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SOME MAJORISATION TYPE DISCRETE INEQUALITIES FOR CONVEX FUNCTIONS

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(*communicated by P. Bullen*)

Abstract. Some majorisation type discrete inequalities for convex functions are established. Two applications are also provided.

1. Introduction

For fixed $n \geqslant 2$, let

$$\mathbf{x} = (x_1, \dots, x_n), \ \mathbf{y} = (y_1, \dots, y_n)$$

be two $n-$ tuples of real numbers. Let

$$\begin{aligned} x_{[1]} &\geqslant x_{[2]} \geqslant \cdots \geqslant x_{[n]}, \quad y_{[1]} \geqslant y_{[2]} \geqslant \cdots \geqslant y_{[n]}, \\ x_{(1)} &\leqslant x_{(2)} \leqslant \cdots \leqslant x_{(n)}, \quad y_{(1)} \leqslant y_{(2)} \leqslant \cdots \leqslant y_{(n)} \end{aligned}$$

be their ordered components.

DEFINITION 1. The $n-$ tuple $\mathbf{y}$ is said to *majorise* $\mathbf{x}$ (or $\mathbf{x}$ is *to be majorised* by $\mathbf{y}$, in symbols $\mathbf{y} \succ \mathbf{x}$), if

$$\sum_{i=1}^{m} x_{[i]} \leqslant \sum_{i=1}^{m} y_{[i]} \quad \text{holds for} \quad m = 1, 2, \dots, n-1; \tag{1.1}$$

and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i. \tag{1.2}$$

The following theorem is well-known in the literature as the *Majorisation Theorem* and a convenient reference for its proof is Marshall and Olkin [1, p. 11]. This result is due to Hardy, Littlewood and Pólya [2, p. 75] and can also be found in Karamata [3]. For a discussion concerning the matter of priority see Mitrinović [4, p.169].

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THEOREM 1. *Let I be an interval in $\mathbb{R}$, and let $\mathbf{x}, \mathbf{y}$ be two $n-$tuples such that $x_i, y_i \in I$ $(i = 1, \ldots, n)$. Then*

$$\sum_{i=1}^{n} \phi(x_i) \leqslant \sum_{i=1}^{n} \phi(y_i) \tag{1.3}$$

holds for every continuous convex function $\phi : I \to \mathbb{R}$ iff $y \succ x$ holds.

The following theorem is a weighted version of Theorem 1. It can be regarded as a generalisation of the majorisation theorem and is given in Fuchs [5].

THEOREM 2. *Let $\mathbf{x}, \mathbf{y}$ be two decreasing $n-$tuples and let $\mathbf{p} = (p_1, \ldots, p_n)$ be a real $n-$tuple such that*

$$\sum_{i=1}^{k} p_i x_i \leqslant \sum_{i=1}^{k} p_i y_i \quad \textit{for} \quad k = 1, \ldots, n-1, \tag{1.4}$$

and

$$\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i y_i. \tag{1.5}$$

Then for every continuous convex function $\phi : I \to \mathbb{R}$ we have

$$\sum_{i=1}^{n} p_i \phi(x_i) \leqslant \sum_{i=1}^{n} p_i \phi(y_i). \tag{1.6}$$

Another result of this type was obtained by Bullen, Vasić and Stanković [6].

THEOREM 3. *Let $\mathbf{x}, \mathbf{y}$ be two decreasing $n-$tuples and $\mathbf{p}$ be a real $n-$tuple. If*

$$\sum_{i=1}^{k} p_i x_i \leqslant \sum_{i=1}^{k} p_i y_i \quad \text{for} \quad k = 1, \ldots, n-1, n; \tag{1.7}$$

holds, then (1.6) holds for every continuous increasing convex function $\phi : I \to \mathbb{R}$. If $\mathbf{x}, \mathbf{y}$ are increasing $n-$tuples and the reverse inequality in (1.7) holds, then (1.6) holds for every decreasing convex function $\phi : I \to \mathbb{R}$.

For a simple proof of Theorem 2 and Theorem 3, see [7, p. 323 – 324].

