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ON RELATIVE GEOMETRIC INEQUALITIES

A. CERDÁN, U. SCHNELL AND S. SEGURA GOMIS

(*communicated by C. Bandle*)

Abstract. Let E be a subset of a convex, open, bounded, planar set G. Let $P(E,G)$ be the relative perimeter of E (the length of the boundary of E contained in G). We obtain relative geometric inequalities comparing the relative perimeter of E with the relative diameter of E and with its relative inradius. We prove the existence of both extremal sets and maximizers for these inequalities and describe the geometric properties of them. We also give a characterization of planar convex sets of constant width in terms of the geometric constant corresponding to the relative diameter.

1. Introduction

The oldest example of a relative geometric inequality arises from Dido's problem. The legend on the foundation of Carthage says that when Dido –the sister of King Pygmalion– fled from Tyre, she landed at the North Africa shore and asked the local chief for a tract by the sea as large as could be contained in an oxhide. The chief graciously agreed and provided her with a large hide which she cut into very thin strips and tied together to form a long string. She was thus faced with the following situation:

Let H be the Euclidean open half plane, ∂H be the straight line which is the frontier of H (the *shoreline*) and Γ be a string of fixed length (the *relative boundary* or the *free boundary*); "maximize the area bounded by Γ and ∂H".

The classical isoperimetric inequality in the plane and a reflection argument show that this area will be maximal if Γ is a half circle:

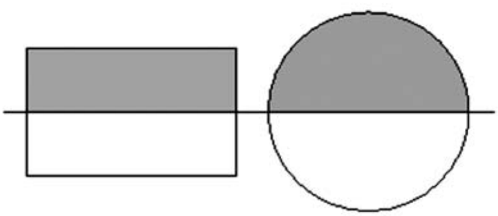

Figure 1. Dido's problem.

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We can write the above result as

$$P(\Gamma)^2 \geqslant 2\pi A$$

or equivalently

$$\frac{A}{P(\Gamma)^2} \geqslant \frac{1}{2\pi}, \tag{1}$$

where equality holds if and only if Γ is a half circle.

Generally speaking, we say that if G is an open set in the Euclidean plane $\mathbb{R}^2$ and E is a subset of G with non–empty interior and rectifiable boundary such that E as well as its complement $G \backslash E$ is connected, a *relative geometric inequality* is an inequality of the type

$$\frac{\mu(E)}{P(E,G)^\alpha} \leqslant C. \tag{2}$$

Here, $\mu(E)$ is some geometric measure, $P(E,G)$ is the *relative perimeter*, i. e. the length of the *relative boundary* or the *free boundary* ($\partial E \cap G$), and $\alpha \in (0, \alpha_0]$, where α_0 is the real positive number which makes the ratio $\mu(E)/P(E)$ invariant under dilatations ($P(E)$ is the perimeter of E).

Small discs with radius $\varepsilon \to 0$ contained in G show that it is necessary for a relative geometric inequality to hold that $\alpha \leqslant \alpha_0$.

If $\alpha > \alpha_0$ the above example shows that using part of the boundary of G does not improve the ratio; so for greater values of the exponent the relative inequality has not meaning. The only inequality that has meaning in this case is the absolute inequality. For instance for the diameter this inequality was proved by Rosenthal and Szasz [5] in 1917.

The *geometric constant* $C(G,\alpha)$ relative to given μ, G and α is the smallest number C for which (2) holds, i.e.

$$C(G,\alpha) = \sup\left\{ \frac{\mu(E)}{P(E,G)^\alpha} \;:\; E \subset G \right\}$$

If we consider the relative geometric problem for an open set G, an *extremal set* is a subset $E_0 \subset G$ such that among all subsets $E \subset G$ with $\mu(E) = \mu(E_0)$, it has minimal value for the relative perimeter: $P(E,G) \geqslant P(E_0,G)$.

With the same assumptions as above, a subset $E_0 \subset G$ is said to be a *maximizer* (with respect to given μ, G and α) if

$$C(G,\alpha) = \frac{\mu(E_0)}{P(E_0,G)^\alpha}$$

(i.e., the equality sign holds in (2)).

Extremal sets and maximizers do not always exist. When both of them exist, obviously every maximizer is an extremal set, but the converse is not true in general. An assumption which usually guarantees the existence of maximizers is the boundedness of G.

Dido's problem is a relative geometric problem in which $G = H$, $\alpha = 2$ and $\mu(E) = A(E)$ is the area of E. In this case, all extremal sets are maximizers.

If we use the same assumptions but take $\alpha = 1$, there are extremal sets (half circles) but there is no maximizer (H is not bounded).

In the case where $\mu(E) = A(E)$, relative geometric problems are called *relative isoperimetric problems* and there are a lot of known results about them (see for instance [1], [2] and [4]). There are also results when G is not a planar set but a subset of the general n-dimensional Euclidean space $\mathbb{R}^n$ (see for instance [4]).

