t \leqslant M$ and let $p > 0$. We put a function $f_p(t,\lambda)$ as follows:

$$f_p(t,\lambda) := \lambda t + (1-\lambda) - pt^{\lambda}, \tag{2.9}$$

where $\lambda \in [0,1]$. Then we have by the differential calculation of $f_{p,\lambda}(t)\ (=f_p(t,\lambda))$ for t

$$f'_{p,\lambda}(t) = \lambda - p\lambda t^{\lambda-1} = \lambda(1-pt^{\lambda-1})$$

and

$$f''_{p,\lambda}(t) = -p\lambda(\lambda-1)t^{\lambda-2}\ (\geqslant 0).$$

It implies that $f_{p,\lambda}(t)$ is a convex function, and hence the maximum of $f_{p,\lambda}(t)$ for $t \in [m,M]$ takes at the extreme point $t = m$ or M. So we have for every $\lambda \in [0,1]$

$$\max_{m \leqslant t \leqslant M} f_{p,\lambda}(t) = \max\left\{f_{p,\lambda}(m), f_{p,\lambda}(M)\right\} \left(= \max\left\{f_p(m,\lambda), f_p(M,\lambda)\right\}\right).$$

Furthermore, let $f_{p,t}(\lambda) = f_p(t,\lambda)$ in (2.9). By the same method as in the proof of Theorem 2.3, we want to determine the maximums of $f_{p,m}(\lambda)$ and $f_{p,M}(\lambda)$ for $\lambda \in [0,1]$. We sketch it.

Let p hold the condition (i) (i.e., $p \in I_m$ for $m \geqslant 1$). We calculate the maximum of $f_{p,m}(\lambda)$ for $\lambda \in [0,1]$. An equation $f'_{p,m}(\lambda) = 0$ has a unique solution $\lambda = \lambda_{p,m}$. The condition $p \in I_m$ is equivalent to the condition $\lambda_{p,m} \in [0,1]$. By the strict concavity of $f_{p,m}(\lambda)$, a maximum of $f_{p,m}(\lambda)$ takes at $\lambda = \lambda_{p,m}$, and so we have

$$\max_{0 \leqslant \lambda \leqslant 1} f_{p,m}(\lambda) = f_{p,m}(\lambda_{p,m}) = L(1,m)\log\frac{S(m)}{p}.$$

Next we calculate the maximum of $f_{p,M}(\lambda)$. Since we have $p \in I_m \subset I_M$ by $M > m \geqslant 1$ and Lemma 2.2, a unique solution $\lambda = \lambda_{p,M}$ of the equation $f'_{p,M}(\lambda) = 0$ for λ is included in $[0,1]$. By the strict concavity of $f_{p,M}(\lambda)$ we have

$$\max_{0 \leqslant \lambda \leqslant 1} f_{p,M}(\lambda) = f_{p,M}(\lambda_{p,M}) = L(1,M)\log\frac{S(M)}{p}.$$

Therefore we have for every $t \in [m,M]$ and every $\lambda \in [0,1]$

$$f_p(t,\lambda) \leqslant \max\{f_p(m,\lambda), f_p(M,\lambda)\} \leqslant \max\left\{L(1,m)\log\frac{S(m)}{p}, L(1,M)\log\frac{S(M)}{p}\right\}.$$

By the same method as in the above case (i), we have in the case (ii) – (iv)

$$pt^{\lambda} + \max\left\{L(1,m)\log\frac{S(m)}{p}, L(1,M)\log\frac{S(M)}{p}\right\} \geqslant \lambda t + (1-\lambda),$$

and so by functional calculus we have the desired inequality (2.8). □

As the special case of Theorem 2.6, we have the following converse ratio and difference inequalities of the Young operator inequality (1.3):

COROLLARY 2.7. *Let A be a positive operator with $0 < m \leqslant A \leqslant M$ and $m < M$. Then the following ratio and difference inequalities hold for every $\lambda \in [0,1]$:*

[*Ratio inequality*]

$$\max\{S(m),S(M)\}A^{\lambda} \geqslant \lambda A+(1-\lambda)\ (\geqslant A^{\lambda}), \tag{2.10}$$

[*Difference inequality*]

$$\max\{L(1,m)\log S(m), L(1,M)\log S(M)\} \geqslant \lambda A+(1-\lambda)-A^{\lambda}\ (\geqslant 0). \tag{2.11}$$

Proof. By Theorem 2.6 and moreover Remark 2.4 and the proof of Theorem 2.6, we have for every $p>0$

$$pA^{\lambda}+\max\left\{L(1,m)\log\frac{S(m)}{p}, L(1,M)\log\frac{S(M)}{p}\right\} \geqslant \lambda A+(1-\lambda)\ (\geqslant A^{\lambda}).$$

Since equations $L(1,m)\log\frac{S(m)}{p}=0$ and $L(1,M)\log\frac{S(M)}{p}=0$ have unique solutions $p=S(m)$ and $S(M)$, respectively. So we have the desired inequality (2.10).

