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BERNSTEIN–DOETSCH TYPE RESULTS FOR QUASICONVEX FUNCTIONS

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Abstract. In this paper various quasiconvexity notions are considered and compared. The main goal is to show that, under the assumption of upper semicontinuity, Jensen-type quasiconvexity properties are equivalent to the corresponding ordinary quasiconvexity property. The results thus obtained are analogous to the classical theorem of Bernstein and Doetsch for convex functions.

Finally, the connection between approximate Jensen quasiconvexity and approximate quasiconvexity is investigated.

1. Introduction

A real valued function f defined on an interval $I \subseteq \mathbb{R}$ is called *Jensen-convex* (cf. eg. [5], [9]) if it satisfies the inequality

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2} \qquad (x, y \in I).$$

It is said to be *convex* if it fulfils

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y) \qquad (x, y \in I,\ \lambda \in [0,1]).$$

Obviously, a convex function $f : I \to \mathbb{R}$ is also Jensen-convex. The classical theorem of F. Bernstein and G. Doetsch [1] states that if a Jensen-convex function $f : I \to \mathbb{R}$ is locally bounded above at a point in I, then it is also convex (cf. [8], too). In connection with the stability theory of functional equations, C. T. Ng and K. Nikodem [6] extended this result showing that if ε is a nonnegative real number and a function $f : I \to \mathbb{R}$ is ε-Jensen-convex, that is, it satisfies

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2} + \varepsilon \qquad (x, y \in I)$$

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and is locally bounded above at a point in I then it is also 2ε-convex, that is, it fulfils

$$f(\lambda x+(1-\lambda)y) \leqslant \lambda f(x)+(1-\lambda)f(y)+2\varepsilon \qquad (x,y\in I,\ \lambda\in[0,1]).$$

The aim of the present paper is to investigate similar properties for quasiconvex functions.

Throughout the paper we call a function $f : I \to \mathbb{R}$ *quasiconvex*, *strictly quasiconvex*, *Jensen-quasiconvex* or *strictly Jensen-quasiconvex* if it satisfies

$$\begin{aligned} f(u) &\leqslant \max\{f(x),f(y)\} \qquad (x,y,u\in I,\ x\leqslant u\leqslant y),\\ f(u) &< \max\{f(x),f(y)\} \qquad (x,y,u\in I,\ x< u< y),\\ f\left(\frac{x+y}{2}\right) &\leqslant \max\{f(x),f(y)\} \qquad (x,y\in I),\\ f\left(\frac{x+y}{2}\right) &< \max\{f(x),f(y)\} \qquad (x,y\in I), \end{aligned}$$

respectively (cf. [2], [9]). It is a trivial consequence of these definitions that quasiconvexity implies Jensen-quasiconvexity and strict quasiconvexity implies strict Jensen-quasiconvexity. On the other hand, (differing from the Bernstein-Doetsch theorem), it is easy to see that Jensen-quasiconvexity and local boundedness (even boundedness) do not imply quasiconvexity. Indeed, the function $f : [0,1] \to \{0,1\}$, defined by

$$f(x) = \begin{cases} 0, & \text{if} \quad x\in\mathbb{Q}\cap[0,1],\\ 1, & \text{if} \quad x\in[0,1]\setminus\mathbb{Q}, \end{cases}$$

is Jensen-quasiconvex and bounded on $[0,1]$ but it is not quasiconvex.

In the second part of the paper we prove that under the assumption of upper semicontinuity (instead of boundedness), Bernstein-Doetsch type results are valid for quasiconvex and strictly quasiconvex functions, too. More generally, using *strict mean*s, that is functions $M : I\times I \to \mathbb{R}$ satisfying

$$\min\{x,y\} < M(x,y) < \max\{x,y\} \qquad (x,y\in I,\ x\neq y),$$

and

$$M(x,x) = x \qquad (x\in I),$$

we will be able to show that if I is an interval, $M : I\times I\to\mathbb{R}$ is a strict mean, continuous in both variables and $f : I\to\mathbb{R}$ is an upper semicontinuous function, then the quasiconvexity of f is equivalent to

$$f(M(x,y)) \leqslant \max\{f(x),f(y)\} \qquad (x,y\in I),$$

furthermore, its strict quasiconvexity is equivalent to

$$f(M(x,y)) < \sup_{[x,y]} f \qquad (x,y\in I,\ x\neq y).$$

In the third part of the paper, giving some (counter)examples, we verify that there are no analogous results for ε-Jensen-quasiconvex functions. Our examples given there also show that the Jensen-quasiconvexity inequality is not stable in the sense introduced by Hyers and Ulam ([4]), different from the quasiconvexity setting (cf. [7]).