The aim of this paper is to obtain some relative geometric inequalities when other geometric magnitudes different from the area are considered, determining the existence and the geometric characteristics of the maximizers. In particular, we solve some relative geometric problems for the diameter and the inradius for general planar convex sets G, and give for some interesting examples of sets (discs, squares, ellipses, ...) complete determination of the maximizers and the extremal sets. These quantities discussed have interesting applications; for instance, for urban subdivisions in town planning there are law regulations that state minimal requirements both for the inradius and for the diameter of the small portions; the bounds that we obtain give control of these quantities in terms of the relative perimeter which is also another interesting magnitude for urban subdivision (for small relative perimeter costs decrease). There are also several geometric applications: for instance we also give a characterization of planar convex sets of constant width depending on the geometric constant corresponding to the diameter.

2. Relative geometric inequalities concerning the diameter and the relative perimeter of a subset of a planar convex set

Throughout this section let G be a planar open bounded set and E a subset of G with non–empty interior and rectifiable boundary such that E as well as its complement $G\backslash E$ is connected. Let $P(E,G)$ be the relative perimeter of E.

The relative diameter of E with respect to G is defined as $d_G(E) = \min\{D(E), D(G\backslash E)\}$, where $D(E)$ is the usual diameter, i.e. $D(E) = sup\{d(x,y), x, y \in E\}$.

For this problem we have $\alpha_0 = 1$.

REMARK 1. *The free boundary of E with respect to G is a path γ joining two boundary points of G and $P(E,G)$ is the length of γ. It is clear that $C(G,\alpha) \geqslant C(G,1) \geqslant 1$ for any G.*

PROPOSITION 1. *There are sets G with $C(G,\alpha) = \infty$, for any $\alpha \in (0,1]$.*

Proof. Obviously it suffices to consider the case $\alpha = 1$. Let $G = \{(x,y) : x > 0, y > 0, |(x,y)| > 1, |(x,y+1)| < 2\}$.

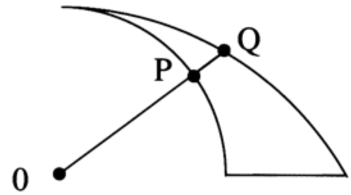

Figure 2. $C(G,\alpha)$ *is not always bounded.*

Further let $P=(\cos\varphi,\sin\varphi)$ and $Q=\lambda P$ be the intersection of $0P$ and the circle with center $(0,-1)$ and radius 2. Elementary calculation yields $\lambda=\sqrt{4-\cos^2\varphi}-\sin\varphi$. Let E be the subset of G which is bounded by the segment $\overline{PQ}$ and the two circular arcs from $(0,1)$ two P and from $(0,1)$ two Q, respectively.

It follows $P(E,G)=\lambda-1=\sqrt{4-\cos^2\varphi}-\sin\varphi-1$. Further

$$D^2=|(0,1)-Q|^2=|(\lambda\cos\varphi,\lambda\sin\varphi-1)|^2=\lambda^2+1-2\lambda\sin\varphi.$$

Consequently

$$\left(\frac{d_G(E)}{P(E,G)}\right)^2=\frac{\lambda^2+1-2\lambda\sin\varphi}{\lambda^2+1-2\lambda}=1+2\cdot\lambda\cdot\frac{1-\sin\varphi}{(\lambda-1)^2}.$$

Applying the law of de l'Hopital shows that $d_G(E)/P(E,G)\to\infty$ $(\varphi\to\pi/2)$. □

On the other hand, for convex sets G, $C(G,\alpha)$ is bounded.

For this let $s(G,u)$ be the maximal length of a chord parallel to u and let $s(G)=\min_u s(G,u)$ be the minimal length of the maximal chord.

LEMMA 1. *Let* $G\subset\mathbb{R}^2$ *be convex and* $E\subset G$. *Then*

$$\frac{d_G(E)}{P(E,G)^\alpha}\leqslant D(G)^{1-\alpha}\cdot\left(1+\left(\frac{D(G)}{s(G)}\right)^2\right)^{\alpha/2}.$$

Proof. Without restriction let $d:=d_G(E)=D(E)$. Further let $P,Q\in E$ with $|P-Q|=d$ and let $\overline{RT}$ be the maximal chord of G orthogonal to PQ. The distance of P to RT is at most $D:=D(G)$. Since G is convex the parallel chord through Q has length $q\geqslant d/D\cdot|R-T|\geqslant d/D\cdot s(G)$.

Without restriction we can assume that $Q\in\mathrm{conv}\{\mathrm{P,R,T}\}$. Q divides the intersection of the chord with $\mathrm{conv}\{\mathrm{P,R,T}\}$ into two parts of length a and b.