REMARK 1. It is known that (see for details [7, p. 324]) the conditions (1.4) and (1.5) are not necessary for (1.6) to hold. However, when the components of $\mathbf{p}$ are all nonnegative, then (1.4) and (1.5) (respectively (1.7)) are necessary for (1.6) to hold.

In the present paper we establish some discrete inequalities for convex functions in terms of the subdifferential ∂f and apply them in obtaining sufficient conditions for the inequality (1.6) to hold.

2. The Results

Suppose that I is an interval of real numbers with interior $\overset{\circ}{I}$ and $f : I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\overset{\circ}{I}$ and has finite left and right derivatives at each point of $\overset{\circ}{I}$. Moreover, if $x, y \in \overset{\circ}{I}$ and $x < y$ then $D^{-}f(x) \leqslant D^{+}f(x) \leqslant D^{-}f(y) \leqslant D^{+}f(y)$, which shows that both $D^{-}f$ and $D^{+}f$ are nondecreasing functions on $\overset{\circ}{I}$. It is also well known (see for example [8, p.271 – 272]) that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \to \mathbb{R}$, the *subdifferential* of f denoted ∂f is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\overset{\circ}{I}\right) \subset \mathbb{R}$ and

$$f(x) \geqslant f(a) + (x - a)\varphi(a) \tag{2.1}$$

for any x and $a \in I$. It is also well known that if f is convex on I, then ∂f is nonempty, $D^{-}f,\ D^{+}f \in \partial f$ and if $\varphi \in \partial f$, then

$$D^{-}f(x) \leqslant \varphi(x) \leqslant D^{+}f(x) \tag{2.2}$$

for every $x \in \overset{\circ}{I}$. In particular, φ is a nondecreasing function.

If f is differentiable convex on $\overset{\circ}{I}$, then $\partial f = \{f'\}$.

The following result in terms of the subdifferential of the convex function f, holds.

THEOREM 4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I, $x_i, y_i \in \overset{\circ}{I}$ $(i = 1, \dots, n)$ and $p_i \geqslant 0$ $(i = 1, \dots, n)$. If $\varphi \in \partial f$, then we have the inequality*

$$\sum_{i=1}^{n} p_i f(x_i) - \sum_{i=1}^{n} p_i f(y_i) \geqslant \sum_{i=1}^{n} p_i x_i \varphi(y_i) - \sum_{i=1}^{n} p_i y_i \varphi(y_i). \tag{2.3}$$

If f is strictly convex on I and $p_i > 0$ $(i = 1, \dots, n)$, then the equality holds in (2.3) iff $x_i = y_i$ $(i = 1, \dots, n)$.

Proof. If we apply (2.1) for the selection $x = x_i,\ a = y_i$ $(i = 1, \dots, n)$ we may write

$$f(x_i) - f(y_i) \geqslant (x_i - y_i)\varphi(y_i) \tag{2.4}$$

for any $i = 1, \dots, n$.

Multiplying (2.4) by $p_i \geqslant 0$ $(i = 1, \dots, n)$ and summing over i from 1 to n we may deduce (2.3). The case of equality for strictly convex functions follows by the fact that we have equality for such a function in (2.4) iff $x_i = y_i$ $(i = 1, \dots, n)$.

REMARK 2.

a) If one chooses in Theorem 4; $y_1 = \cdots = y_n = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i$, then from (2.3) we deduce Jensen's inequality

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \geqslant f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right), \quad x_i \in I, \quad p_i \geqslant 0 \quad \left(i = \overline{1, n}\right). \tag{2.5}$$

b) If one chooses in Theorem 5: $x_1 = \cdots = x_n = \frac{1}{P_n}\sum_{i=1}^{n} p_i y_i$, then from (2.4) we may deduce the following counterpart of Jensen's inequality firstly obtained by Dragomir and Ionescu in 1994 [9] for differentiable functions and proved in the present form by C.P. Niculescu in [10]

$$\begin{aligned} 0 &\leqslant \frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(y_i\right) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i y_i\right) \\ &\leqslant \frac{1}{P_n}\sum_{i=1}^{n} p_i y_i \varphi\left(y_i\right) - \frac{1}{P_n}\sum_{i=1}^{n} p_i y_i \cdot \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi\left(y_i\right). \end{aligned} \tag{2.6}$$

Some other particular cases of interest of the above theorem are included in the corollaries below.

