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A GENERAL FRAMEWORK FOR THE SOLVABILITY OF A CLASS OF NONLINEAR VARIATIONAL INEQUALITIES

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Abstract. Based on a general framework for the auxiliary problem principle involving continuously m-Frechet-differentiable $(m \geqslant 2)$ mappings, the approximation-solvability of the following class of nonlinear variational inequality problems (NVIP) involving the generalized partially relaxed monotone mappings is presented.

Find an element $x^* \in K$ such that

$$\langle T(x^*), \eta(x, x^*)\rangle + f(x) - f(x^*) \geqslant 0 \quad \text{for all} \quad x \in K,$$

where $T : K \to \mathbf{R}^n$ is a mapping from a nonempty closed invex subset K of $\mathbf{R}^n$ into $\mathbf{R}^n$, $\eta : K \times K \to \mathbf{R}^n$ is a mapping, and $f : K \to \mathbf{R}$ is a continuous invex function on K. The general class of the auxiliary problems principle is described as follows: for a given iterate $x^k \in K$ and for a parameter $\rho > 0$, determine x^{k+1} such that

$$\langle \rho T(x^k) + h'(x^{k+1}) - h'(x^k), \eta(x, x^{k+1})\rangle + \rho[f(x) - f(x^{k+1})] \geqslant 0 \quad \text{for all} \quad x \in K,$$

where $h : K \to R$ is continuously Frechet-differentiable on K.

1. Introduction

Verma [5, 8, 9] introduced the general class of partially relaxed monotone mappings and applied them to the approximation-solvability of several classes of nonlinear variational inequalities in different space settings. In the context of numerical computations, it has been an open question: how can one come up with some adaptive linesearch rule that would work under the partial relaxed monotonicity condition? The main obstacle is the way this condition evolves in the analysis, that is, it always involves a unknown solution point. As a result it differs from how, for example, strong monotonicity and Lipschitz continuity conditions, are usually applied to other projection methods. Recently Argyros and Verma [1] developed a general framework for the auxiliary problem principle in the context of the solvability of a general class of nonlinear variational inequalities. Here our aim is to apply this general framework for the auxiliary problem principle to the approximation-solvability of a general class of nonlinear variational inequalities involving a class of generalized partially relaxed monotone mappings. The results thus obtained complement the earlier works of Cohen [3], Verma [5, 8, 9] on

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the approximation-solvability of nonlinear variational inequalities in different space settings.

Let $T : K \to \mathbf{R}^n$ be any mapping from K, a nonempty closed invex subset of $\mathbf{R}^n$, into $\mathbf{R}^n$. Let $f : K \to \mathbf{R}$ be a continuous invex function on K. We consider a class of nonlinear variational inequality problems (abbreviated as NVIP) involving generalized partially relaxed monotone mappings as follows: find an element $x^* \in K$ such that

$$\langle T(x^*), \eta(x,x^*)\rangle + f(x) - f(x^*) \geqslant 0 \quad \text{for all} \quad x \in K. \tag{1.1}$$

For $\eta(x,x^*) = x - x^*$, we have:
Find an element $x^* \in K$ such that

$$\langle T(x^*), x - x^*\rangle + f(x) - f(x^*) \geqslant 0 \quad \text{for all} \quad x \in K. \tag{1.2}$$

Let $||x||_B$ denote the norm induced by the positive definite matrix B, defined by

$$||x||_B = \langle Bx, x\rangle^{\frac{1}{2}}.$$

And let $||x||_2$ denote the standard Euclidean norm on $\mathbf{R}^n$ with respect to the dot product $\langle \cdot, \cdot \rangle$.

A subset K of a linear space X is said to be invex if there exists a function $\eta : K \times K \to X$ such that whenever $x, y \in K$ and $t \in [0,1]$, it follows that $tx + (1-t)\eta(y,x) \in K$.

A function $f : K \to \mathbf{R}$ is called *invex* if whenever $x, y \in K$ and $t \in [0,1]$, we have

$$f[x + t\eta(y,x)] \leqslant (1-t)f(x) + tf(y).$$

A mapping $T : K \to \mathbf{R}^n$ is said to be η-γ-μ *generalized partially relaxed monotone* (GPRM) if for all $x, y, z \in K$, we have

$$\langle T(x) - T(y), \eta(z,y)\rangle \geqslant (-\gamma)||z - x||^2 + \mu||x - y||^2,$$

where $\gamma, \mu > 0$ are constants. Clearly, it implies that

$$\langle T(x) - T(y), \eta(z,y)\rangle \geqslant (-\gamma)||z - x||^2,$$

that is, we have the following implication:

the η-γ-μ-generalized partially relaxed monotonicity
$\Downarrow$
the η-γ-generalized relaxed monotonicity

The generalized partial relaxed monotonicity is more general than the other notions of strong monotonicity and cocoercivity. For more details on partial relaxed monotonicity and cocoercivity, we recommend [2, 5, 9, 10].