$P(E,G)$ is at least the sum of the distances from Q to the lines PR and PT and so

$$P(E,G)\geqslant\frac{ad}{a^2+d^2}+\frac{bd}{b^2+d^2}.$$

Minimizing this function under the condition $a+b=q$ gives $P(E,G)\geqslant d\cdot q/\sqrt{q^2+d^2}$

and so

$$\frac{d_G(E)}{P(E,G)^\alpha} \leqslant \frac{d}{d^\alpha}\sqrt{1+(d/q)^2}^\alpha \leqslant D^{1-\alpha}\cdot\left(1+\left(\frac{d}{q}\right)^2\right)^{\alpha/2}$$
$$\leqslant D^{1-\alpha}\cdot\left(1+\left(\frac{D}{s(G)}\right)^2\right)^{\alpha/2}.$$

□

LEMMA 2. *Let P,A,B be three points and $d>0$. Then there is a point Q with $|P-Q|=d$ and minimal value of $|A-Q|+|B-Q|$. Further the angles $\sphericalangle AQP$ and $\sphericalangle BQP$ are congruent.*

Proof. The existence of Q follows from a standard compactness argument. Q is the solution of the following optimization problem: minimize $|x-A|+|x-B|$ under the restriction $|x-P|=d$. The corresponding Lagrange function is $|x-A|+|x-B|+\lambda(|x-P|-d)$. It follows that Q satisfies the equation $0=(Q-A)/|Q-A|+(Q-B)/|Q-B|+\lambda\cdot(Q-P)/|Q-P|$. Hence there is a μ such that $Q-P=\mu((Q-A)/|Q-A|+(Q-B)/|Q-B|)$. This means that P lies on the bisector of the angle $\sphericalangle AQB$. □

LEMMA 3. *Let $G\subset\mathbb{R}^2$ be a bounded set with diameter $D(G)$. Then for each $d\in(0,D(G))$ there is an extremal set $E\subset G$ with $d_G(E)=d$.*

Proof. Let $0<d<D(G)$ and $p_0:=\inf\{P(E,G) : E\subset G, d_G(E)=d\}$. There is a sequence $E_n\subset G$ with $P(E_n,G)\to p_0$. The sequence $\{\overline{E_n}\}$ of compact sets has a converging subsequence with limit $E\subset G$. It follows $d_G(E)=d$ and $P(E,G)=p_0$ and hence E is an extremal set. □

The boundary of extremal sets consists of a segment an a part of the boundary of G:

THEOREM 1. *Let $G\subset\mathbb{R}^2$ be convex and bounded and $E\subset G$ extremal. Then the free boundary of E is a segment.*

Proof. Let $E\subset G$ be an extremal set with $d_G(E)=d$ and minimal $P(E,G)\leqslant d$. Let γ be the free boundary of E with endpoints A and B. There are points $Q,Q'\in\gamma$ in the half planes corresponding to the line AB with maximal distance to AB. Since E is extremal it follows that Q and Q' together with points P and P' on the opposite sides form diameters of E and $G\backslash E$ respectively, i. e. $|P-Q|=D(E)=: d_1$ and $|P'-Q'|=D(G\backslash E)=: d_2$ (without restriction let $d_2\geqslant d_1=d$).

From the extremality of E it follows that γ consists of the three line segments $\overline{AQ}, \overline{QQ'}$ and $\overline{Q'B}$.

(1) If $d_1<d_2$ then $Q'=B$ (else we can decrease $P(.,G)$ without changing d_G). Let h_A,h_B be supporting lines to G through A and B respectively. Since E is extremal we can assume that h_A is perpendicular to AQ and h_B is perpendicular to BQ

and h_A and h_B intersect on the side of γ which contains E. Then Q has minimal sum of distances to h_A and h_B among all points in G with distance d_1 to P. It is easy to see that then $Q = A$ or $Q = B$ and so γ is a segment.

(2) Now let $d_1 = d_2 = d$. As in case (1) we can assume that $Q \neq A$ and $Q' \neq B$. Let S be the intersection of the two circles with centers P and P' and radius d. A, B, Q, Q' are contained in $S \cap G$. Let h_Q be the line through Q perpendicular to PQ and $h_{Q'}$ the line through Q' perpendicular to $P'Q'$. Further let h_A be the line through A perpendicular to QA and h_B the line through B perpendicular to $Q'B$. Since E is extremal it follows that h_A and h_B are supporting lines for G.

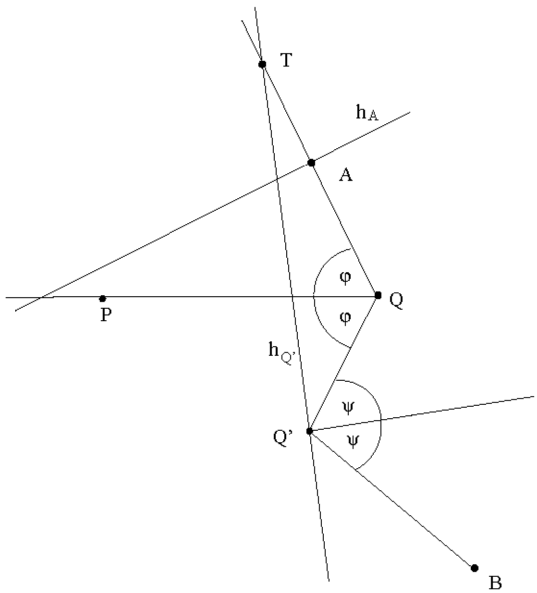

Figure 3. PQ and $P'Q'$ bisect angles AQQ' and BQQ'.

