Mathematical
Inequalities
& Applications
Volume 7, Number 2 (2004), 161–164

# COUNTING SETS WITH EXCEPTIONS

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(*communicated by A. Laforgia*)

*Abstract.* Let $S \subseteq \mathbb{N}$ be a set of integers, $S_x \overset{\text{def}}{=} \cap[0,x]$, $X \overset{\text{def}}{=} \sharp S_x$ and let $\tilde{S} \subseteq S$ be the "expectional" set, $\tilde{S}_x \overset{\text{def}}{=} \tilde{S} \cap [0,x]$, $E \overset{\text{def}}{=} \sharp \tilde{S}_x$. An upper bound for the fraction of subsets of $S_x$ having $N$ elements and intersecting $K$ times at least the set $\tilde{S}$ is proved when $N,E,X \to \infty$.

## 1. Introduction and results

Let $S \subseteq \mathbb{N}$ be a set of integers, $S_x \overset{\text{def}}{=} S \cap [0,x]$, $X \overset{\text{def}}{=} \sharp S_x$ and let $\tilde{S} \subseteq S$ be the "exceptional" set, $\tilde{S}_x \overset{\text{def}}{=} \tilde{S} \cap [0,x]$, $E \overset{\text{def}}{=} \sharp \tilde{S}_x$. The combinatorial problem we face is: estimate the fraction of subsets of $S_x$ having $N$ elements and intersecting $K$ times at least the "bad" set $\tilde{S}$, for $N,E,X \to \infty$. Since $\binom{X}{N}$ is the number of different subsets of $S_x$ which have $N$ elements and since

$$\mathcal{N}(X,N,M) \overset{\text{def}}{=} \sharp\{B \subseteq S_x : \sharp B = N,\ \sharp(B \cap \tilde{S}) = M\} = \binom{X-E}{N-M}\binom{E}{M},$$

$$\tilde{\mathcal{N}}(X,N,K) \overset{\text{def}}{=} \sharp\{B \subseteq S_x : \sharp B = N,\ \sharp(B \cap \tilde{S}) \leqslant K\} = \sum_{j=0}^{K} \mathcal{N}(X,N,j),$$

the exact statement of the problem is: find non-trivial upper bounds for

$$1 - \binom{X}{N}^{-1} \tilde{\mathcal{N}}(X,N,K), \quad \text{when } N,E,X \to \infty.$$

In [2] the following argument is used to approach the problem: suppose $M < \min\{N,E\}$ and $N+E < X$, then

$$\begin{aligned}\mathcal{N}(X,N,M) &= \binom{X-E}{N-M}\binom{E}{M} = \binom{X}{N}\binom{N}{M}\frac{E!}{(E-M)!}\frac{(X-N)!}{(X-N+M-E)!}\frac{(X-E)!}{X!}\\ &= \binom{X}{N}\binom{N}{M}\frac{[E\cdots(E+1-M)]}{[X\cdots(X+1-M)]}\frac{[(X-N)\cdots(X-N+1-(E-M))]}{[(X-M)\cdots(X+1-E)]}\\ &\leqslant \binom{X}{N}\binom{N}{M}\left(\frac{E}{X}\right)^M \leqslant \binom{X}{N}2^N\left(\frac{E}{X}\right)^M,\end{aligned}$$

*Mathematics subject classification* (2000): 05A20.
*Key words and phrases*: exceptional set.

hence, summing over $K \leqslant M$, the following upper bound arises for the number of subsets of $S_x$ with $N$ elements and intersecting $K$ times at least the "bad" set $\tilde{S}$:

$$\binom{X}{N} - \tilde{\mathcal{N}}(X,N,K) = \sum_{M=K+1}^{N} \mathcal{N}(X,N,M) \ll \binom{X}{N} c^N \Big(\frac{E}{X}\Big)^K, \quad \text{for some } c > 1. \tag{1}$$

Actually, it is immediate to verify that $\binom{X}{N}^{-1}\binom{X-E}{N-M}\binom{E}{M} = \binom{X}{E}^{-1}\binom{X-N}{E-M}\binom{N}{M}$, so that the problem shows a symmetry for $N \longleftrightarrow E$; in particular, from 1 also we get

$$\binom{X}{N} - \tilde{\mathcal{N}}(X,N,K) \ll \binom{X}{N} c^E \Big(\frac{N}{X}\Big)^K, \quad \text{for some } c > 1. \tag{2}$$

The dependence of these bounds on the factors $E/X$ and $N/X$ is suitable for the applications of [2] but the presence of the exponential factors $c^N$ and $c^E$ makes unuseful these results when the growth of $N$ and $E$ with $X$ is sufficiently strong. The aim of this note is to prove the following different upper bound.

