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COUNTEREXAMPLES TO A MATRIX EXPONENTIAL INEQUALITY

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(*communicated by R. Horn*)

Abstract. Counterexamples are constructed for a matrix exponential inequality conjectured in [Linear and Multilinear Algebra 47 (2000) 249–258].

1. Main Result

The following inequality is well-known for the exponential function

$$|e^z| \leqslant e^{|z|}$$

where $|z|$ denotes the absolute value of a complex number z. In [2], So and Thompson generalized this scalar inequality to the matrix inequality in Theorem 1.1 with the absolute value of a matrix $|X|$ defined as $(X^*X)^{1/2}$, and the partial ordering of Hermitian matrices $X \geqslant Y$ defined as $X - Y$ being positive semi-definite.

THEOREM 1.1. *For each $n \times n$ complex matrix A, there exist unitary matrices $U_1, \ldots, U_n$ such that*

$$|e^A| \leqslant \frac{1}{n}\sum_{i=1}^{n} U_i^* e^{|A|} U_i.$$

Of course, the U_i s depend on A. For instance, if A is normal then all the U_i s can be taken to be the identity matrix, and so we have $|e^A| \leqslant e^{|A|}$. However this matrix inequality is not true in general. Nonetheless, So and Thompson [2] conjectured that the sum in Theorem 1.1 can be reduced to only two terms instead of n terms. In this paper we construct counterexamples to this conjecture. Indeed, we show the following.

THEOREM 1.2. *For each $n \geqslant 3$ and each $k = 1, 2, \ldots n-1$, there exists an $n \times n$ complex matrix A such that*

$$|e^A| \leqslant \frac{1}{k}\sum_{i=1}^{k} U_i^* e^{|A|} U_i$$

is false for all unitary matrices $U_1, \ldots, U_k$.

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2. Proofs

Given an $n \times n$ matrix X, let $s_1(X) \geqslant s_2(X) \geqslant \cdots \geqslant s_n(X)$ denote the ordered singular values of X. Let $\|X\| \equiv s_1(X)$ denote the spectral norm of a matrix X. If X has real eigenvalues only, order them so that $\lambda_1(X) \geqslant \lambda_2(X) \geqslant \cdots \geqslant \lambda_n(X)$. Note that $\lambda_i(|X|) = s_i(X)$ for $i = 1, 2, \ldots, n$. Also note that if $X \geqslant Y$ then $\lambda_i(X) \geqslant \lambda_i(Y)$ for all $i = 1, 2, \ldots, n$. For proofs of these basic results and more, see [1]. To avoid trivial discussion, we assume that n is a fixed positive integer greater than 2 for the rest of the paper.

LEMMA 2.1. *Let A be an $n \times n$ matrix and let $k \in \{1, 2, \ldots, n-1\}$. If there exist unitary matrices $U_1, \ldots, U_k$ such that*

$$|e^A| \leqslant \frac{1}{k} \sum_{i=1}^{k} U_i^* e^{|A|} U_i,$$

then $s_{k+1}(e^A) \leqslant e^{s_2(A)}$. Consequently, $s_n(e^A) \leqslant e^{s_2(A)}$.

Proof. Weyl's inequality [1, Theorem 4.3.7] ensures that

$$\begin{aligned} \lambda_{k+1}\left(\sum_{i=1}^{k} U_i^* e^{|A|} U_i\right) &= \lambda_{k+1}\left(\left[\sum_{i=1}^{k-1} U_i^* e^{|A|} U_i\right] + U_k^* e^{|A|} U_k\right) \\ &\leqslant \lambda_k\left(\sum_{i=1}^{k-1} U_i^* e^{|A|} U_i\right) + \lambda_2\left(U_k^* e^{|A|} U_k\right) \\ &= \lambda_k\left(\sum_{i=1}^{k-1} U_i^* e^{|A|} U_i\right) + e^{s_2(A)}. \end{aligned}$$

Now, by induction, we have

$$\lambda_{k+1}\left(\sum_{i=1}^{k} U_i^* e^{|A|} U_i\right) \leqslant k\, e^{s_2(A)}.$$

Hence,

$$\begin{aligned} s_{k+1}(e^A) &= \lambda_{k+1}\left(|e^A|\right) \\ &\leqslant \frac{1}{k}\ \lambda_{k+1}\left(\sum_{i=1}^{k} U_i^* e^{|A|} U_i\right) \\ &\leqslant \frac{1}{k}\ \left[k\, e^{s_2(A)}\right] \\ &= e^{s_2(A)} \end{aligned}$$