Further we conclude from Lemma 2 that $\measuredangle AQP = \measuredangle PQQ' =: \varphi$ and $\measuredangle BQ'P' = \measuredangle P'Q'Q =: \psi$. Let $a = |A - Q|, q = |Q - Q'|$ and $b = |Q' - B|$. Let $Z = \{X : |X - A| + |X - Q'| \leqslant a + q\}$ the ellipse with foci A and Q' and axes $(a+q)/2$ and $\sqrt{aq}\cos\varphi$, and $K = \{X : |X - P| \leqslant d\}$ the circle with center P and radius d.

From the extremality of E we conclude that there is a neighborhood U of Q such that $U \cap Z \subset U \cap K$. It follows that d is greater or equal than the radius of curvature of ∂Z in Q. Hence

$$d \geqslant \frac{2aq}{(a+q) \cdot \cos\varphi}. \tag{3}$$

On the other hand P is on the same side of h_A as Q and so $\measuredangle PAQ$ is acute and hence $d \leqslant a/\cos\varphi$. Together with (3) we obtain $a \geqslant q$.

Now consider the triangle with vertices Q, Q' and T, where T is the intersection point of the lines $h_{Q'}$ and QA. From $|T - Q| > |A - Q| = a \geqslant q$ it follows $\measuredangle QQ'T > \measuredangle Q'TQ$ and so $\pi/2 - \psi > \pi - 2\varphi - (\pi/2 - \psi)$ or equivalently $\psi < \varphi$.

But the same considerations with interchanging the roles of A and B, Q and Q', φ and ψ shows that also $\varphi < \psi$ which is a contradiction. □

An immediate consequence is the following Corollary.

COROLLARY 1. *Let $G \subset \mathbb{R}^2$ be convex. Then*

$$C(G,\alpha) = \sup\left\{\frac{d_G(E)}{P(E,G)^\alpha} : E = G \cap H^+,\ H^+ \text{ half plane}\right\},$$

i. e. it suffices to consider sets E whose free boundary is a line segment.

THEOREM 2. *Let $G \subset \mathbb{R}^2$ be convex. Then*

$$C(G,\alpha) = \sup\left\{\frac{d_G(E)}{P(E,G)^\alpha} : E \subset G,\ D(E) = D(G\backslash E)\right\}.$$

Proof. We consider a subset $E \subset G$ with $d_1 := D(E) < D(G\backslash E) =: d_2$. It suffices to give a subset E' with $D(E') = D(G\backslash E')$ such that $\frac{d_G(E')}{P(E',G)^\alpha} \geqslant \frac{d_G(E)}{P(E,G)^\alpha}$.

Because of Corollary 1 we can assume that the free boundary of E is a line segment $\overline{AB}$. Further let $P,Q \in E$ with $|P-Q| = d_1$ and $P',Q' \in G\backslash E$ with $|P'-Q'| = d_2$. Let a,b be supporting lines to G through A and B, respectively. If a and b are parallel or intersect in the half plane with respect to AB which contains $G\backslash E$, then AB can be translated into a parallel line such that $d_1 = d_2$ without decreasing d_G and without increasing $P(.,G)$.

So we can assume that a and b intersect in the half plane with respect to AB which contains E.

Q is one of the points A,B: Let g be the line parallel to AB through Q and $\tilde{A},\tilde{B}$ its points of intersection with $\overline{PA}$ and $\overline{PB}$, respectively. Then

$$|P-Q| \leqslant \max\{|P-\tilde{A}|, |P-\tilde{B}|\} \leqslant \max\{|P-A|, |P-B|\}.$$

In the following we assume that $Q = A$.

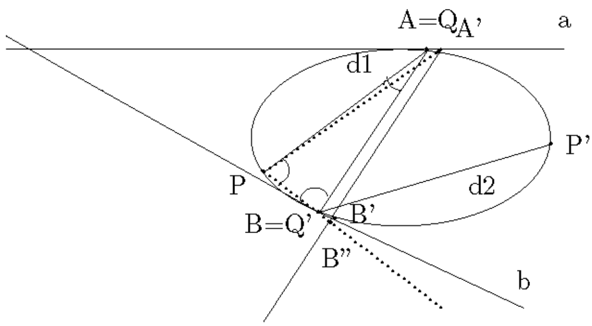

Figure 4. $D(E') = D(G/E')$ *and* $\frac{d_G(E')}{P(E',G)^\alpha} \geqslant \frac{d_G(E)}{P(E,G)^\alpha}$.

