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QUASI–VARIATIONAL EQUATION

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(*communicated by A. M. Fink*)

Abstract. The main result proved in this article is the following. There is an $\overline{x} \in X$ such that $g(\overline{x}) \in C(\overline{x})$ and $\sup_{y \in C(\overline{x})} \Psi(\overline{x}, y) = \Psi(\overline{x}, g(\overline{x}))$ where $C : X \to 2^Y$ is a correspondence; (g is a function defined over X in Y) and a function Ψ defined over $X \times Y$ in $\mathbb{R}$; X and Y are different sets. This results generalizes the quasi-variational inequation. According to this result, we show the existence of generalized Berge strong equilibrium for a constrained non cooperative game.

1. Introduction and preliminary results

Considering the importance of the quasi-variational inequation in different areas such as mathematics, optimal control theory, mathematical economics and more generally non linear optimization, cited in Arrow and Debreu [1], Aubin [3], Aubin and Ekeland [5], Mosco [11], Shafer and Sonnenshein [14], etc.

Many researchers attempted to generalize this inequality by weakening the conditions for the existence of at least one solution. Among these researchers, one can cite the papers of Shih and Tan [13], Tian and Zhou ([15], [16]) and Zhou and Chen [17].

Given two nonempty subsets X, Y of a space E and F respectively, a function Ψ defined over $X \times Y$ in $\mathbb{R}$, a function g defined over X in Y and a correspondence C defined over X in 2^Y.

We study in this article the existence of $\overline{x} \in X$ such that:

$$g(\overline{x}) \in C(\overline{x}) \quad \text{and} \quad \sup_{y \in C(\overline{x})} \Psi(\overline{x}, y) = \Psi(\overline{x}, g(\overline{x})). \tag{1.1}$$

This paper is organized as follows. Section 2 derives a theorem about the quasi-variational equation. In Section 3, we show how to apply this result.

Let us first introduce some notation and definitions.

Consider X a nonempty subset of a metrical space E, Y a nonempty subset of a locally convex space F. Let 2^Y be the set of all the parts of Y. A correspondence $C : X \to 2^Y$ is said to be upper semi-continuous over X if the set $\{x \in X \text{ such that } C(x) \cap A \neq \emptyset\}$ is closed in X, for all closed set A in Y [18], it is said to be closed if the corresponding

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graph is closed in $X \times Y$; i.e., set $\{(x,y) \in X \times Y \text{ such that } y \in C(x)\}$ is closed in $X \times Y$ [2]. A function $f : Y \to \mathbb{R}$ is said to be upper semi-continuous over Y if $\forall y_0 \in Y$, $\forall \lambda > f(y_0)$, there is a neighborhood v of y_0 such that $\forall y \in v$, $\lambda \geqslant f(y)$; f is said to be continuous over Y if f and $-f$ are upper semi-continuous over Y. We say that f is quasi-concave (resp. quasi-convex) over Y if for any y_1, y_2 in Y and for any $\theta \in [0,1]$, we have $\min\{f(y_1), f(y_2)\} \leqslant f(\theta y_1 + (1-\theta)y_2)$ ($-f$ is quasi-concave). We say that correspondence $C : Y \to 2^Y$ is upper hemi-continuous over Y if for any $p \in Y^*$, function $x \mapsto \sigma(C(x),p) = \sup_{y \in C(x)} \langle p, y \rangle$ is upper semi-continuous Y, where $\langle p, y \rangle = p(y)$ and Y^* is the set of continuous linear forms of Y.

We denote by $\overline{A}$ the closure of set A and by ∂A its border. Given Y_0 a nonempty subset of Y and $y \in Y_0$, we denote by $H_{Y_0}(y)$, $Z_{Y_0}(y)$, $T_{Y_0}(y)$ and $N_{Y_0}(y)$ the following subsets: $H_{Y_0}(y) = \bigcup_{h>0} [Y_0 - y]/h$, $T_{Y_0}(y) = \overline{H_{Y_0}(y)}$, $Z_{Y_0}(y) = [T_{Y_0}(y) + y] \cap Y$ and $N_{Y_0}(y) = \{p \in Y^* \text{ such that } \langle p, v \rangle \leqslant 0,\ \forall v \in T_{Y_0}(y)\}$.

DEFINITION 1.1. *A quasi-variational equation is the following system (1.1).*

DEFINITION 1.2. *We say that correspondence $C : X \to 2^E$ satisfies (X is assumed to be convex) :*

1) *the tangential condition if*

$$\forall x \in X,\ C(x) \cap T_X(x) \neq \emptyset \tag{1.2}$$

2) *the dual tangential condition if*

$$\forall x \in X, \forall p \in N_X(x), \textit{ then } \sigma(C(x), -p) \geqslant 0. \tag{1.3}$$

We will use the following results :

LEMMA 1.1. [4] *The tangential condition (1.2) implies the dual tangential condition (1.3).*

LEMMA 1.2. [7] *Let $C : E \to 2^F$ be a correspondence where E and F are metrical spaces. If the graph of C is compact, then C is upper semi-continuous over E.*

LEMMA 1.3. [18] (Separation Theorem) *Consider K a nonempty convex and closed subset of a locally convex space X. If x_0 does not belong to K, there is a continuous linear form $p \in X^*$ such that $\langle -p, x_0 \rangle > \sigma(K, -p)$.*

2. The results

In the following theorem, we establish a sufficient condition for the existence of a solution of the quasi-variational equation (1.1).