Consequently, $s_n(e^A) \leqslant s_{k+1}(e^A) \leqslant e^{s_2(A)}$. □

Let N be the $n \times n$ upper triangular part of $J - I$, where J is the $n \times n$ matrix with all entries equal to 1, and I is the $n \times n$ identity matrix. Since N is upper triangular with all diagonal entries equal to 0, e^{-N} is upper triangular with all diagonal entries equal to 1. Hence the spectral radius of e^{-N}, $\rho(e^{-N})$, is 1. Moreover $N+N^*=J-I$, so $\lambda_1(N+N^*)=n-1$ and $\lambda_2(N+N^*)=\cdots=\lambda_n(N+N^*)=-1$.

LEMMA 2.2. *There exists a positive integer* r_0 *such that* $s_n(e^{r_0N})>e^{-r_0/2}$.

Proof. Gelfand's Theorem [1, Corollary 5.7.10] ensures that $\lim_{r\to\infty}\|(e^{-N})^r\|^{1/r}=\rho(e^{-N})=1$. Since $e^{1/2}>1$, there exists an integer r_0 such that $\|e^{-r_0N}\|^{1/r_0}<e^{1/2}$. Hence

$$[s_n(e^{r_0N})]^{-1}=s_1(e^{-r_0N})=\|e^{-r_0N}\|<e^{r_0/2}. \qquad \square$$

LEMMA 2.3. *Let* r_0 *be the positive integer in Lemma 2.2. Then there exists a positive real number* t_0 *such that*

$$[t_0+\ln s_n(e^{r_0N})]^2>t_0^2+t_0\lambda_2(r_0N+r_0N^*)+\lambda_1(r_0^2NN^*).$$

Proof. By Lemma 2.2, $s_n^2(e^{r_0N})>e^{-r_0}=e^{\lambda_2(r_0N+r_0N^*)}$ since $\lambda_2(N^*+N)=-1$. It follows that

$$2\ln s_n(e^{r_0N})>\lambda_2(r_0N+r_0N^*).$$

Hence there exists a positive real number t_0 such that

$$2t_0\ln s_n(e^{r_0N})+[\ln s_n(e^{r_0N})]^2>t_0\lambda_2(r_0N+r_0N^*)+\lambda_1(r_0^2NN^*).$$

Consequently,

$$[t_0+\ln s_n(e^{r_0N})]^2>t_0^2+t_0\lambda_2(r_0N+r_0N^*)+\lambda_1(r_0^2NN^*). \qquad \square$$

THEOREM 2.4. *For each* $n\geqslant 3$, *there exists an* $n\times n$ *matrix* A *such that*

$$s_n(e^A)>e^{s_2(A)}. \tag{1}$$

Proof. Let r_0 and t_0 be the positive integer and the real positive number obtained in Lemmas 2.2 and 2.3 respectively, and take $A=t_0I+r_0N$. Then

$$\begin{aligned}
\left[\ln(s_n(e^A))\right]^2 &= \left[\ln(s_n(e^{t_0I+r_0N}))\right]^2\\
&= \left[\ln(s_n(e^{t_0}e^{r_0N}))\right]^2\\
&= \left[\ln(e^{t_0}s_n(e^{r_0N}))\right]^2\\
&= \left[t_0+\ln(s_n(e^{r_0N}))\right]^2\\
&> t_0^2+t_0\lambda_2(r_0N+r_0N^*)+\lambda_1(r_0^2NN^*) \qquad \text{by Lemma 2.3}\\
&\geqslant t_0^2+\lambda_2\left(t_0(r_0N+r_0N^*)+r_0^2NN^*\right) \qquad \text{by Weyl's inequality}\\
&= \lambda_2\left(t_0^2I+t_0(r_0N+r_0N^*)+r_0^2NN^*\right)\\
&= \lambda_2\left((t_0I+r_0N)(t_0I+r_0N)^*\right)\\
&= \lambda_2(AA^*)\\
&= s_2^2(A)
\end{aligned}$$

Hence $s_n(e^A) > e^{s_2(A)}$. $\square$

Proof of Theorem 1.2. Let A satisfy the inequality (1). Then Lemma 2.1 ensures that A is a matrix with the property asserted in Theorem 1.2. $\square$

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REFERENCES

[1] R. HORN AND C. JOHNSON, *Matrix Analysis*, Cambridge University Press, 1988.
[2] W. SO AND R. C. THOMPSON, *Singular values of matrix exponentials*, Linear and Multilinear Algebra **47** (2000) 249–258.

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