There is a chord $\overline{A'B'}$ of G parallel to $\overline{AB}$ on the side of AB opposite to P, which divides G into two subsets E' and $G\backslash E'$ with equal diameter $d_1' = d_2' \geqslant d_1$. Let $\varphi = \sphericalangle APB$ and $\psi = \sphericalangle PBA$.

From $|P-Q| \geqslant |Q-B|$ it follows $\varphi \leqslant \psi$, in particular $\varphi \leqslant \pi/2$. Let B'' be the intersection point of $A'B'$ with PB and $\varphi' = \sphericalangle A'PB''$. Obviously $\varphi' \leqslant \varphi \leqslant \pi/2$. From the sinus theorem it follows

$$\begin{aligned}\frac{d_G(E')}{P(E',G)^\alpha} &\geqslant d_G(E')^{1-\alpha}\cdot\left(\frac{|P-A'|}{|A'-B'|}\right)^\alpha \geqslant d_G(E')^{1-\alpha}\cdot\left(\frac{|P-A'|}{|A'-B''|}\right)^\alpha \\ &= d_G(E')^{1-\alpha}\cdot\left(\frac{\sin\psi}{\sin\varphi'}\right)^\alpha \geqslant d_G(E')^{1-\alpha}\cdot\left(\frac{\sin\psi}{\sin\varphi}\right)^\alpha \\ &= d_G(E')^{1-\alpha}\cdot\left(\frac{|P-A|}{|A-B|}\right)^\alpha = d_G(E')^{1-\alpha}\cdot\frac{d_G(E)^\alpha}{P(E,G)^\alpha} \\ &\geqslant \frac{d_G(E)}{P(E,G)^\alpha}.\end{aligned}$$

□

For each $0 < \alpha \leqslant 1$ there are maximizers.

COROLLARY 2. *Let $G \subset \mathbb{R}^2$ be convex. Then there is a convex set $E_0 \subset G$ such that*

$$\frac{d_G(E_0)}{P(E_0,G)^\alpha} = C(G,\alpha).$$

Proof. Let $\{E_n\}$ be a sequence of subsets of G such that $\frac{d_G(E_n)}{P(E_n,G)^\alpha} \to C(G,\alpha)$. By Corollary 1 we can assume that E_n is the intersection of G with a half plane generated by a chord $\overline{AB}$. By Theorem 2 we can assume that $D(E_n) = D(G\backslash E_n)$, for all n. In particular we have $D(E_n) \geqslant D(G)/2$.

From Lemma 1 it follows that $P(E_n,G) \geqslant c > 0$ for all n, where c is a constant depending only on G. According to Bolzano–Weierstrass the sequences $\{A_n\}$ and $\{B_n\}$ have convergent subsequences. We can assume that already $A_n \to A_0$ and $B_n \to B_0$. Let E_0 be the intersection of G with the half plane generated by $\overline{A_0B_0}$. Then E_0 is a convex set with $D(E_0) = D(\lim E_n) = \lim D(E_n)$ and $D(G\backslash E_0) = D(\lim G\backslash E_n) = \lim D(E_n)$ and so $d_G(E_0) = \lim d_G(E_n)$. Further

$$P(E_0,G) = |A_0 - B_0| = \lim |A_n - B_n| = \lim P(E_n,G) > 0.$$

Hence E_0 is nonempty and

$$\frac{d_G(E_0)}{P(E_0,G)^\alpha} = \lim \frac{d_G(E_n)}{P(E_n,G)^\alpha} = C(G,\alpha).$$

□

COROLLARY 3. *Let $G \subset \mathbb{R}^2$ be a centrally symmetric convex set. Then there is a convex set $E_0 \subset G$ with $G \backslash E_0 = -E_0$ such that*

$$\frac{d_G(E_0)}{P(E_0, G)^\alpha} = C(G, \alpha).$$

Proof. Because of Theorem 2 and Corollary 2 there is a convex set $E \subset G$ with $\frac{d_G(E)}{P(E,G)^\alpha} = C(G, \alpha)$ and $D(E) = D(G \backslash E)$. It suffices to conclude a contradiction from the assumption that $0 \notin \overline{AB}$. If this is the case then without restriction let $0 \in \text{intG} \backslash \text{E}$. Let $P, Q \in E$ with $|P - Q| = D(E)$. We can assume that Q is contained in $\overline{AB}$ (else $\overline{AB}$ can be translated parallel towards P without decreasing $d_G/P(.,G)^\alpha$). Hence, without loss of generality, let $Q = A$.