THEOREM 2.1. *If :*

1) *X is a nonempty compact subset of a metrical space E*
2) *Y is a nonempty convex and compact subset of a locally convex separated space F*

3) *$g : X \to Y$ is a continuous function over X such that:*
   3.1) *$g(X)$ is convex over Y*
4) *C is an upper hemi-continuous correspondence over X in 2^Y with nonempty, convex and closed values such that for any $g(x) \in \partial g(X)$, $[C(x) - g(x)] \cap T_{g(X)}(g(x)) \neq \emptyset$*
5) *function $\Psi : X \times Y \to \mathbb{R}$ satisfies:*
   5.1) *function $(x, y) \mapsto \Psi(x, y)$ is continuous over $X \times Y$*
   5.2) *for any $x \in X$, function $y \mapsto \Psi(x, y)$ is quasi-concave over Y*
   5.3) *for any $g(x) \in \partial g(X)$, for any $y \in Y$ and for any $p \in Y^*$, there is a $w \in Z_{g(X)}(g(x))$ such that* $\begin{cases} 5.3.1)\ \Psi(x,y) \leqslant \Psi(x,w) \\ 5.3.2)\ \langle p, y\rangle \leqslant \langle p, w\rangle \end{cases}$
6) *set $\left\{ x \in X \text{ such that } \alpha(x) = \sup_{y \in C(x)} \Psi(x, y) \leqslant \Psi(x, g(x)) \right\}$ is closed.*

Than there is an $\overline{x} \in X$ such that

$$g(\overline{x}) \in C(\overline{x}) \text{ and } \sup_{y \in C(\overline{x})} \Psi(\overline{x}, y) = \Psi(\overline{x}, g(\overline{x})).$$

REMARK 2.1. *Condition 5.3) in Theorem 2.1 implies $\forall g(x) \in \partial g(X)$, $\forall p \in Y^*$, we have $\sup_{y \in Y} [\Psi(x, y) + \langle p, y\rangle] \leqslant \sup_{z \in Z_{g(X)}(g(x))} [\Psi(x, z) + \langle p, z\rangle]$.*

REMARK 2.2. *Condition 5.3) in Theorem 2.1 is sufficient if g is surjective.*

REMARK 2.3. *Condition 6) in Theorem 2.1 is true if we assume furthermore that correspondence C is lower semi-continuous over X.*

Proof. Assume that the conclusion of Theorem 2.1 is false; i.e. $\forall x \in X$, we have

$$g(x) \notin C(x) \text{ or } \sup_{y \in C(x)} \Psi(x, y) > \Psi(x, g(x)). \tag{2.1}$$

Let

$$V_0 = \left\{ x \in X \text{ such that } \sup_{y \in C(x)} \Psi(x, y) > \Psi(x, g(x)) \right\}.$$

According to Separation Theorem and considering the fact that $C(x)$ is nonempty, convex and closed, $g(x) \notin C(x)$ implies $\forall g(x) \in g(X)$, $\exists p \in Y^*$ such that

$$\langle -p, g(x)\rangle - \sigma(C(x), -p) > 0$$

where $\sigma(C(x), p) = \sup_{y \in C(x)} \langle p, y\rangle$ is a supporting function of $C(x)$.

Let

$$V_p = \{x \in X \text{ such that } \langle -p, g(x)\rangle - \sigma(C(x), -p) > 0\}.$$

Assumptions 3), 4), 5.1) and 6) of Theorem 2.1 implies that sets V_0, V_p and $p \in Y^*$ are open.

Claim (2.1) implies that $X \subset V_0 \cup \bigcup_{p\in Y^*} V_p$. Since X is compact, it is possible to cover it by a finite number n of its subsets. Let $\{h_i\}_{i=\overline{0,n}}$ be a partition unity subordinate to $\{V_0, V_{p_1}, ..., V_{p_n}\}$; i.e., we have $\begin{cases} \forall x \in X,\ \sum_{i=0}^{n} h_i(x) = 1 \\ \forall i = \overline{1,n},\ supp\ h_i \subset V_{p_i} \text{ and } supp\ h_0 \subset V_0. \end{cases}$

Introduce the following function

$$\Phi : X \times Y \to \mathbb{R}$$

defined by $(x, y) \longmapsto \Phi(x, y) = h_0(x)\Psi(x, y) + \sum_{i=1}^{n} h_i(x)\langle p_i, y - g(x)\rangle$.

Function $(x, y) \longmapsto \Phi(x, y)$ is continuous over $X \times Y$. We now show that there is an $\overline{x} \in X$ such that

$$\sup_{y\in Y}\Phi(\overline{x}, y) = \Phi(\overline{x}, g(\overline{x})).$$

Assume

$$\forall x \in X,\ \exists y \in Y \text{ such that } \Phi(x, y) > \Phi(x, g(x)). \tag{2.2}$$

Construct the following sets :

$$\theta_y = \{x \in X \text{ such that } \Phi(x, y) > \Phi(x, g(x))\},\ y \in Y.$$

Then, $\forall y \in Y,\ \theta_y$ is open, $X \subset \bigcup_{y\in Y} \theta_y$. Since X is compact, it can covered by a finite number r of its subsets. Let $\{l_j\}_{j=\overline{1,r}}$ be a partition unity subordinate to $\{\theta_{y_1}, ..., \theta_{y_r}\}$; i.e. we have $\forall x \in X$, $\sum_{j=1}^{r} l_j(x) = 1$ and $\forall j = \overline{1,r},\ supp\ \ l_j \subset \theta_{y_j}$.