2. Characterizations of quasiconvexity

In our first result, we characterize strict quasiconvexity.

THEOREM 1. *Let $I \subseteq \mathbb{R}$ be an interval, $M : I \times I \to \mathbb{R}$ be a strict mean which is continuous in both variables. An upper semicontinuous function $f : I \to \mathbb{R}$ is strictly quasiconvex if and only if*

$$f(M(x,y)) < \sup_{[x,y]} f \qquad (x, y \in I,\ x \neq y). \tag{1}$$

Proof. If $f : I \to \mathbb{R}$ is strictly quasiconvex then, using the fact that $u = M(x,y) \in]x,y[$, we have

$$f(M(x,y)) < \max\{f(x), f(y)\} \leqslant \sup_{[x,y]} f,$$

that is, (1) holds.

To prove the other part of our statement, suppose that f satisfies (1) but it is not strictly quasiconvex. Then there exist $x_0, y_0, u_0 \in I$ such that $x_0 < u_0 < y_0$ and

$$f(u_0) \geqslant \max\{f(x_0), f(y_0)\}. \tag{2}$$

Since f is upper semicontinuous, it attains its supremum in $[x_0, y_0]$. Obviously, we may assume that this supremum is attained at u_0. We have now two cases:

$$u_0 \in]x_0, M(x_0, y_0)]$$

or

$$u_0 \in [M(x_0, y_0), y_0[\,.$$

In the first case, the continuity of M in the second variable implies the existence of $y^* \in]x_0, y_0]$ such that $u_0 = M(x_0, y^*)$. Then

$$f(u_0) = f(M(x_0, y^*)) < \sup_{[x_0, y^*]} f \leqslant \sup_{[x_0, y_0]} f = f(u_0),$$

which is a contradiction. In the second case, using the continuity of M in the second variable, a contradiction can be obtained similarly. □

It is immediate to derive the following consequence of Theorem 1.

COROLLARY 1. *Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ be an upper semicontinuous function n be a positive integer and $0 = \lambda_0 \leqslant \ldots \leqslant \lambda_n = 1$. Then f is strictly quasiconvex if and only if*

$$f\left(\frac{x+y}{2}\right) < \max_{0 \leqslant i \leqslant n} f(\lambda_i x + (1-\lambda_i) y) \qquad (x, y \in I,\ x \neq y).$$

Now we offer a characterization of quasiconvexity. Clearly, (1) automatically holds with nonstrict inequality sign. Therefore, the right hand side of (1) has to be replaced by a smaller expression.

THEOREM 2. *Let* $I \subseteq \mathbb{R}$ *be an interval,* $M : I \times I \to \mathbb{R}$ *be a strict mean which is continuous in both variables. An upper semicontinuous function* $f : I \to \mathbb{R}$ *is quasiconvex if and only if*

$$f(M(x,y)) \leqslant \max\{f(x), f(y)\} \qquad (x,y \in I). \tag{3}$$

Proof. Obviously, the quasiconvexity of f implies (3).