Further, G has a supporting line through B which is orthogonal to AB (else you can fix d_G and improve $P(E, G) = |A - B|$ locally). It follows that G is contained in the strip generated by the two parallel supporting lines through B and $-B$. We have $\sphericalangle PA(-B) \geqslant \pi/2$ and consequently $|P - (-B)| > D(E)$. This leads to the contradiction

$$D(E) = D(G \backslash E) \geqslant |(-P) - B| = |P + B| > D(E).$$

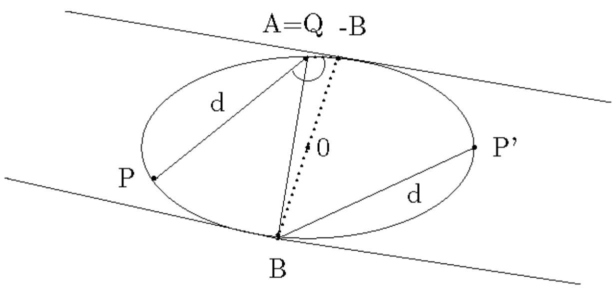

Figure 5. If $0 \notin \overline{AB}$ then we conclude a contradiction.

□

EXAMPLES. (For all examples the optimal E is an intersection of G with a half plane H^+ containing 0 in its boundary.)

(1) If G is a circular disc then $C(G, 1) = 1$.

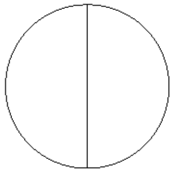

Figure 6. Maximizer of a circular disc.

(2) If G is a square then $C(G, 1) = \frac{1}{2}\sqrt{3 + \sqrt{5}} = 1.144\ldots$:

Let $G = \operatorname{conv}\{\pm(1,1), \pm(1,-1)\}$. Further let $A = (1,x)$ and $B = -A$ and E_x the corresponding set. Then $d_G(E_x) = \sqrt{4+(x+1)^2}$ and $P(E_x, G) = 2\sqrt{1+x^2}$. Standard methods show that $d_G(E_x)/P(E_x, G)$ attains its maximum $\sqrt{3+\sqrt{5}}$ in $x = \sqrt{5}-2$ (A divides the edge $\overline{(-1,1),(1,1)}$ in the golden ratio).

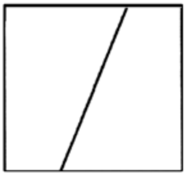

Figure 7. Maximizer of a square.

(3) If G is an ellipse with axis lengths a and b then $C(G,1) = \frac{1}{2}\left(\frac{a}{b} + \frac{b}{a}\right)$.

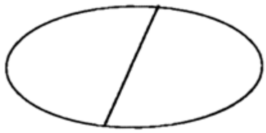

Figure 8. Maximizer of an ellipse with axis lengths a and b.

For some particular sets G, extremal sets can be determined:

EXAMPLES. (1) If G is a circular disc, extremal sets are the circular segments. All extremal sets are maximizers.

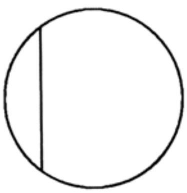

Figure 9. Extremal set of a circular disc.

(2) If G is a square, extremal sets are represented in the Figure 10.

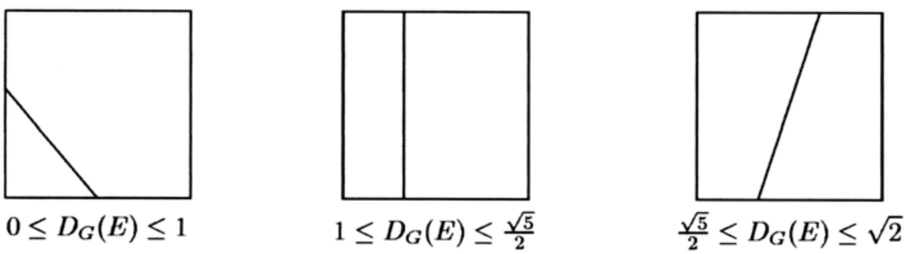

Figure 10. Extremal sets.

An interesting question is whether the circular disc is the only convex set with $C(G,1) = 1$. In the following theorem we show that there is a whole class of convex bodies with this property.

For this we first need the following notation.

A chord $\overline{AB}$ of a convex set G is called a normal in A if there is a supporting hyperplane for G through A which is orthogonal to $\overline{AB}$.

The following characterization of convex sets of constant width is very well–known (see for instance [3]).

PROPOSITION 2. *Let G be convex. Then the following statements are equivalent*

(i) G is of constant width.

(ii) Any chord $\overline{AB}$ of G which is normal in A is also normal in B.

(iii) For any pair H_1, H_2 of parallel supporting planes for G there are points $A \in H_1, B \in H_2$ such that $\overline{AB}$ is orthogonal to H_1, H_2.