Consider the following correspondence :

$$M : X \to 2^Y$$

defined by $x \longmapsto M(x) = \left\{ y \in Y \text{ such that } \max_{\lambda\in S} \sum_{i=1}^{r} \lambda_i \Phi(x, y_i) \leqslant \Phi(x, y) \right\}$ where $S = \left\{ \lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^r \text{ such that } \sum_{i=1}^{r} \lambda_i = 1,\ \lambda_i \geqslant 0\ \forall i = \overline{1,r} \right\}$.

We now show that correspondence M is upper semi-continuous over X, with nonempty, convex and closed values in Y and satisfying: $\forall g(x) \in \partial g(X)$, $\exists u \in X$, $\exists \alpha > 0$ such that $\alpha g(u) + (1-\alpha)g(x) \in M(x)$.

1. $\forall x \in X,\ M(x) \neq \emptyset$.

Consider an $x \in X$, function $\lambda \mapsto \sum_{i=1}^{r} \lambda_i \Phi(x, y_i))$ is linear over $\mathbb{R}^r$. Therefore, it is continuous over the compact set S and according to the Theorem of Weierstrass [2], $\exists \overline{\lambda} \in S$ such that $\max_{\lambda\in S} \sum_{i=1}^{r} \lambda_i \Phi(x, y_i) = \sum_{i=1}^{r} \overline{\lambda}_i \Phi(x, y_i) \leqslant \sum_{i=1}^{r} \overline{\lambda}_i \max_{i=\overline{1,r}} \Phi(x, y_i) = \Phi(x, y_{i_0})$. Therefore, $y_{i_0} \in M(x)$, which implies $M(x) \neq \emptyset$.

2. $\forall x \in X,\ M(x)$ is closed in Y.

Consider $x \in X$ and $z \in \overline{M(x)}$. There is a sequence $\{z_k\}_{k\geqslant 1}$ of elements of $M(x)$ which converges towards z.
As a consequence, the fact that $\forall k \geqslant 1,\ z_k \in M(x)$ implies

$$\forall k \geqslant 1,\ \max_{\lambda\in S}\sum_{i=1}^{r}\lambda_i\Phi(x,y_i) \leqslant \Phi(x,z_k) \tag{2.3}$$

Taking into account condition 5.1) of Theorem 2.1 and the fact that $p_i \in Y^*$, $i=\overline{1,r}$ with $k \to +\infty$, we obtain

$$\max_{\lambda\in S}\sum_{i=1}^{r}\lambda_i\Phi(x,y_i) \leqslant \Phi(x,z)$$

Therefore, $z \in M(x)$; i.e. $M(x)$ is closed.

3. $\forall x \in X,\ M(x)$ is convex in Y.

Let $x \in X$ and $\overline{z},\ \overline{\overline{z}}$ be elements of $M(x)$ and $\theta \in [0,1]$. We now show that $\theta\overline{z} + (1-\theta)\overline{\overline{z}} \in M(x)$.

Since $\overline{z}$ and $\overline{\overline{z}}$ are elements of $M(x)$, we have

$$\max_{\lambda\in S}\sum_{i=1}^{r}\lambda_i\Phi(x,y_i) \leqslant \Phi(x,\overline{z}) \quad \text{and} \quad \max_{\lambda\in S}\sum_{i=1}^{r}\lambda_i\Phi(x,y_i) \leqslant \Phi(x,\overline{\overline{z}}).$$

Therefore,

$$\max_{\lambda\in S}\sum_{i=1}^{r}\lambda_i\Phi(x,y_i) \leqslant \min\left\{\Phi(x,\overline{z}),\ \Phi(x,\overline{\overline{z}})\right\}. \tag{2.4}$$

Taking into account condition 5.2) of Theorem 2.1 and the fact that $p_i \in Y^*$, $i=\overline{1,r}$ and the inequality (2.4), we obtain

$$\max_{\lambda\in S}\sum_{i=1}^{r}\lambda_i\Phi(x,y_i) \leqslant \Phi(x,\theta\overline{z}+(1-\theta)\overline{\overline{z}}),\ \forall\theta \in [0,1],$$

i.e., $\theta\overline{z} + (1-\theta)\overline{\overline{z}} \in M(x)$.

4. M is upper semi-continuous over X.

According to Lemma 1.2, it is sufficient to show that the graph of M is closed in the compact set $X \times Y$. Let $(x,z) \in \overline{\text{Graph}(M)}$. There is a sequence $\{(x_k,z_k)\}_{k\geqslant 1}$ of elements of $\text{Graph}(M)$ which converges toward (x,z). Therefore, $\forall k \geqslant 1,\ z_k \in M(x_k)$; i.e., $\forall k \geqslant 1,\ \max_{\lambda\in S}\sum_{i=1}^{r}\lambda_i\Phi(x_k,y_i) \leqslant \Phi(x_k,z_k)$.