THEOREM. *Suppose $N + E < X$, then*

$$\binom{X}{N} - \tilde{\mathcal{N}}(X,N,K) \leqslant \binom{X}{N} \frac{NE}{(K+1)X}, \quad \textit{for every } K \leqslant \min\{N,E\}.$$

The conclusion of this theorem is non-trivial, i.e., better than (1)-(2), when

$$\frac{NE}{(K+1)X} \ll \min\Big\{c^N\Big(\frac{E}{X}\Big)^K, c^E\Big(\frac{N}{X}\Big)^K\Big\},$$

i.e., when

$$\max\Big\{\frac{K-1}{N}\ln\Big(\frac{X}{E}\Big) + \frac{1}{N}\ln\Big(\frac{N}{K+1}\Big), \frac{K-1}{E}\ln\Big(\frac{X}{N}\Big) + \frac{1}{E}\ln\Big(\frac{E}{K+1}\Big)\Big\} \to 0, \quad \text{for } N,E,X \to \infty.$$

This condition simplifies to

$$\max\Big\{\frac{K}{N}\ln\Big(\frac{X}{E}\Big), \frac{K}{E}\ln\Big(\frac{X}{N}\Big)\Big\} \to 0, \quad \text{for} \quad N,E,X \to \infty, \quad K \geqslant 1,$$

and in particular it is satisfied when $\frac{\ln X}{N/K}$ and $\frac{\ln X}{E/K} \to 0$.

## 2. Proof of the Theorem

The proof is quite elementary, nevertheless we think it has some interest. The first step gives a different representation of $\tilde{\mathcal{N}}(X,N,K)$.

PROPOSITION 1. *Let $c_0 \stackrel{\text{def}}{=} \binom{X-E}{N}$ and $(a)_j \stackrel{\text{def}}{=} \Gamma(a+j)/\Gamma(a) = a(a+1)(a+2)\cdots(a+j-1)$, then for $N+E < X$*

$$\mathcal{N}(X,N,j) = \frac{(-N)_j(-E)_j}{(X-N-E+1)_j}\,\frac{c_0}{j!}. \tag{3}$$

*Proof.* Let $c_j \stackrel{\text{def}}{=} \mathcal{N}(X,N,j)$, then

$$\frac{c_{j+1}}{c_j} = \binom{X-E}{N-j-1}\binom{E}{j+1}\Big/\left[\binom{X-E}{N-j}\binom{E}{j}\right] = \frac{N-j}{X-N-E+j+1}\,\frac{E-j}{j+1}$$

so that

$$\begin{aligned} c_{j+1} &= c_0\,\frac{\prod_{u=1}^{j+1}(N+1-u)}{\prod_{u=1}^{j+1}(X-N-E-u)}\,\frac{\prod_{u=1}^{j+1}(E+1-u)}{(j+1)!} \\ &= c_0\,\frac{(-1)^{j+1}(-N)_{j+1}(-1)^{j+1}(-E)_{j+1}}{(X-N-E+1)_{j+1}}\,\frac{1}{(j+1)!} \\ &= c_0\,\frac{(-N)_{j+1}(-E)_{j+1}}{(X-N-E+1)_{j+1}}\,\frac{1}{(j+1)!}, \end{aligned}$$

which is the claim.

We recall the following proposition giving the value at $x=1$ of the hypergeometric function (see [1], Theorem 2.2.2).