2. General auxiliary problem principle

This section deals with the approximation-solvability of the NVIP (1.1) based on a general framework for the existing auxiliary problem principle (APP) introduced by Cohen [3] and later generalized by Verma [5]. This general framework for the auxiliary problem principle (Gapp) is stated as follows:

GAPP 2.1. *For a given iterate x^k, determine an x^{k+1} such that (for $k \geqslant 0$)*

$$\langle \rho T(x^k) + h'(x^{k+1}) - h'(x^k), \eta(x, x^{k+1}) \rangle + \rho[f(x) - f(x^{k+1})] \geqslant 0 \quad \text{for all } x \in K, \quad (2.1)$$

where K is an invex subset of $\mathbf{R}^n$, $h : \mathbf{R}^n \to \mathbf{R}$ is continuously Frechet-differentiable, and $\rho > 0$, a parameter.

For $\eta(x, y) = x - y$ in Gapp 2.1, we arrive at:

GAPP 2.2. *For a given iterate x^k, compute an x^{k+1} such that*

$$\langle \rho T(x^k) + h'(x^{k+1}) - h'(x^k), x - x^{k+1} \rangle + \rho[f(x) - f(x^{k+1})] \geqslant 0 \quad \text{for all} \quad x \in K.$$

Next, we recall some auxiliary results crucial to the approximation-solvability of the NVIP (1.1).

Let $h : E_2 \to \mathbf{R}$ be a continuously Fre'chet-differentiable mapping. It follows that $h'(x) \in L(E_2, \mathbf{R})$ – the space of bounded linear operators from E_2 into $\mathbf{R}$. From now on, we denote the real number $h'(x)(y)$ by $\langle h'(x), y \rangle$ for $x, y \in E_2$.

LEMMA 2.1 [1]. *Let K be a non-empty invex subset of $\mathbf{R}^n$. Suppose that the following assumptions hold:*

(i) There exist an $x^* \in K$ and numbers $\alpha \geqslant 0$ and $r \geqslant 0$ such that for all $x \in K, t \in [0, 1]$, we have

$$\langle h'(x^* + t\eta(x, x^*)) - h'(x^*), \eta(x, x^*) \rangle \geqslant t\alpha||\eta(x, x^*)||^2,$$

where $h : K \to \mathbf{R}$ is a continuously Frechet-differentiable mapping, and $\eta : K \times K \to \mathbf{R}$ satisfies:

$$||\eta(x, x^*)|| \leqslant r, \quad \text{and} \quad ||\eta(x, x^*)|| \geqslant ||x - x^*||.$$

(ii) The set S_0 defined by

$$S_0 = \{(h, \eta) : h'(x^* + t(x - x^*))(x - x^*) \geqslant \langle h'(x^* + t\eta(x, x^*)), \eta(x, x^*) \rangle \}$$

is nonempty.

(iii) The set

$$K_0 = \overline{\cup}(x^*, r) = \{x \in K : ||x - x^*|| \leqslant r\}.$$

Then, for all $x \in K_0$ and $(h, \eta) \in S_0$, the following estimate holds

$$h(x) - h(x^*) - \langle h'(x^*), \eta(x, x^*) \rangle \geqslant \frac{\alpha}{2}||x - x^*||^2.$$

LEMMA 2.2 [1]. *Let E_1 and E_2 be two Banach spaces and K be a nonempty invex subset of E_1. Suppose that the following assumptions hold:*

(i) There exist an $x^* \in K$ and numbers $\delta \geqslant 0$, $q \geqslant 0$ such that for all $x \in K_1$ and $t \in [0,1]$, we have

$$\langle h'(x^* + t\eta(x,x^*)) - h'(x^*), \eta(x,x^*)\rangle \leqslant t\delta||\eta(x,x^*)||^2,$$

where $h : K \to \mathbf{R}$ is a continuously Frechet-differentiable mapping, and $\eta : K \times K \to E_2$ satisfies:

$$||\eta(x,x^*)|| \leqslant ||x - x^*|| \leqslant q.$$

(ii) The set S_1 defined by

$$S_1 = \{(h,\eta) : h'(x^* + t(x - x^*))(x - x^*) \leqslant \langle h'(x^* + t\eta(x,x^*)), \eta(x,x^*)\rangle\}$$

is nonempty.