Taking into account condition 5.1) of Theorem 2.1 and $p_i \in Y^*$, $i=\overline{1,r}$ with $k \to \infty$, we obtain $\max_{\lambda\in S}\sum_{i=1}^{r}\lambda_i\Phi(x,y_i) \leqslant \Phi(x,z)$; i.e., $z \in M(x)$, then $(x,z) \in Graph(M)$; in other words, $Graph(M)$ is closed.

5. $\forall g(x) \in \partial g(X)$, $\exists\alpha > 0,\ \exists u \in X$ such that $\alpha g(u) + (1-\alpha)g(x) \in M(x)$.

Let $g(x) \in \partial g(X)$. It is shown in 1 that $\forall x \in X$, $\exists y_{i_0} \in Y$ such that

$$\max_{\lambda \in S} \sum_{i=1}^{r} \lambda_i \Phi(x, y_i) \leqslant \Phi(x, y_{i_0}) \tag{2.5}$$

(In particular, (2.5) remains true for any $x \in X$ such that $g(x) \in \partial g(X)$). Condition 5.3) of Theorem 2.1, implies $\exists \alpha > 0$, $\exists u \in X$ such that $\Phi(x, y_{i_0}) \leqslant \Phi(x, \alpha g(u) + (1-\alpha) g(x))$ with $\alpha g(u) + (1-\alpha) g(x) \in Y$. Therefore,

$$\max_{\lambda \in S} \sum_{i=1}^{r} \lambda_i \Phi(x, y_i) \leqslant \Phi(x, \alpha g(u) + (1-\alpha) g(x)),$$

i.e., $\alpha g(u) + (1-\alpha) g(x) \in M(x)$.

We now show that $\overline{x} \in X$ such that $g(\overline{x}) \in M(\overline{x})$.

Assume that the conclusion is not true; i.e., $\forall x \in X,\ g(x) \notin M(x)$.

We have $\forall x \in X,\ g(x) \notin M(x)$, which implies, according to Lemma 1.3 and the fact that $M(x)$ is nonempty, convex and closed, $\forall x \in X$, there is a $q \in Y^*$ such that

$$\langle -q, g(x) \rangle > \sigma(M(x), -q).$$

Define

$$\Delta_q = \{x \in X \text{ such that } \langle -q, g(x) \rangle - \sigma(M(x), -q) > 0\}.$$

The continuity of function g and the upper semi-continuity of M implies that subsets Δ_q are open, $\forall q \in Y^*$.

We have $X \subset \bigcup_{q \in Y^*} \Delta_q$. Furthermore, since X is compact, it can be covered by a finite number m of its subsets. Let $\{f_j\}_{j=\overline{1,m}}$ be a partition unity subordinate to this cover.

Consider the following function

$$\Sigma : g(X) \times g(X) \to \mathbb{R}$$

defined by $(g(u), g(v)) \mapsto \Sigma(g(u), g(v)) = \sum_{j=1}^{m} f_j(u) \langle q_j, g(v) - g(u) \rangle$.

Function Σ is continuous with respect to the first variable, and quasi-concave with respect to the second variable. Since $g(X)$ is compact and convex. According to the inequality of Ky Fan [9], there is a $g(\overline{u}) \in g(X)$ such that

$$\forall g(v) \in g(X),\ \sum_{j=1}^{m} f_j(\overline{u}) \langle q_j, g(v) - g(\overline{u}) \rangle \leqslant 0,$$

which implies $\overline{q} = \sum_{j=1}^{m} f_j(\overline{u}) q_j$ belongs to *cône normal* $N_{g(X)}(g(\overline{u}))$. According to Lemma 1.1 and taking into account 5, we derive

$$\sigma(M(\overline{u}), -\overline{q}) \geqslant \langle -\overline{q}, g(\overline{u}) \rangle.$$

If $f_j(\overline{u}) > 0$, then $j \in supp\, f_j \subset \Delta_{q_j}$; i.e., $\langle -q_j, g(\overline{u})\rangle > \sigma(M(\overline{u}), -q_j)$, therefore,

$$\langle -\overline{q}, g(\overline{u})\rangle \leqslant \sigma(M(\overline{u}), -\overline{q}) = \sigma(M(\overline{u}), -\sum_{j=1}^{m} f_j(\overline{u})q_j)$$

$$\leqslant \sum_{j=1}^{m} f_j(\overline{u})\sigma(M(\overline{u}), -q_j) < \sum_{j=1}^{m} f_j(\overline{u})\langle -q_j, g(\overline{u})\rangle = \langle -\overline{q}, g(\overline{u})\rangle ,$$

which is impossible.

As a consequence, there is an $\widetilde{x} \in X$ such that $g(\widetilde{x}) \in M(\widetilde{x})$; i.e.,

$$\max_{\lambda \in S} \sum_{i=1}^{r} \lambda_i \Phi(\widetilde{x}, y_i) \leqslant \Phi(\widetilde{x}, g(\widetilde{x})).$$

Thus $\forall \lambda \in S,\ \sum_{i=1}^{r} \lambda_i \Phi(\widetilde{x}, y_i) \leqslant \Phi(\widetilde{x}, g(\widetilde{x}))$.