COROLLARY 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval $\mathring{I}$, $x_i, y_i \in I$ $(i = 1, \ldots, n)$, $p_i \geqslant 0$ $(i = 1, \ldots, n)$ with $P_n > 0$. If $(x_i - y_i)_{i=\overline{1,n}}$ is nondecreasing (nonincreasing), $(y_i)_{i=\overline{1,n}}$ is nondecreasing (nonincreasing) and $\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i y_i$, then*

$$\sum_{i=1}^{n} p_i f\left(x_i\right) \geqslant \sum_{i=1}^{n} p_i f\left(y_i\right). \tag{2.7}$$

If f is strictly convex on $\mathring{I}$, $p_i > 0$ $(i = 1, \ldots, n)$, the equality holds in (2.7) iff $x_i = y_i$ $(i = 1, \ldots, n)$.

Proof. From (2.3) we may write for $\varphi \in \partial f$ that

$$\sum_{i=1}^{n} p_i f\left(x_i\right) - \sum_{i=1}^{n} p_i f\left(y_i\right) \geqslant \sum_{i=1}^{n} p_i \left(x_i - y_i\right) \varphi\left(y_i\right) \tag{2.8}$$

with equality for f strictly convex and $p_i > 0$ $(i = 1, \ldots, n)$ iff $x_i = y_i$ $(i = 1, \ldots, n)$.

Using Čebyšev's inequality for synchronous sequences

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i b_i \geqslant \frac{1}{P_n}\sum_{i=1}^{n} p_i a_i \cdot \frac{1}{P_n}\sum_{i=1}^{n} p_i b_i, \tag{2.9}$$

where $p_i \geqslant 0$ $(i = 1, \ldots, n)$ with $P_n > 0$, $(a_i - a_j)(b_i - b_j) \geqslant 0$ for any $i, j \in \{1, \ldots, n\}$ (the equality holds in (2.9) for $p_i > 0$ $(i = 1, \ldots, n)$ iff $(a_i)_{i=\overline{1,n}}$ or $(b_i)_{i=\overline{1,n}}$ is constant), we may write that

$$\begin{aligned} \sum_{i=1}^{n} p_i \left(x_i - y_i\right) \varphi\left(y_i\right) &\geqslant \frac{1}{P_n}\sum_{i=1}^{n} p_i \left(x_i - y_i\right) \cdot \sum_{i=1}^{n} p_i \varphi\left(y_i\right) \\ &= 0 \end{aligned} \tag{2.10}$$

since $(x_i - y_i)_{i=\overline{1,n}}$, $(\varphi(y_i))_{i=\overline{1,n}}$ are synchronous. The equality holds in (2.10) for $p_i > 0$ $(i = 1, \ldots, n)$ iff $(x_i - y_i)_{i=\overline{1,n}}$ is constant or $(\varphi(y_i))_{i=\overline{1,n}}$ is constant.

If $x_i = y_i$ for each $i \in \{1, \ldots, n\}$, then (2.7) holds with equality. Combining (2.8) with (2.10) we deduce (2.7).

Now, assume that $p_i > 0$ $(i = 1, \ldots, n)$, f is strictly convex on $\overset{\circ}{I}$ and there is a $i_0 \in \{1, \ldots, n\}$ so that $x_{i_0} \neq y_{i_0}$. Then

$$f(x_{i_0}) - f(y_{i_0}) > (x_{i_0} - y_{i_0}) \varphi(y_{i_0})$$

showing that

$$\sum_{i=1}^{n} p_i f(x_i) - \sum_{i=1}^{n} p_i f(y_i) > \sum_{i=1}^{n} p_i (x_i - y_i) \varphi(y_i) \geqslant 0,$$

and thus $\sum_{i=1}^{n} p_i f(x_i) \neq \sum_{i=1}^{n} p_i f(y_i)$. Consequently, equality holds in (2.7) iff $x_i = y_i$ for each $i \in \{1, \ldots, n\}$.