On the other hand, by putting $p=1$ (which is included in every interval defined by (2.1)) the inequality (2.8) suffices for the inequality (2.11). □

We remark that the value $\max\{S(m),S(M)\}$ in (2.10) is determined as

$$\max\{S(m),S(M)\} = \begin{cases} S(M) & \text{if } mM\geqslant 1 \\ S(m) & \text{if } 1>mM>0 \end{cases}$$

by Lemma 2.1.

In the following theorem, we show a converse inequality of the Young operator inequality (1.4), that is, it is the operator version of (2.3):

THEOREM 2.8. *Let A and B be positive invertible operators with $0<m\leqslant A,B\leqslant M$ and $h=\frac{M}{m}>1$. Suppose that p be a positive number in I_h. Then the inequality*

$$p(B\,\sharp_{\lambda}\,A)+hL(m,M)\log\frac{S(h)}{p} \geqslant B\,\triangledown_{\lambda}\,A\ (\geqslant B\,\sharp_{\lambda}\,A) \tag{2.12}$$

holds for every $\lambda\in[0,1]$.

Proof. In Theorem 2.6, we replace A with $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$. Then we have $\frac{m}{M}\leqslant B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\leqslant\frac{M}{m}$, i.e., $\frac{1}{h}\leqslant B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\leqslant h$. Moreover $p\in I_h$ is corresponding to a condition (ii) in Theorem 2.6 $\left(\text{i.e., } p\in I_{\frac{1}{\frac{1}{h}}} \text{ and } h\cdot\frac{1}{h}=1>\frac{1}{h}>0\right)$. Hence we have for every $\lambda\in[0,1]$

$$p(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\lambda}+L(1,h)\log\frac{S(h)}{p} \geqslant \lambda B^{-\frac{1}{2}}AB^{-\frac{1}{2}}+(1-\lambda)$$

by Lemma 2.1 ($S(t)=S(\frac{1}{t})$ for $t>0$) and the property of the mean ($L(1,s)>L(1,t)$ for $s>t>0$), and hence we have

$$pB^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\lambda}B^{\frac{1}{2}}+L(1,h)\left(\log\frac{S(h)}{p}\right)B \geqslant \lambda A+(1-\lambda)B.$$

So we have the desired inequality (2.12) by $M \geqslant B \geqslant m > 0$. $\square$

As the special case of Theorem 2.8, we have the following converse ratio and difference inequalities [10] of the Young operator inequality (1.4):

COROLLARY 2.9. *Let A and B be positive invertible operators with $0 < m \leqslant A, B \leqslant M$ and $h = \frac{M}{m} > 1$. Then the following ratio and difference inequalities hold for every $\lambda \in [0,1]$*
[*Ratio inequality*]

$$S(h)(B \sharp_\lambda A) \geqslant B \bigtriangledown_\lambda A \ (\geqslant B \sharp_\lambda A), \tag{2.13}$$

[*Difference inequality*]

$$hL(m,M)\log S(h) \geqslant B \bigtriangledown_\lambda A - B \sharp_\lambda A \ (\geqslant 0). \tag{2.14}$$

Proof. Since we have $S(h), 1 \in I_h$ from Lemma 2.2, the inequality (2.12) suffices for the inequalities (2.13) and (2.14) by putting $p = S(h)$ and 1, respectively. $\square$

From Theorems 2.6 and 2.8, we see that the logarithmic mean and Specht's ratio play an important role in a converse inequalities of the Young operator inequalities (1.3) and (1.4).

3. Applications of the converse Young inequalities

In this section, as applications of the converse Young operator inequalities, we consider converse inequalities of the well-known Hölder-McCarthy inequality and inequalities related to concavity of the logarithmic function.

We cite the well-known Hölder-McCarthy inequality [4]:

THE HÖLDER-MCCARTHY INEQUALITY. *Let A be a positive operator. Then the inequality*

$$\langle Ax, x\rangle^{\lambda} \geqslant \langle A^{\lambda}x, x\rangle \tag{3.1}$$

holds for every $\lambda \in [0,1]$ and every unit vector $x \in H$.