Let $\widetilde{\lambda} = (l_1(\widetilde{x}), ..., l_r(\widetilde{x}))$. We have $\widetilde{\lambda} \in S$ since $l_i(\widetilde{x}) \geqslant 0$ and $\sum_{i=1}^{r} l_i(\widetilde{x}) = 1$. We have

$$\sum_{i=1}^{r} l_i(\widetilde{x}) \Phi(\widetilde{x}, y_i) \leqslant \Phi(\widetilde{x}, g(\widetilde{x})) \tag{2.6}$$

Consider set $J = \{i = 1, ..., r \text{ such that } l_i(\widetilde{x}) > 0\}$. By construction, $J \neq \emptyset$. Note that $\sum_{i=1}^{r} l_i(\widetilde{x}) \Phi(\widetilde{x}, y_i) = \sum_{i \in J} l_i(\widetilde{x}) \Phi(\widetilde{x}, y_i)$.

We have $\forall i \in J$, $l_i(\widetilde{x}) > 0$, therefore $\widetilde{x} \in supp\, l_i \subset \theta_{y_i}$, i.e. $\forall i \in J,\ \Phi(\widetilde{x}, y_i) > \Phi(\widetilde{x}, g(\widetilde{x}))$. We then have $\sum_{i \in J} l_i(\widetilde{x}) \Phi(\widetilde{x}, y_i) > \sum_{i \in J} l_i(\widetilde{x}) \Phi(\widetilde{x}, g(\widetilde{x})) = \Phi(\widetilde{x}, g(\widetilde{x}))$, i.e. $\Phi(\widetilde{x}, g(\widetilde{x})) < \Phi(\widetilde{x}, g(\widetilde{x}))$, which is impossible. We can thus conclude that there is an $\widehat{x} \in X$ such that

$$\sup_{y \in Y} \Phi(\widehat{x}, y) = \Phi(\widehat{x}, g(\widehat{x}))$$

or $\forall y \in Y$,

$$h_0(\widehat{x})\Psi(\widehat{x}, y) + \sum_{i=1}^{n} h_i(\widehat{x})\langle p_i, y - g(\widehat{x})\rangle \leqslant h_0(\widehat{x})\Psi(\widehat{x}, g(\widehat{x})). \tag{2.7}$$

If $h_0(\widehat{x}) = 0$, we have $\sum_{i=1}^{n} h_i(\widehat{x}) = 1$. Therefore, (2.7) becomes

$$\sum_{i=1}^{n} h_i(\widehat{x})\langle p_i, y - g(\widehat{x})\rangle \leqslant 0,\ \forall y \in Y. \tag{2.8}$$

Inequality (2.8) implies $\overline{p} = \sum_{i=1}^{n} h_i(\widehat{x}) p_i$ belongs to *normal cone* $N_{g(X)}(g(\widehat{x}))$. According to Lemma 1.1 and condition 4) of Theorem 2.1, we have

$$\sigma(C(\widehat{x}), -\overline{p}) \geqslant \langle -\overline{p}, g(\widehat{x})\rangle . \tag{2.9}$$

The fact that $h_i(\widehat{x}) > 0$, $i = \overline{1,n}$, implies $\widehat{x} \in supp\ h_i \subset V_{p_i}$; i.e.,

$$\langle -p_i, g(\widehat{x})\rangle > \sigma(C(\widehat{x}), -p_i)$$

We have

$$\sigma(C(\widehat{x}), -\overline{p}) = \sigma(C(\widehat{x}), -\sum_{i=1}^{n} h_i(\widehat{x})p_i) \leqslant \sum_{i=1}^{n} h_i(\widehat{x})\sigma(C(\widehat{x}), -p_i)$$
$$< \sum_{i=1}^{n} h_i(\widehat{x})\langle -p_i, g(\widehat{x})\rangle = \langle -\overline{p}, g(\widehat{x})\rangle,$$

which is in contradiction with inequality (2.9). We can then conclude $h_0(\widehat{x}) > 0$.

Relation $h_0(\widehat{x}) > 0$ implies that $\widehat{x} \in \text{supp}\, h_0 \subset V_0$. Therefore,

$$\sup_{y \in C(\widehat{x})} \Psi(\widehat{x}, y) > \Psi(\widehat{x}, g(\widehat{x})).$$

Since function $y \mapsto \Psi(\widehat{x}, y)$ is continuous over the compact $C(\widehat{x})$, it follows that, according to Theorem of Weierstrass [3], there is a $\widehat{y} \in C(\widehat{x})$ such that $\sup_{y \in C(\widehat{x})} \Psi(\widehat{x}, y) = \Psi(\widehat{x}, \widehat{y})$. Therefore:

$$h_0(\widehat{x})\Psi(\widehat{x}, \widehat{y}) > h_0(\widehat{x})\Psi(\widehat{x}, g(\widehat{x})). \tag{2.10}$$