REMARK 3. Obviously if $(y_i)_{i=\overline{1,n}}$ and $(x_i - y_i)_{i=\overline{1,n}}$ are nondecreasing (nonincreasing) so is $(x_i)_{i=\overline{1,n}}$.

COROLLARY 2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a nondecreasing convex function on I, $x_i, y_i \in \overset{\circ}{I}$, $p_i \geqslant 0$ $(i = 1, \ldots, n)$ and $P_n > 0$. If $(y_i)_{i=\overline{1,n}}$ is nondecreasing (nonincreasing), $(x_i - y_i)_{i=\overline{1,n}}$ is nondecreasing (nonincreasing) and $\sum_{i=1}^{n} p_i x_i \geqslant \sum_{i=1}^{n} p_i y_i$, then (2.7) holds true. If f is strictly convex and $p_i > 0$ $(i = 1, \ldots, n)$, then the equality holds in (2.7) iff $x_i = y_i$ $(i = 1, \ldots, n)$.*

Proof. Since f is nondecreasing, it follows that for $\varphi \in \partial f$, $\varphi \geqslant 0$ showing that

$$\sum_{i=1}^{n} p_i \varphi(y_i) \geqslant 0.$$

Using (2.10), the fact that, by the hypothesis we have

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i (x_i - y_i) \geqslant 0$$

and (2.8), we deduce that $\sum_{i=1}^{n} p_i f(x_i) - \sum_{i=1}^{n} p_i f(y_i) \geqslant 0.$

The case of equality may be proven in a similar way as in the proof of Corollary 1. We omit the details.

The above result incorporated in Corollary 1 may be improved in the following manner by the use of Čebyšev's refinement embodied in the following lemma.

LEMMA 1. *If a_i, b_i $(i = 1, \ldots, n)$ are synchronous sequences, then we have the* *inequalities*

$$
\begin{aligned}
& \frac{1}{P_n}\sum_{i=1}^{n} p_i a_i b_i - \frac{1}{P_n}\sum_{i=1}^{n} p_i a_i \cdot \frac{1}{P_n}\sum_{i=1}^{n} p_i b_i \qquad (2.11)\\
\geqslant\ & \max\left\{\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i \left|a_i\right| b_i - \frac{1}{P_n}\sum_{i=1}^{n} p_i \left|a_i\right| \cdot \frac{1}{P_n}\sum_{i=1}^{n} p_i b_i\right|,\right.\\
& \left|\frac{1}{P_n}\sum_{i=1}^{n} p_i \left|a_i b_i\right| - \frac{1}{P_n}\sum_{i=1}^{n} p_i \left|a_i\right| \cdot \frac{1}{P_n}\sum_{i=1}^{n} p_i b_i\right|,\\
& \left.\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i \left|b_i\right| - \frac{1}{P_n}\sum_{i=1}^{n} p_i a_i \cdot \frac{1}{P_n}\sum_{i=1}^{n} p_i \left|b_i\right|\right|\right\}\\
\geqslant\ & 0
\end{aligned}
$$

for any $p_i \geqslant 0$ *with* $P_n > 0$.

For a proof of this fact, see for example [11] or [12]. The following result holds.

COROLLARY 3. *With the assumptions of Corollary 1, one has the inequality*

$$
\begin{aligned}
& \frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(x_i\right) - \frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(y_i\right) \qquad (2.12)\\
\geqslant\ & \max\left\{\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i \left|x_i - y_i\right| \varphi\left(y_i\right) - \frac{1}{P_n}\sum_{i=1}^{n} p_i \left|x_i - y_i\right| \cdot \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi\left(y_i\right)\right|,\right.\\
& \left|\frac{1}{P_n}\sum_{i=1}^{n} p_i \left|\left(x_i - y_i\right)\varphi\left(y_i\right)\right| - \frac{1}{P_n}\sum_{i=1}^{n} p_i \left|x_i - y_i\right| \cdot \frac{1}{P_n}\sum_{i=1}^{n} p_i \left|\varphi\left(y_i\right)\right|\right|,\\
& \left.\left|\frac{1}{P_n}\sum_{i=1}^{n} p_i \left(x_i - y_i\right)\left|\varphi\left(y_i\right)\right|\right|\right\}\\
\geqslant\ & 0,
\end{aligned}
$$

where $\varphi \in \partial f$.