THEOREM 3. *Let $G \subset \mathbb{R}^2$ be convex. Then $C(G,1) = 1$ if and only if G is of constant width.*

Proof. (1) Let G be of constant width. We will show that the assumption $C(G,1) > 1$ leads to a contradiction. By Corollary 2 there is a set $E \subset G$ with $d_G(E)/P(E,G) = C(G,1) > 1$. By Theorem 1 we can assume that E is the intersection of G with a half plane generated by a chord $\overline{AB}$. Let $P, Q \in E$ with $|P-Q| = D(E)$ and $P', Q' \in G\backslash E$ with $|P'-Q'| = D(G\backslash E)$. Since $d_G(E) > P(E,G)$ we can assume that P and P' are not contained in $\overline{AB}$.

From the optimality of E it follows that $\overline{PQ}$ and $\overline{P'Q'}$ are normal in P and P', respectively. According to Proposition 2 the chords $\overline{PQ}$ and $\overline{P'Q'}$ of E are also normal in Q and Q'. In particular both diameters are equal to the constant width $w = D(E)$. We have $|A-B| < w$ and because of Proposition 2 $\overline{AB}$ is normal in no endpoint A, B. So these points cannot be moved on $\partial(G)$ without changing $d_G(E)$.

Hence we can assume that $Q = A$ and $Q' = B$. Then we have

$$G\backslash E \subset T := \{Z : |Z-P| \leqslant w, |Z-A| \leqslant w\} \cap H^+,$$

where H^+ is the half plane defined by $\overline{PA}$ which contains B. T has diameter w and it is attained for three pairs of points (P,A), (P,S) and (A,S), where S is the intersection point of the two circular arcs defining T. Since $A \neq B$ we have $Q' = B = P$ or $B = S$. In both cases we obtain $P(E,G) = |A-B| = w$ which is a contradiction.

(2) Now let $C(G,1) = 1$. Let h_1, h_2 be two parallel supporting lines for G with normal vector u. Every chord in direction u divides G into two parts E_1, E_2. According to the intermediate value theorem there is a chord $\overline{AB}$ in direction u such that $D(E_1) = D(E_2)$. By assumption $\overline{AB}$ is a diameter of E_1 as well as of E_2. It follows that $C = E_1 \cup E_2$ is contained in the strip between the lines h_A, h_B parallel to h_1, h_2 through A and B, respectively. Hence $\overline{AB}$ is a chord as required in Proposition 2 (iii). Since h_1, h_2 were chosen arbitrary it follows from Proposition 2 that C is of constant width. $\square$

3. Relative geometric inequalities concerning the inradius and the relative perimeter of a subset of a planar convex set

Throughout this section let G be a planar open bounded convex set and $\mathscr{E}$ the family of subsets $E \subset G$ with non–empty interior and rectifiable boundary such that E as well as its complement $G \backslash E$ is connected.

The relative inradius of E with respect to G is defined as $\rho_G(E) = \min\{\rho(E), \rho(G\backslash E)\}$.

For this problem again $\alpha_0 = 1$ and for $\alpha \in (0,1]$ let

$$C(G,\alpha) = sup\left\{\frac{\rho_G(E)}{P(E,G)^\alpha}\ :\ E \in \mathscr{E}\right\}.$$

REMARK 2. *It is clear that* $C(G,1) \leqslant \frac{1}{2}$. *(Equality is attained for many sets, for instance a rectangle with edges* a *and* b, $b \geqslant 2a$*).*

LEMMA 4. *Let* $G \subset \mathbb{R}^2$ *be convex and* $\rho_0 = \sup\{\rho_G(E)\ :\ E \in \mathscr{E}\}$. *Then for each* $\rho \in (0,\rho_0)$ *there is an extremal set* $E \in \mathscr{E}$ *with* $\rho_G(E) = \rho$.

Proof. Let $0 < \rho < \rho_0$ and $p_0 := \inf\{P(E,G)\ :\ E \in \mathscr{E}, \rho_G(E) = \rho\}$. There is a sequence $E_n \in \mathscr{E}$ with $P(E_n,G) \to p_0$. The sequence $\{\overline{E_n}\}$ of compact sets has a converging subsequence with limit $E_0 \subset G$. ρ is continuous and $P(\cdot,G)$ is lower semi–continuous. It follows $\rho_G(E_0) = \rho$ and $P(E_0,G) \leqslant p_0$ and hence E_0 is an extremal set. □

THEOREM 4. *Let* $G \subset \mathbb{R}^2$ *be convex and let* $E \in \mathscr{E}$ *be an extremal set with respect to* $C(G,\alpha)$ *and with inradius* $\rho \in (0,\rho_0)$. *Then* E *and* $G\backslash E$ *contain balls* B_1, B_2 *with radius* ρ *interior and tangent to* ∂G.