If $\sum_{i=1}^{n} h_i(\widehat{x}) = 0$, (2.7) becomes $h_0(\widehat{x})\Psi(\widehat{x}, y) \leqslant h_0(\widehat{x})\Psi(\widehat{x}, g(\widehat{x}))$, $\forall y \in Y$, which is in contradiction with inequality (2.10). Therefore $\sum_{i=1}^{n} h_i(\widehat{x}) > 0$. Let $K = \{i = 1, ..., n \text{ such that } h_i(\widehat{x}) > 0\}$. We have $K \neq \emptyset$. In fact, $\sum_{i=1}^{n} h_i(\widehat{x}) > 0$. If $i \in K$, we have $\widehat{x} \in supp\ h_i \subset V_{p_i}$; i.e.,

$$\langle -p_i, g(\widehat{x})\rangle > \sigma(C(\widehat{x}), -p_i)$$

We have

$$\langle -\overline{p}, \widehat{y}\rangle \leqslant \sigma(C(\widehat{x}), -\overline{p}) = \sigma(C(\widehat{x}), -\sum_{i=1}^{n} h_i(\widehat{x})p_i)$$
$$\leqslant \sum_{i=1}^{n} h_i(\widehat{x})\sigma(C(\widehat{x}), -p_i) < \sum_{i=1}^{n} h_i(\widehat{x})\langle -p_i, g(\widehat{x})\rangle = \langle -\overline{p}, g(\widehat{x})\rangle,$$

thus

$$\sum_{i=1}^{n} h_i(\widehat{x})\langle p_i, \widehat{y} - g(\widehat{x})\rangle > 0. \tag{2.11}$$

Inequalities (2.10) and (2.11) imply

$$h_0(\widehat{x})\Psi(\widehat{x}, \widehat{y}) + \sum_{i=1}^{n} h_i(\widehat{x})\langle p_i, \widehat{y} - g(\widehat{x})\rangle > h_0(\widehat{x})\Psi(\widehat{x}, g(\widehat{x})),$$

which is in contradiction with (2.7).

We can then conclude that there is an $\overline{x} \in X$ such that

$$g(\overline{x}) \in C(\overline{x}) \text{ and } \sup_{y \in C(\overline{x})} \Psi(\overline{x}, y) = \Psi(\overline{x}, g(\overline{x})).$$

□

REMARK 2.4. *If F is finished dimension, condition Y is compact in Theorem 2.1 can be replaced with Y is bounded.*

The quasi-variational inequality [3] is straightforward from Theorem 2.1.

COROLLARY 2.1. [3] (Quasi-variational inequality Theorem) *Let*

1) *$E = F$: separated locally convex space*
2) *$X = Y$: a non empty, convex and compact part of E*
3) *C is a upper hemi-continuous correspondence from X to 2^X with nonempty, convex and closed values.*
4) *Consider function $\Psi : X \times X \to \mathbb{R}$ such that*
   4.1) *function $(x, y) \mapsto \Psi(x, y)$ is continuous over $X \times X$*
   4.2) *for any $x \in X$, function $y \mapsto \Psi(x, y)$ is quasi-concave over X*
5) *set $\left\{ x \in X \text{ such that } \alpha(x) = \sup_{y \in C(x)} \Psi(x, y) \leqslant \Psi(x, x) \right\}$ is closed in X.*

Then there is an $\overline{x} \in X$ such that

$$\overline{x} \in C(\overline{x}) \text{ and } \sup_{y \in C(\overline{x})} \Psi(\overline{x}, y) = \Psi(\overline{x}, \overline{x}).$$

Proof. Let us introduce the following identity function $x \mapsto g(x) = x$, $\forall x \in X$. Function g is continuous and satisfies condition 3.1) of Theorem 2.1.

Let us show that correspondence C satisfies condition 4) of Theorem 2.1. Since $C(x) \subset X$ and $X \subset T_{g(X)}(g(x))$, $\forall x \in X$, we deduce that $[C(x) - x] \cap T_{g(X)}(g(x)) \neq \emptyset$, $\forall x \in X$.

We now show that function Ψ satisfies condition 5.3) of Theorem 2.1. Let $x \in X$ and $y \in X$, $\alpha = 1$ and $u = y$. We then have $y = \alpha u + (1 - \alpha)x$, therefore $\Psi(x, y) = \Psi(x, \alpha u + (1 - \alpha)x)$ and $\forall p \in X^*$, $\langle p, y \rangle = \langle p, \alpha u + (1 - \alpha)x \rangle$.

We can then conclude that all conditions in Theorem 2.1 are satisfied and consequently $\exists \overline{x} \in X$ such that $\overline{x} \in C(\overline{x})$ and $\sup_{y \in C(\overline{x})} \Psi(\overline{x}, y) = \Psi(\overline{x}, \overline{x})$. □

From the proof of Theorem 2.1, we deduce following g-maximum equality theorem and g-fixed-point Theorem [12].