(iii) The set

$$K_1 = \overline{\text{U}}(x^*, q) \subset K.$$

Then, for all $x \in K_1$ and $(h,\eta) \in S_1$, the following estimate holds

$$h(x) - h(x^*) - \langle h'(x^*), \eta(x,x^*)\rangle \leqslant \frac{\delta}{2}||x - x^*||^2.$$

We are just about ready to present, based on the Gapp 2.1, the approximation-solvability of the NVIP (1.1).

THEOREM 2.1. *Let $T : K \to \mathbf{R}^n$ be η-γ-generalized partially relaxed monotone from a nonempty closed invex subset K of $\mathbf{R}^n$ into $\mathbf{R}^n$. Let $f : K \to \mathbf{R}$ be proper, invex and lower semicontinuous on K and $h : K \to \mathbf{R}$ be continuously Fre'chet-differentiable on K. Suppose that there exist an $x^* \in K$ and non-negative numbers α, δ, r and q such that for all $t \in [0,1]$ and $x \in K_0 \cap K_1$, we have*

$$\langle h'(x^* + t\eta(x,x^*)) - h'(x^*), \eta(x,x^*)\rangle \geqslant t\alpha||x - x^*||^2, \tag{2.2}$$

and

$$\langle h'(x^* + t\eta(x,x^*)) - h'(x^*), \eta(x,x^*)\rangle \leqslant t\delta||x - x^*||^2, \tag{2.3}$$

where

$$K_0 = \{x \in K : ||x - x^*|| \leqslant r\}, \qquad K_1 = \{x \in K : ||x - x^*|| \leqslant q\},$$

and $\eta : K \times K \to \mathbf{R}^n$ satisfies the following assumptions:

(i) $\eta(x,y) + \eta(y,x) = 0$ and $||\eta(x,y)|| \leqslant ||x - y||$.

(ii) For each fixed $y \in K$, map $x \to \eta(y,x)$ is sequentially continuous from the weak topology to the weak topology in the second variable.

(iii) η is s-expanding.

(iv) The set S defined by

$$S = \{(h,\eta) : h'(x^* + t(x - x^*))(x - x^*) \geqslant \langle h'(x^* + t\eta(x,x^*)), \eta(x,x^*)\rangle\}$$

is nonempty.

If in addition, $x^* \in K$ is any fixed solution of the NVIP (1.1) and

$$0 < \rho < \frac{\alpha}{2\gamma},$$

then the sequence $\{x^k\}$, generated by Gapp 2.1, converges strongly to x^*.

Proof. To show the sequence $\{x^k\}$ converges to x^*, a solution of the NVIP (1.1), we first define a function Λ^* by

$$\Lambda^*(x) = h(x^*) - h(x) - \langle h'(x), \eta(x^*, x)\rangle .$$

Then, by Lemma 2.1, we have

$$\Lambda^*(x) = h(x^*) - h(x) - \langle h'(x), \eta(x^*, x)\rangle \geqslant \frac{\alpha}{2}||x^* - x||^2 \quad \text{for} \quad x \in K, \tag{2.4}$$

where x^* is any fixed solution of the NVIP (1.1). It follows that

$$\Lambda^*(x^{k+1}) = h(x^*) - h(x^{k+1}) - \langle h'(x^{k+1}), \eta(x^*, x^{k+1})\rangle . \tag{2.5}$$

Now we can express

$$\begin{aligned}
&\Lambda^*(x^k) - \Lambda^*(x^{k+1}) \\
&\quad = h(x^{k+1}) - h(x^k) - \langle h'(x^k), \eta(x^{k+1}, x^k)\rangle + \langle h'(x^{k+1}) - h'(x^k), \eta(x^*, x^{k+1})\rangle \\
&\quad \geqslant \frac{\alpha}{2}||x^{k+1} - x^k||^2 + \langle h'(x^{k+1}) - h'(x^k), \eta(x^*, x^{k+1})\rangle \\
&\quad \geqslant \frac{\alpha}{2}||x^{k+1} - x^k||^2 + \rho\langle T(x^k), \eta(x^{k+1}, x^*)\rangle + \rho[f(x^{k+1}) - f(x^*)] \\
&\quad \text{for} \quad x = x^* \quad \text{in} \quad (2.1).
\end{aligned} \tag{2.6}$$