The boundary of E *consists of at most five parts:*

- *two straight line segments tangent to* B_1, B_2 *and orthogonal to* ∂G*;*
- *two circular arcs of* ∂B_1 *and* ∂B_2*;*
- *a straight line segment tangent to* B_1 *and* B_2 *joining the two circular arcs.*

Proof. Let E be an extremal set with relative inradius ρ. Then there are balls $B_1 \subset E$ and $B_2 \subset G\backslash E$ with radius ρ. The relative boundary of E with respect to G is a geodesic in $G\backslash(B_1 \cup B_2)$. Hence it consists of straight line segments and arcs of ∂B_1 or ∂B_2. Since every part is a geodesic segment there is at most one connected arc in each of the boundaries of B_1, B_2.

From the minimality of $P(E,G)$ it follows by local arguments that the line segments joining ∂G and ∂B_1 (or ∂B_2) are orthogonal to ∂G in one endpoint and tangent to B_1 (or B_2) in the (possible) other endpoint. Analogously the possible line segment joining ∂B_1 and ∂B_2 is tangent to the circular arcs. □

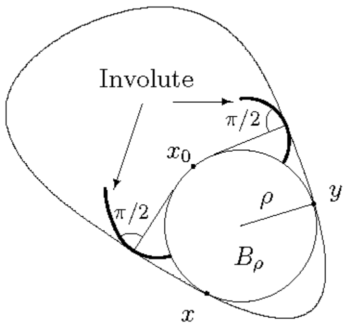

Figure 11. Extremal set with a three arcs boundary.

REMARK 3. *Remark (1) As Figure 12 shows the boundary of an extremal set may have five arcs. If $\rho_G(E) \neq \rho_G(G \backslash E)$ then it consists only of three parts: the circular arc of only one inball and two line segments.*

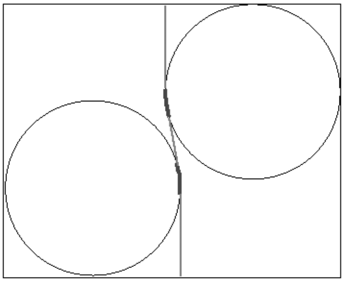

Figure 12. Extremal set with a five arcs boundary.

(2) Let $E \subset G$ and let φ be the angle between the supporting lines through the endpoints of the free boundary. Then $P(E,G) \geqslant \rho_G(E) \cdot (2+\varphi)$.

THEOREM 5. *Let $G \subset \mathbb{R}^2$ be convex. Then for each $\alpha \in (0,1]$ there is an extremal set $E_\alpha \subset G$ which is also a maximizer.*

Proof. There is a sequence of extremal sets $\{E_n\}$ with $E_n \in \mathscr{E}$ and $\rho_G(E_n)/P(E_n,G)^\alpha \to C(G,\alpha)$. The sequence $\{\overline{E_n}\}$ of compact convex sets has a converging subsequence with limit $E_0 \subset G$.

If $\rho(E_0) > 0$ then E_0 is a maximizer since ρ is continuous and $P(\cdot, G)$ is lower semi–continuous.

We still have to exclude the case $\rho(E_0) = 0$. In this case E_0 is a point $x_0 \in \partial G$. If $\alpha < 1$ then $\rho_G(E_n)/P(E_n,G)^\alpha \leqslant P(E_n,G)^{1-\alpha}/2 \to 0$, for $n \to \infty$, which is a contradiction.

Hence let $\alpha = 1$. Let $\varphi = 2\pi \cdot \lim\limits_{\epsilon \to 0} A(G \cap B_\epsilon(x_0))/A(B_\epsilon(x_0))$ be the internal angle of G at x_0. From the above remark it follows that $P(E_n,G) \geqslant \rho_G(E_n)(2+\varphi)$ for infinitely many n and so $C(G,1) \leqslant 1/(\varphi+2)$.

On the other hand we can choose a ball $B \subset G$ with (sufficiently small) radius ρ which touches both arcs of ∂G in points y_0 and z_0. Let h_y, h_z be the supporting lines to G through y_0 and z_0, let C be the cone which is formed by h_y and h_z and ψ be the angle of C in its apex. Further let E be the subset of G containing B whose relative boundary consists of an arc of ∂B and two line segments orthogonal to ∂C and tangent to ∂B. Then $P(E,G) \leqslant \rho \cdot (2+\psi)$.

From $\psi \leqslant \varphi$ we conclude that $\rho_G(E)/P(E,G) \geqslant 1/(\psi+2) \geqslant C(G,1)$. Hence E is a maximizer. □

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A. Cerdan
Departamento de Análisis Matemático
Universidad de Alicante
Campus de San Vicente del Raspeig
E-03080-Alicante, Spain
e-mail: aacs@alu.ua.es

U. Schnell
Mathematical Institute
University of Applied Sciences Zittau/Görlitz
D–02763 Zittau, Germany
e-mail: USchnell@hs-zigr.de

S. Segura Gomis
Departamento de Análisis Matemático
Universidad de Alicante
Campus de San Vicente del Raspeig
E-03080-Alicante, Spain
e-mail: Salvador.Segura@ua.es