COROLLARY 2.2. [12] (g-maximum equality Theorem) *Let :*

1) *X is a non empty, compact subset of a metrical space E*
2) *Y is a non empty, convex and compact subset of a separated locally convex space F*
3) *$g : X \to Y$ is a function continuous over X such that:*
   3.1) *$g(X)$ is convex in Y*

4) *Let function* $\Psi : X \times Y \to \mathbb{R}$ *be such that*
   4.1) *function* $(x,y) \mapsto \Psi(x,y)$ *is continuous over* $X \times Y$
   4.2) *for any* $x \in X$*, function* $y \mapsto \Psi(x,y)$ *is quasi-concave over* Y

   4.3) $\forall g(x) \in \partial g(X)$ *and* $\forall y \in Y$*,* $\exists z \in Z_{g(X)}(g(x))$ *such that* $\Psi(x,y) \leqslant \Psi(x,z)$.
   Then there is an $\overline{x} \in X$ *such that*

$$\sup_{y \in Y} \Psi(\overline{x}, y) = \Psi(\overline{x}, g(\overline{x})).$$

REMARK 2.5. *Corollary 2.2 (* g *-maximum equality Theorem) is a generalization of the minimax inequality (Ky fan [9]).*

COROLLARY 2.3. (g -fixed-point Theorem) *Let* X *and* Y *be a nonempty compact subset of a metrical space* E *and a nonempty, convex and compact subset of a separated locally convex space* F*, respectively. Let* $g : X \to Y$ *be a continuous function over* X *and* C *a correspondence upper hemi-continuous over* X *into* 2^Y *with nonempty, convex and closed values and satisfying:*
1) $g(X)$ *is convex in* Y
2) *for any* $g(x) \in \partial g(X)$*,* $[C(x) - g(x)] \cap T_{g(X)}(g(x)) \neq \emptyset$.
   Then there is an $\overline{x} \in X$ *such that* $g(\overline{x}) \in C(\overline{x})$.

REMARK 2.6. *The corollary 2.3. (* g *-fixed-point Theorem) generalize the fixed point theorems of Kakutani and Browder [18].*

3. Application

In this section, we examine the existence of the generalized Berge strong equilibrium for a constrained non cooperative game.

Let $I = \{1, 2, ..., n\}$ be the finite set of players, $X = \prod_{i \in I} X_i$ the set of issues, X_i the strategy set of player i; $X_i \subset E_i$. Consider the following correspondences of constraints: $S_{I-i} : X_i \to 2^{X_{I-i}}$; $I - i = I - \{i\}$ and vector $f = (f_1, f_2, ..., f_n)$ where $f_i : X \longrightarrow \mathbb{R}$ is the payoff function of player i. Let x and x_{I-i} be elements of X and X_{I-i}, respectively.

We then obtain the following constrained non cooperative game:

$$G = (X_i,\ S_{I-i},\ f_i)_{i \in I}. \tag{3.1}$$

NOTATION 3.1. $I - i = I - \{i\} = \{1, ..., i-1, i+1, ..., n\}$, $X_{I-i} = \prod_{j \in I-i} X_j$, $\widehat{X} = \prod_{i \in I} \prod_{j \in I-i} X^j_{I-i}$, $E = \prod_{i \in I} \prod_{j \in I-i} E^j_i$ *where* $X^j_{I-i} = X_{I-i}$, $E_i = E^j_i$, $\forall i \in I, \forall j \in I - i$, $x_{I-i} \in X_{I-i}$, $\widehat{x} \in \widehat{X}$ *and* $|I|$ *is the cardinal of set* I.

DEFINITION 3.1. *An issue* $\overline{x} \in X$ *is said to be generalized Berge strong equilibrium(GBSE) of game* G *(3.1) if*

$$\forall i \in I,\ \forall j \in I - i,\ f_j(\overline{x}_i, y_{I-i}) \leqslant f_j(\overline{x}),\ \forall y_{I-i} \in S_{I-i}(\overline{x}_i). \tag{3.2}$$

REMARK 3.1. *If* $S_{I-i}(x_i) \equiv X_{I-i}$, $\forall i \in I$ *then game* G *is a game in normal form* $G = (X_i, f_i)_{i\in I}$ *([6], [10], [12]).*

Consider the following functions

$$g : X \to \widehat{X}$$

defined by $x \mapsto g(x) = ((\overbrace{x_{I-1}, \ldots, x_{I-1}}^{|I|-1 \text{ times}}), \ldots, (\overbrace{x_{I-n}, \ldots, x_{I-n}}^{|I|-1 \text{ times}}))$

$$C : X \to 2^{\widehat{X}}$$

defined by $x \mapsto C(x) = ((\overbrace{S_{I-1}(x_1), \ldots, S_{I-1}(x_1)}^{|I|-1 \text{ times}}), \ldots, (\overbrace{S_{I-n}(x_n), \ldots, S_{I-n}(x_n)}^{|I|-1 \text{ times}}))$

$$F : X \times \widehat{X} \longrightarrow \mathbb{R}$$

defined by $(x, \widehat{y}) \longrightarrow F(x, \widehat{y}) = \sum_{i\in I}\sum_{j\in I-i} f_j(x_i, y^j_{I-i})$
where $\widehat{X} = \prod_{i\in I}\prod_{j\in I-i} X^j_{I-i}$, $\widehat{y} \in \widehat{X}$.

LEMMA 3.1. *The following propositions are true:*
1) *function* g *is sequentially continuous over* X,
2) *if for any* $i \in I$, X_i *is convex and compact, then* $g(X)$ *also is convex and compact.*

Proof. By construction of function g and set $\widehat{X}$, the first proposition is straightforward. The compactness of set $g(X)$ stems from the compactness of $\widehat{X}$ (Tychonoff theorem). Concerning the convexity of $g(X)$, it is sufficient to verify that function g is linear. □

On the basis of Theorem 2.1, we establish conditions on sets X and $\widehat{X}$, function f_i and correspondence C to make sure of at least one GBSE for the game (3.1) .