To prove the other part of the statement, suppose that f satisfies (3) but it is not quasiconvex, that is, there exist $x_0, y_0, u_0 \in I$, such that $x_0 < u_0 < y_0$ and

$$f(u_0) > \max\{f(x_0), f(y_0)\}.$$

The function f is upper semicontinuous, therefore, it attains its supremum in $[x_0, y_0]$. We may assume that u_0 is the smallest element in $[x_0, y_0]$ where this supremum is attained. Let us consider the set

$$H = \{u \in [x_0, y_0] \mid f(u) < \sup_{[x_0,y_0]} f\}.$$

It is easy to see, that H is open in $[x_0, y_0]$, $[x_0, u_0[\subseteq H$ and $y_0 \in H$. Moreover, inequality (3) implies that $M(x,y) \in H$ for $x, y \in H$. In the following we show that H is dense in $[x_0, y_0]$. If not then there exists an open interval $]\alpha, \beta[\subset [x_0, y_0]$ which does not intersect H. We may assume that $]\alpha, \beta[$ is a maximal subinterval with this property. Then there exist sequences $x_n, y_n \in H$ such that $x_n \nearrow \alpha$, $y_n \searrow \beta$ as $n \to \infty$. Since M is a strict mean, we have $\alpha < M(\alpha, \beta) < \beta$, so, $\alpha < M(\alpha, y_n) < \beta$, if n is large enough. Similarly, for some k, $\alpha < M(x_k, y_n) < \beta$, that is, $]\alpha, \beta[\cap H \neq \emptyset$, which is a contradiction. Thus, H is dense in $[x_0, y_0]$. Due to this property, there is a sequence $y_n \in H$ such that $y_n \searrow u_0$. We have $x_0 < M(x_0, u_0) < u_0$, therefore, $x_0 < M(x_0, y_n) < u_0$, if n is large enough. On the other hand,

$$M(x_0, y_n) < u_0 < M(u_0, y_n) < y_n,$$

thus, by the continuity of M in its first variable, there exists an $x^* \in [x_0, u_0[$ for which $M(x^*, y_n) = u_0$. We have $[x_0, u_0[\subseteq H$, so, $x^* \in H$, therefore, $u_0 = M(x^*, y_n) \in H$, which contradicts the choice of u_0. The contradiction obtained yields that f is quasiconvex. □

The above theorem directly yields the following results.

COROLLARY 2. *A real valued, upper semicontinuous function* f *defined on an interval* I *is quasiconvex if and only if it is Jensen-quasiconvex.*

COROLLARY 3. *Let* $I \subseteq \mathbb{R}$ *be an interval and* $\lambda : I \times I \to]0,1[$ *be continuous in both variables. If a function* $f : I \to \mathbb{R}$ *is upper semicontinuous and satisfies*

$$f(\lambda(x,y)x + (1 - \lambda(x,y)y)) \leqslant \max\{f(x), f(y)\} \qquad (x,y \in I),$$

then it is quasiconvex.

3. Approximate Jensen-quasiconvexity

THEOREM 3. *Let $I \subseteq \mathbb{R}$ be an interval, $\varepsilon > 0$ and $\delta \geqslant 0$ be real numbers. There exists an upper semicontinuous function $f : I \to \mathbb{R}$ which is ε-Jensen-quasiconvex, that is, it fulfils the inequality*

$$f\left(\frac{x+y}{2}\right) \leqslant \max\{f(x), f(y)\} + \varepsilon \qquad (x, y \in I) \tag{4}$$

but it is not δ-quasiconvex, i.e., it does not satisfy

$$f(u) \leqslant \max\{f(x), f(y)\} + \delta \qquad (x, y, u \in I,\ x < u < y).$$

Proof. To prove our statement, we construct a function satisfying the properties above. Without loss of generality, we may assume that $I = [0,1]$, $\varepsilon = 1$ and $\delta = n-1$, where n is a positive integer. We show that the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & \text{if} \quad x \in]0, \frac{1}{2^n+1}[\quad \text{or} \quad x \in]\frac{2^n}{2^n+1}, 1[, \\ n, & \text{if} \quad (2^n+1)x \quad \text{is an integer,} \\ n-p, & \text{otherwise, where } p \text{ is the greatest integer for which } 2^p \mid [(2^n+1)x] \end{cases}$$

is upper semicontinuous on $[0,1]$, fulfils the inequality

$$f\left(\frac{x+y}{2}\right) \leqslant \max\{f(x), f(y)\} + 1 \qquad (x, y \in [0,1]) \tag{5}$$

but it does not satisfy

$$f(u) \leqslant \max\{f(x), f(y)\} + n - 1 \qquad (x, y, u \in [0,1],\ x < u < y). \tag{6}$$

(Here $[a]$ denotes the integer part of the real number a.)