A similar result may be stated under the hypothesis of Corollary 2.We omit the details.

Recall now the following result due to Biernacki [13] (for a generalisation, see Burkill and Mirsky [14]).

LEMMA 2. *Let* $p_i > 0\ \ (i = 1, \ldots, n)$ *and* $(x_i)_{i=\overline{1,n}}$, $(y_i)_{i=\overline{1,n}}$ *be two sequences which are monotonic nondecreasing (nonincreasing) in mean, i.e.,*

$$
\frac{1}{P_k}\sum_{i=1}^{k} p_i x_i \underset{(\geqslant)}{\leqslant} \frac{1}{P_{k+1}}\sum_{i=1}^{k+1} p_i x_i,\quad k = 1, \ldots, n-1
$$

and

$$
\frac{1}{P_k}\sum_{i=1}^{k} p_i y_i \underset{(\geqslant)}{\leqslant} \frac{1}{P_{k+1}}\sum_{i=1}^{k+1} p_i y_i,\quad k = 1, \ldots, n-1.
$$

Then

$$\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j y_j \geqslant \frac{1}{P_n}\sum_{j=1}^{n} p_j x_j \cdot \frac{1}{P_n}\sum_{j=1}^{n} p_j y_j. \tag{2.13}$$

If one sequence is monotonic nondecreasing in mean and the other is monotonic nonincreasing in mean, then the reverse inequality holds in (2.13).

It is well known that any monotonic nondecreasing (nonincreasing) sequence is monotonic nondecreasing (nonincreasing) in mean, for any positive weights sequences $(p_i)_{i=\overline{1,n}}$. Consequently, the above inequality also holds if one sequence is monotonic and the other is monotonic in mean in the same sense.

We may now state another result for convex functions.

THEOREM 5. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I, $x_i,\ y_i \in \overset{\circ}{I}$ $(i = 1,\ldots,n)$, $p_i > 0$ $(i = 1,\ldots,n)$ and $\varphi \in \partial f$. If $(y_i)_{i=\overline{1,n}}$ is monotonic nondecreasing (nonincreasing) and $(x_i - y_i)_{i=\overline{1,n}}$ is monotonic nondecreasing (nonincreasing) in mean by raport of $(p_i)_{i=\overline{1,n}}$, then we have the inequality:*

$$\sum_{i=1}^{n} p_i f(x_i) - \sum_{i=1}^{n} p_i f(y_i) \geqslant \frac{1}{P_n}\left(\sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} p_i y_i\right)\sum_{i=1}^{n} p_i \varphi(y_i). \tag{2.14}$$

If f is strictly convex, then equality holds in (2.14) iff $x_i = y_i$ $(i = 1,\ldots,n)$.

Proof. We know, by Theorem 4, that

$$\sum_{i=1}^{n} p_i f(x_i) - \sum_{i=1}^{n} p_i f(y_i) \geqslant \sum_{i=1}^{n} p_i (x_i - y_i)\varphi(y_i). \tag{2.15}$$

Applying Lemma 2 for $(x_i - y_i)_{i=\overline{1,n}}$ and $(\varphi(y_i))_{i=\overline{1,n}}$ we have

$$\sum_{i=1}^{n} p_i (x_i - y_i)\varphi(y_i) \geqslant \frac{1}{P_n}\sum_{i=1}^{n} p_i (x_i - y_i)\sum_{i=1}^{n} p_i \varphi(y_i). \tag{2.16}$$

Now combining (2.15) with (2.16), we deduce (2.14). The equality case may be proved as in Corollary 1.