The above Hölder-McCarthy inequality is extended by using the geometric mean as follows:

THE EXTENDED HÖLDER-MCCARTHY INEQUALITY. *Let A and B be positive invertible operators. Then the inequality*

$$\langle Bx, x\rangle \sharp_\lambda \langle Ax, x\rangle \geqslant \langle B \sharp_\lambda Ax, x\rangle \tag{3.2}$$

holds for every $\lambda \in [0,1]$ and every unit vector $x \in H$.

Without depending on $\lambda \in [0,1]$, we show converse inequalities of the Hölder-McCarthy inequality (3.1) by Theorem 2.6 as follows:

THEOREM 3.1. *Let A be a positive operator with $0 < m \leqslant A \leqslant M$ and $m < M$. Suppose that p(, m and M) be a positive number satisfying one of the conditions* (i) – (iv) *in Theorem 2.6. Then the inequality*

$$p\langle A^{\lambda}x, x\rangle + \max\left\{L(1,m)\log\frac{S(m)}{p}, L(1,M)\log\frac{S(M)}{p}\right\} \geqslant \langle Ax, x\rangle^{\lambda} \ (\geqslant \langle A^{\lambda}x, x\rangle) \tag{3.3}$$

holds for every $\lambda \in [0,1]$ *and every unit vector* $x \in H$.

Furthermore the following ratio and difference inequalities hold for every $\lambda \in [0,1]$ *and every unit vector* $x \in H$:

[*Ratio inequality*]

$$\max\{S(m), S(M)\}\langle A^{\lambda}x, x\rangle \geqslant \langle Ax, x\rangle^{\lambda} \ (\geqslant \langle A^{\lambda}x, x\rangle), \tag{3.4}$$

[*Difference inequality*]

$$\max\{L(1,m)\log S(m), L(1,M)\log S(M)\} \geqslant \langle Ax, x\rangle^{\lambda} - \langle A^{\lambda}x, x\rangle \ (\geqslant 0). \tag{3.5}$$

Proof. By Theorem 2.6 and the Young inequality (1.1), we have for a given $p > 0$

$$\begin{aligned} p\langle A^{\lambda}x, x\rangle + \max\left\{L(1,m)\log\frac{S(m)}{p}, L(1,M)\log\frac{S(M)}{p}\right\} \\ \geqslant \lambda\langle Ax, x\rangle + (1-\lambda) \geqslant \langle Ax, x\rangle^{\lambda}, \end{aligned}$$

which is just the desired inequality (3.3). By the same method as in the above, (3.4) and (3.5) are easily checked from Corollary 2.7. □

By the same method as in Theorem 3.1, we have the following converse inequalities of the extended Hölder-McCarthy inequality (3.2) by Theorem 2.8, independent of $\lambda \in [0,1]$:

THEOREM 3.2. *Let* A *and* B *be positive invertible operators with* $0 < m \leqslant A, B \leqslant M$ *and* $h = \frac{M}{m} > 1$. *Suppose that* p *be a positive number in* I_h. *Then the inequality*

$$p\langle B \sharp_{\lambda} Ax, x\rangle + hL(m,M)\log\frac{S(h)}{p} \geqslant \langle Bx, x\rangle \sharp_{\lambda} \langle Ax, x\rangle \ (\geqslant \langle B \sharp_{\lambda} Ax, x\rangle) \tag{3.6}$$

holds for every $\lambda \in [0,1]$ *and every unit vector* $x \in H$.

Furthermore the following ratio and difference inequalities hold for every $\lambda \in [0,1]$ *and every unit vector* $x \in H$:

[*Ratio inequality*]

$$S(h)\langle B \sharp_{\lambda} Ax, x\rangle \geqslant \langle Bx, x\rangle \sharp_{\lambda} \langle Ax, x\rangle \ (\geqslant \langle B \sharp_{\lambda} Ax, x\rangle), \tag{3.7}$$

[*Difference inequality*]

$$hL(m,M)\log S(h) \geqslant \langle Bx, x\rangle \sharp_{\lambda} \langle Ax, x\rangle - \langle B \sharp_{\lambda} Ax, x\rangle \ (\geqslant 0). \tag{3.8}$$

Proof. By Theorem 2.8 and the Young inequality (1.2), we have for a given $p \in I_h$

$$\begin{aligned} p\langle B \sharp_{\lambda} Ax, x\rangle + hL(m,M)\log\frac{S(h)}{p} \geqslant \langle B\nabla_{\lambda}Ax, x\rangle = \lambda\langle Ax, x\rangle + (1-\lambda)\langle Bx, x\rangle \\ \geqslant \langle Bx, x\rangle \sharp_{\lambda} \langle Ax, x\rangle. \end{aligned}$$

So we have the desired inequality (3.6).