Obviously, f is upper semicontinuous on $[0,1]$ and does not satisfy (6). In order to prove the validity of (5), we consider three cases.

I. If $x, y \in [0,1]$ and $(2^n+1)x$ or $(2^n+1)y$ is an integer, then $\max\{f(x), f(y)\} = n$, which implies (5).

II. In the case when $x, y \in \left]0, \frac{1}{2^n+1}\right[$ or $x, y \in \left]\frac{2^n}{2^n+1}, 1\right[$, we have $f\left(\frac{x+y}{2}\right) = 0$; if $x \in \left]0, \frac{1}{2^n+1}\right[$ or $y \in \left]\frac{2^n}{2^n+1}, 1\right[$, then $f\left(\frac{x+y}{2}\right) = 1$, that is, in these cases, (5) holds. Let now $x \in \left]0, \frac{1}{2^n+1}\right[$, $y \in \left]\frac{1}{2^n+1}, \frac{2^n}{2^n+1}\right[$ and suppose that $(2^n+1)y$ is not an integer. There exists a non-negative integer p and a positive, odd integer q such that $[(2^n+1)y] = 2^p q$, and, by the definition of f, $f(y) = n-p$. If $p = 0$, then (5) is valid. If $p > 0$, we get

$$\left[(2^n+1)\frac{x+y}{2}\right] = \left[\frac{2^p q + \{(2^n+1)y\} + (2^n+1)x}{2}\right] = 2^{p-1}q,$$

thus, $f\left(\frac{x+y}{2}\right) = n-p+1$, which implies (5). In the case when $x \in \left]\frac{2^n}{2^n+1}, 1\right[$, $y \in \left]\frac{1}{2^n+1}, \frac{2^n}{2^n+1}\right[$, we can argue similarly.

III. Finally, let $x,y \in \left]\frac{1}{2^n+1},\frac{2^n}{2^n+1}\right[$ such that $(2^n+1)x$ and $(2^n+1)y$ are not integers. There exist non-negative integers $p,\bar{p}$ and positive, odd integers $q,\bar{q}$ for which $(2^n+1)x = 2^p q$ and $(2^n+1)y = 2^{\bar{p}}\bar{q}$. By the definition of f, we have $f(x) = n-p$ and $f(y) = n-\bar{p}$. In the case when $p=0$ or $q=0$ inequality (5) holds trivially, therefore, we may assume, that p and $\bar{p}$ are positive, furthermore, that $\bar{p} \geqslant p$. We have

$$\left[(2^n+1)\frac{x+y}{2}\right] = \left[\frac{2^p q + \{(2^n+1)x\} + 2^{\bar{p}}\bar{q} + \{(2^n+1)y\}}{2}\right] = 2^{p-1}(q + 2^{\bar{p}-p}\bar{q}),$$

therefore, $f\left(\frac{x+y}{2}\right) \leqslant n-p+1$, which implies that (5) is valid in this case as well. □

REMARK. Using the functions defined in Theorem 3, it can be easily verified that the Jensen-quasiconvexity inequality is not stable in the sense of Hyers-Ulam ([3], [4]), that is, if $I \subseteq \mathbb{R}$ is an interval and $\varepsilon > 0$ and $\delta \geqslant 0$ are real numbers then there exists a function $f : I \to \mathbb{R}$ which satisfies (4), but for every Jensen-quasiconvex function $g : I \to \mathbb{R}$, there exists an $x \in I$ such that $|f(x) - g(x)| \geqslant \delta$. In fact, assuming also here that $I = [0,1]$, $\varepsilon = 1$ and $\delta = n-1$, the function defined in the proof above satisfies inequality (5), but, it is easy to see that, for every Jensen-quasiconvex function $g : [0,1] \to \mathbb{R}$, there exists an $x \in [0,1]$ for which $|f(x) - g(x)| \geqslant n-1$.

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