If we replace x by x^{k+1} in (1.1) and combine with (2.6), we obtain

$$\begin{aligned}
&\Lambda^*(x^k) - \Lambda^*(x^{k+1}) \\
&\quad \geqslant \frac{\alpha}{2}||x^{k+1} - x^k||^2 + \rho\langle T(x^k), \eta(x^{k+1}, x^*)\rangle - \rho\langle (T(x^*), \eta(x^{k+1}, x^*)\rangle \\
&\quad = \frac{\alpha}{2}||x^{k+1} - x^k||^2 + \rho\langle T(x^k) - T(x^*), \eta(x^{k+1}, x^*)\rangle .
\end{aligned}$$

Since T is η-γ-generalized partially relaxed monotone, it implies that

$$\begin{aligned}
\Lambda^*(x^k) - \Lambda^*(x^{k+1}) &\geqslant \frac{\alpha}{2}||x^{k+1} - x^k||^2 - \rho\gamma||x^{k+1} - x^k||^2 \\
&= \frac{1}{2}[\alpha - 2\rho\gamma]||x^{k+1} - x^k||^2 \quad \text{for} \quad \alpha - 2\rho\gamma > 0,
\end{aligned} \tag{2.7}$$

that is,

$$\Lambda^*(x^k) - \Lambda^*(x^{k+1}) \geqslant \frac{1}{2}[\alpha - 2\rho\gamma]||x^{k+1} - x^k||^2 \quad \text{for} \quad \alpha - 2\rho\gamma > 0. \tag{2.8}$$

Now if $x^{k+1}=x^k$, then x^k is a solution to the NVIP (1.1). If not, then left hand side of (2.8) is non-negative for $0<\rho<\frac{\alpha}{2\gamma}$, and as a result, the sequence $\{\Lambda^*(x^k)\}$ is a strictly decreasing sequence, that means, the difference of two succeeding terms tends to zero as $k\to\infty$. Hence,

$$\lim_{k\to\infty}||x^{k+1}-x^k||=0.$$

Since $||x^k-x^*||^2\leqslant\frac{2}{\alpha}\Lambda^*(x^k)$ and the sequence $\{\Lambda^*(x^k)\}$ is strictly decreasing, it implies that the sequence $\{x^k\}$ is bounded. Thus, there exists a cluster point x' and hence there exists a subsequence of the sequence $\{x^k\}$ that converges strongly to x'. If we take the limit in (2.1), x' is a solution of the NVIP (1.1).

In order to prove that the entire sequence $\{x^k\}$ converges to x', we need to replace x^* by x' and to rerun the whole convergence analysis as before for x'. We first define an associate function Λ' by

$$\Lambda'(x^k)=h(x')-h(x^k)-\langle h'(x^k),\eta(x',x^k)\rangle.$$

Then the sequence $\{\Lambda'(x^k)\}$ still strictly decreases, and so by Lemma 2.2, we find

$$\Lambda'(x^k)\leqslant\frac{\delta}{2}||x^k-x'||^2,$$

we can infer that $\Lambda'(x^k)$ tends to zero. On the other hand, by Lemma 2.1, we have

$$\Lambda'(x^k)\geqslant\frac{\alpha}{2}||x^k-x'||^2.$$

In light of the above arguments, we conclude that the entire sequence $\{x^k\}$ converges to x' as $k\to\infty$. This completes the proof.

When $\eta(x,y)=x-y$ in Theorem 2.1, we arrive at:

THEOREM 2.2. *Let $T:K\to\mathbf{R}^n$ be γ-partially relaxed monotone from a nonempty closed convex subset K of $\mathbf{R}^n$ into $\mathbf{R}^n$. Let $f:K\to\mathbf{R}$ be proper, convex and lower semicontinuous on K and $h:K\to\mathbf{R}$ be continuously Fre'chet-differentiable on K. If in addition, $x^*\in K$ is any fixed solution of the NVIP (1.2) and*

$$0<\rho<\frac{\alpha}{2\gamma},$$

then the sequence $\{x^k\}$, generated by Gapp 2.2, converges strongly to x^*.

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