By the same method as in the above, (3.7) and (3.8) are easily checked from Corollary 2.9. □

From Theorems 3.1 and 3.2, we see that the estimations of the converse (extended) Hölder-McCarthy inequalities independent of $\lambda \in [0,1]$ are represented by using Specht's ratio and the logarithmic mean.

Next we consider the estimation of the following inequality which represents the concavity of the logarithmic function: For every positive numbers a and b

$$\lambda \log a + (1-\lambda)\log b \leqslant \log(\lambda a + (1-\lambda)b) \tag{3.9}$$

holds for $\lambda \in [0,1]$.

In the following theorem, we estimate the upper bound in (3.9), in which Specht's ratio appears:

THEOREM 3.3. *Let a and b be positive numbers. Then the inequality*

$$0 \leqslant \log(\lambda a + (1-\lambda)b) - (\lambda \log a + (1-\lambda)\log b) \leqslant \log S\left(\frac{a}{b}\right) \tag{3.10}$$

holds for every $\lambda \in [0,1]$.

Consequently, the inequality

$$0 \leqslant \log(\lambda a + (1-\lambda)) - \lambda \log a \leqslant \log S(a) \tag{3.11}$$

holds for every $\lambda \in [0,1]$.

Proof. Taking logarithmic of both sides of 2.5, the desired inequality (3.10) is obtained. The desired inequality (3.11) is obtained by putting $b = 1$ in (3.10). □

The logarithmic function is well-known as one of the operator concave functions, that is, for positive invertible operators A and B the inequality

$$\lambda \log A + (1-\lambda)\log B \leqslant \log(\lambda A + (1-\lambda)B) \tag{3.12}$$

holds for every $\lambda \in [0,1]$.

We consider the operator version of (3.10), i.e., the converse inequality of (3.12). For it, as the application of Theorem 2.8, we have the following lemma by putting $\langle Ax,x\rangle$ and $\langle Bx,x\rangle$ instead of A and B for every unit vector $x \in H$, respectively:

LEMMA 3.4. *Let A and B be positive operators with $0 < m \leqslant A, B \leqslant M$ and $h = \frac{M}{m} > 1$. Suppose that p be a positive number in I_h. Then the inequality*

$$\begin{aligned} p\langle Ax,x\rangle^{\lambda}\langle Bx,x\rangle^{1-\lambda} + hL(m,M)\log\frac{S(h)}{p} \\ \geqslant \lambda\langle Ax,x\rangle + (1-\lambda)\langle Bx,x\rangle \ (\geqslant \langle Ax,x\rangle^{\lambda}\langle Bx,x\rangle^{1-\lambda}) \end{aligned} \tag{3.13}$$

holds for every $\lambda \in [0,1]$ and every unit vector $x \in H$.

Furthermore the following ratio and difference inequalities hold for every $\lambda \in [0,1]$ and every unit vector $x \in H$:

[*Ratio inequality*]

$$\begin{aligned} S(h)\langle Ax,x\rangle^{\lambda}\langle Bx,x\rangle^{1-\lambda} \\ \geqslant \lambda\langle Ax,x\rangle + (1-\lambda)\langle Bx,x\rangle \ (\geqslant \langle Ax,x\rangle^{\lambda}\langle Bx,x\rangle^{1-\lambda}), \end{aligned} \tag{3.14}$$

[*Difference inequality*]

$$hL(m,M)\log S(h) \geqslant \lambda\langle Ax,x\rangle + (1-\lambda)\langle Bx,x\rangle - \langle Ax,x\rangle^{\lambda}\langle Bx,x\rangle^{1-\lambda} \ (\geqslant 0). \qquad (3.15)$$

Proof. In (2.12), we replace A and B with $\langle Ax,x\rangle$ and $\langle Bx,x\rangle$, respectively. Then we have $\frac{m}{M} \leqslant \frac{\langle Ax,x\rangle}{\langle Bx,x\rangle} \leqslant \frac{M}{m}$, i.e., $\frac{1}{h} \leqslant \frac{\langle Ax,x\rangle}{\langle Bx,x\rangle} \leqslant h$. Hence for $p \in I_h$ the inequality

$$p\langle Ax,x\rangle^{\lambda}\langle Bx,x\rangle^{1-\lambda} + hL(m,M)\log\frac{S(h)}{p} \geqslant \lambda\langle Ax,x\rangle + (1-\lambda)\langle Bx,x\rangle$$

holds for every $\lambda \in [0,1]$ and every unit vector $x \in H$. By the same method as in the above, the inequalities (3.14) and (3.15) are checked from Corollary (2.9). □