THEOREM 3.1. *Assume that* $\forall i \in I$, X_i *is a nonempty, convex and compact part of a separated locally convex space* E_i *and the following conditions are satisfied:*

1) *function* $(x, \widehat{y}) \to F(x, \widehat{y})$ *is continuous over* $X \times \widehat{X}$
2) $\forall x \in X$, *function* $\widehat{y} \mapsto F(x, \widehat{y})$ *is quasi-concave over* $\widehat{X}$
3) $\forall g(x) \in \partial g(X)$, $\forall \widehat{y} \in \widehat{X}$, $\forall \widehat{p} \in \widehat{X}^*$, $\exists \widehat{z} \in Z_{g(X)}(g(x))$ *such that* $F(x, \widehat{y}) \leqslant F(x, \widehat{z})$ *and* $\langle \widehat{p}, \widehat{y} - \widehat{z} \rangle \leqslant 0$
4) *correspondence* C *is upper hemi-continuous with non empty, convex, and closed values satisfying:*

$$\forall g(x) \in \partial g(X), \quad [C(x) - g(x)] \cap T_{g(X)}(g(x)) \neq \emptyset$$

5) *set* $\left\{ x \in X \text{ such that } \alpha(x) = \sup_{\widehat{y} \in C(x)} F(x, \widehat{y}) \leqslant F(x, g(x)) \right\}$ *is closed.*

Then, game G possesses at least one GBSE.

Proof. Taking into account Lemma 3.1 and conditions 1)-5) in Theorem 3.1, we can conclude that all conditions of Theorem are satisfied. Therefore, according to Theorem 2.1, there is an $\overline{x} \in X$ such that
$g(\overline{x}) \in C(\overline{x})$ and $\sup_{\widehat{y} \in C(\overline{x})} F(\overline{x}, \widehat{y}) = F(\overline{x}, g(\overline{x}))$ and as a consequence, $\forall i \in I$, $\forall j \in I - i$, $f_j(\overline{x}_i, y_{I-i}) \leqslant f_j(\overline{x})$, $\forall y_{I-i} \in S_{I-i}(\overline{x}_i)$; i.e., $\overline{x}$ is an GBSE of game G. $\square$

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REFERENCES

[1] K. ARROW AND DEBREU, *Existence of Equilibrium for a Competitive Economy*, Econometrica **22** (1954), 265–290.
[2] J. P. AUBIN, *Initiation à l'Analyse Appliqué*. Masson, Paris, 1994.
[3] J. P. AUBIN, *Mathematical Methods of Game and Economic Theory*, North-Holland, Amesterdam, 1979.
[4] J. P. AUBIN, *l'Analyse Non Linéaire et ses Motivations Économiques*. Masson, Paris, 1984.
[5] J. P. AUBIN AND I. EKLAND, *Applied Nonlinear Analysis*. Wiley Interscience, New York, 1984.
[6] C. BERGE, *Théorie Général des Jeux à $n-$ personnes*, Gauthier Villars, Paris, 1957.
[7] C. BERGE, *Espaces Topologiques, Fonctions multivoque*, Dunod, Paris, 1966.
[8] G. DEBREU, *A Social Equilibrium Existence Theorem* . Proc. Nat. Acad. Sci. U.S.A. **38** (1952).
[9] K. FAN, *Minimax inequality and application*, in Inequalities, III (O. Shisha Ed.) Academic Press, New York, 1972.
[10] M. LARBANI ET R. NESSAH, *Sur l'équilibre fort selon Berge*. RAIRO Oper. Res. **35** (2001), 439–451.
[11] U. MOSCO, *Implicit Variational problems and Quasi-variational Inequalities*, in Lecture Notes in Math., Vol. **543**, pp 83–156, Springer-Verlag, New York / Berlin, 1976.
[12] R. NESSAH, M. LARBANI AND F. YALAOUI, *g-Maximum Equality*. Soumis pour publication dans la revue Journal of Nonlinear and Convex Analysis.
[13] M. H. SHIH AND K. K. TAN, *Generalizatied quasi-variational inequalities in locallyconvex topological vector space*, J. Math. Anal. Appl **108** (1985), 333–343.
[14] W. SHAFER AND H. SONNENSHEIN, *Equilibrium in Abstract Economies without Ordered Preferences*. J. Math. Econom **2** (1975), 345–348.
[15] G. TIAN AND J. ZHOU, *Quasi-variational Inequalities with non-compact sets*. J. Math. Appl. **160** (1991), 583–595.
[16] G. TIAN AND J. ZHOU, *Quasi-variational Inequalities without the Convexity Assumption*, J. Math. Anal. Appl. **172**, 289–299 (1993).
[17] J. X. ZHOU AND G. CHEN, *Diagonal Convexity Conditions for Problems in Convex Analysis and Quasi-variational Inequalities*, J. Math. Appl. **132** (1988), 213–225.
[18] E. ZEIDLER, *Nonlinear Functional Analysis and its Applications, III : Variational Methods and Optimization*. Springer-Verlag, New York, 1985.

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