PROPOSITION. [Gauss] *Let* ${}_2F_1(a,b;c;x) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}$ *be the hypergeometric function, then*

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \textit{for} \quad \Re(c-a-b)>0. \tag{4}$$

As a consequence of (3) and (4) we prove that

PROPOSITION 2. *Assume $N+E<X$, then*

$$\frac{1}{N+1}\sum_{K=0}^{N}\binom{X}{N}^{-1}\tilde{\mathcal{N}}(X,N,K) = 1-\frac{NE}{(N+1)X}.$$

*Proof.* Let $A_K \stackrel{\text{def}}{=} c_0^{-1}\tilde{\mathcal{N}}(X,N,K)$. By (3) we have $A_K = \sum_{j=0}^{K}\frac{(-N)_j(-E)_j}{(X-N-E+1)_j}\frac{1}{j!}$, so that

$$\begin{aligned} \frac{1}{N+1}\sum_{K=0}^{N}A_K &= \frac{1}{N+1}\sum_{j=0}^{N}(N+1-j)\frac{(-N)_j(-E)_j}{(X-N-E+1)_j}\,\frac{1}{j!} \\ &= \sum_{j=0}^{N}\frac{(-N)_j(-E)_j}{(X-N-E+1)_j}\frac{1}{j!} - \frac{1}{N+1}\sum_{j=0}^{N}\frac{(-N)_j(-E)_j}{(X-N-E+1)_j}\,\frac{j}{j!}. \end{aligned}$$

Using the identity $(a)_{j+1} = a(a+1)_j$ in the second sum we get

$$\begin{aligned} \frac{1}{N+1}\sum_{K=0}^{N}A_K &= \sum_{j=0}^{N}\frac{(-N)_j(-E)_j}{(X-N-E+1)_j}\frac{1}{j!} \\ &\quad - \frac{NE}{(N+1)(X-N-E+1)}\sum_{j=0}^{N-1}\frac{(-N+1)_j(-E+1)_j}{(X-N-E+2)_j}\,\frac{1}{j!}. \end{aligned}$$

But $N+E<X$ and $(-a)_j=0$ for $a\in\mathbb{N}$, $j\geqslant a+1$, therefore we can allow $j$ to run up to $\infty$ in both sums, obtaining

$$\frac{1}{N+1}\sum_{K=0}^{N}A_K={}_2F_1(-N,-E;X-N-E+1;1)$$
$$-\frac{NE}{(N+1)(X-N-E+1)}\,{}_2F_1(-N+1,-E+1;X-N-E+2;1).$$

By $(4)$ we conclude

$$\begin{aligned}\frac{1}{N+1}\sum_{K=0}^{N}A_K&=\frac{\Gamma(X-N-E+1)\Gamma(X+1)}{\Gamma(X-N+1)\Gamma(X-E+1)}\\&\quad-\frac{NE}{(N+1)(X-N-E+1)}\,\frac{\Gamma(X-N-E+2)\Gamma(X)}{\Gamma(X-N+1)\Gamma(X-E+1)}\\&=\frac{\Gamma(X-N-E+1)\Gamma(X+1)}{\Gamma(X-N+1)\Gamma(X-E+1)}\left[1-\frac{NE}{(N+1)X}\right]\\&=\frac{(X-N-E)!X!}{(X-N)!(X-E)!}\left[1-\frac{NE}{(N+1)X}\right]=\binom{X}{N}c_0^{-1}\left[1-\frac{NE}{(N+1)X}\right].\end{aligned}$$

Obviously $\binom{X}{N}^{-1}\tilde{\mathcal{N}}(X,N,j)$ is increasing as a function of $j$ when $X,N$ are fixed, therefore

$$\binom{X}{N}^{-1}\tilde{\mathcal{N}}(X,N,K)\geqslant\frac{1}{K+1}\sum_{j=0}^{K}\binom{X}{N}^{-1}\tilde{\mathcal{N}}(X,N,j). \tag{5}$$

Moreover, by Proposition 2 and the evident upper bound $\binom{X}{N}^{-1}\tilde{\mathcal{N}}(X,N,j)\leqslant 1$, we get

$$\sum_{j=0}^{K}\binom{X}{N}^{-1}\tilde{\mathcal{N}}(X,N,j)=N+1-\frac{NE}{X}-\sum_{j=K+1}^{N}\binom{X}{N}^{-1}\tilde{\mathcal{N}}(X,N,j)\geqslant N+1-\frac{NE}{X}-(N-K). \tag{6}$$

Combining $(5)$ and $(6)$ the Theorem follows.

*Acknowledgement.* I wish to thank prof. Perelli who suggested me this problem.

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(Received October 10, 2003)

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