The following lemma gives the upper bound of $(0 \leqslant)\ \log\langle Ax,x\rangle - \langle(\log A)x,x\rangle$ which is a converse difference inequality of Jensen's inequality related to a logarithmic function (cf. [5], [6], [7], [9]):

LEMMA 3.5. *Let A be a positive invertible operator on H with $0 < m \leqslant A \leqslant M$ and $h = \frac{m}{M} > 1$. Then the inequality*

$$(0 \leqslant) \log\langle Ax,x\rangle - \langle(\log A)x,x\rangle \leqslant \log S(h) \qquad (3.16)$$

holds for every unit vector $x \in H$.

Proof. The proof of this lemma is given in [9]. But we give the proof to complete this lemma directly. Since the inequality $\log t \geqslant \alpha_{\log}t + \beta_{\log}$ holds for $m \leqslant t \leqslant M$, we have by functional calculus

$$\langle(\log A)x,x\rangle \geqslant \alpha_{\log}\langle Ax,x\rangle + \beta_{\log},$$

where $\alpha_{\log} = \frac{\log M - \log m}{M-m}$ and $\beta_{\log} = \frac{M\log m - m\log M}{M-m}$. So it is sufficient to see the maximum value of $\log\langle Ax,x\rangle - (\alpha_{\log}\langle Ax,x\rangle + \beta_{\log})$, i.e., the function $f(t) := \log t - (\alpha_{\log}t + \beta_{\log})$ on $t \in [m,M]$. We have $f'(t) = \frac{1}{t} - \alpha_{\log} = 0$ if and only if $t = t_{\log} = \frac{1}{\alpha_{\log}} = L(m,M)$ $(\in [m,M])$. Moreover we easily see $f'(t) > 0$ for $0 < t < t_{\log}$ and $f'(t) < 0$ for $t > t_{\log}$. Hence the maximum value of $f(t)$ on $[m,M]$ is attained for $t = t_{\log}$, and we have as the explicit expression

$$\begin{aligned}\max_{m\leqslant t\leqslant M} f(t) &= f(t_{\log}) = \log\frac{1}{\alpha_{\log}} - (1+\beta_{\log}) \\ &= \log\frac{M-m}{\log M - \log m} - 1 + \frac{m\log M - M\log m}{M-m} \\ &= \log\frac{m}{e\log h^{\frac{1}{h-1}}} + \log\frac{M^{\frac{m}{M-m}}}{m^{\frac{M}{M-m}}} \\ &= \log\frac{m}{e\log h^{\frac{1}{h-1}}} + \log\frac{M^{\frac{m}{M-m}}}{m^{\frac{m}{M-m}}\cdot m} = \log\frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}},\end{aligned}$$

which shows the desired inequality (3.16). □

In the following theorem, we show the converse inequality of (3.12) independent of $\lambda \in [0,1]$ as the operator version of Theorem 3.3:

THEOREM 3.6. *Let A and B be positive invertible operators with $0 < m \leqslant A, B \leqslant M$ and $h = \frac{m}{M} > 1$. Then the inequality*

$$0 \leqslant \log(\lambda A + (1-\lambda)B) - (\lambda \log A + (1-\lambda)\log B) \leqslant 2\log S(h) \tag{3.17}$$

holds for every $\lambda \in [0,1]$.

Proof. Since a logarithmic function is operator monotone, we have the first inequality of (3.17). On the other hand we have

$$\begin{aligned}\langle \log(\lambda A + (1-\lambda)B)x, x\rangle &\leqslant \log\langle (\lambda A + (1-\lambda)B)x, x\rangle \\ &\leqslant \lambda \log\langle Ax, x\rangle + (1-\lambda)\log\langle Bx, x\rangle + \log S(h) \\ &\qquad (\text{by}(3.14)) \\ &\leqslant \lambda\langle (\log A)x, x\rangle + (1-\lambda)\langle (\log B)x, x\rangle + 2\log S(h) \\ &\qquad (\text{by } (3.16)).\end{aligned}$$

Hence the proof of Theorem 3.6 is complete. □

As the operator version of (3.11), we hold the following theorem:

THEOREM 3.7. *Let A be a positive invertible operator with $0 < m \leqslant A \leqslant M$ and $h = \frac{M}{m} > 1$. Then the following inequality holds*

$$0 \leqslant \log(\lambda A + (1-\lambda)) - \lambda \log A \leqslant \log S(h). \tag{3.18}$$

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