Results providing some sufficient conditions for the inequality (2.7) to hold true are embodied in the following corollary.

COROLLARY 4. *Let f, x_i, y_i and p_i $(i = 1,\ldots,n)$ be as in Theorem 5.*

1. *If, in addition, $\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i y_i$, then (2.7) holds.*
2. *If, in addition, f is nondecreasing on $\overset{\circ}{I}$ and $\sum_{i=1}^{n} p_i x_i \geqslant \sum_{i=1}^{n} p_i y_i$, then (2.7) holds.*

If f is strictly convex, then the equality holds in (2.7) iff $x_i = y_i$ $(i = 1,\ldots,n)$.

3. Applications

1. Consider the convex function $f:(0,\infty)\to\mathbb{R}$, $f(x)=-\ln x$. Assume that $x_i,y_i,p_i>0\ \ (i=1,\ldots,n)$. Then by (2.3) we have the inequality

$$\sum_{i=1}^{n}p_i\ln y_i-\sum_{i=1}^{n}p_i\ln x_i\geqslant\sum_{i=1}^{n}p_i\left(\frac{y_i-x_i}{y_i}\right) \tag{3.1}$$

or, equivalently

$$\prod_{i=1}^{n}\left(\frac{y_i}{x_i}\right)^{p_i}\geqslant\exp\left[\sum_{i=1}^{n}p_i\left(\frac{y_i-x_i}{y_i}\right)\right]. \tag{3.2}$$

The equality holds in (3.2) iff $x_i=y_i\ \ (i=1,\ldots,n)$.
If $(x_i-y_i)_{i=\overline{1,n}}$ is nondecreasing (nonincreasing), $(y_i)_{i=\overline{1,n}}$ is nondecreasing (nonincreasing) and $\sum_{i=1}^{n}p_ix_i=\sum_{i=1}^{n}p_iy_i$, then, by Corollary 1, we have

$$\prod_{i=1}^{n}y_i^{p_i}\geqslant\prod_{i=1}^{n}x_i^{p_i}. \tag{3.3}$$

The equality holds in (3.3) iff $x_i=y_i\ \ (i=1,\ldots,n)$.
With the above assumptions and using Corollary 3, we may improve the inequality (3.3) as follows:

$$\begin{aligned}&\frac{1}{P_n}\sum_{i=1}^{n}p_i\ln y_i-\frac{1}{P_n}\sum_{i=1}^{n}p_i\ln x_i \qquad (3.4)\\&\geqslant\max\left\{\left|\frac{1}{P_n}\sum_{i=1}^{n}p_i\left|x_i-y_i\right|\cdot\frac{1}{P_n}\sum_{i=1}^{n}\frac{p_i}{y_i}-\frac{1}{P_n}\sum_{i=1}^{n}p_i\frac{\left|x_i-y_i\right|}{y_i}\right|,\right.\\&\left|\frac{1}{P_n}\sum_{i=1}^{n}p_i\left|\frac{x_i-y_i}{y_i}\right|-\frac{1}{P_n}\sum_{i=1}^{n}p_i\left|x_i-y_i\right|\cdot\frac{1}{P_n}\sum_{i=1}^{n}\frac{p_i}{y_i}\right|,\\&\left.\left|\frac{1}{P_n}\sum_{i=1}^{n}p_i\frac{(x_i-y_i)}{y_i}\right|\right\}\geqslant 0.\end{aligned}$$

If $(y_i)_{i=\overline{1,n}}$ is monotonic nondecreasing (nonincreasing) and $(x_i-y_i)_{i=\overline{1,n}}$ is monotonic nondecreasing (nonincreasing) in mean by rapport of $(p_i)_{i=\overline{1,n}}$ and $\sum_{i=1}^{n}p_ix_i=\sum_{i=1}^{n}p_iy_i$, then (3.3) holds true.

2. Consider the function $f:(0,\infty)\to\mathbb{R}$, $f(x)=x\ln x$. Then $f'(x)=\ln x+1$, $f''(x)=\frac{1}{x}$, showing that f is convex on $(0,\infty)$, monotonic decreasing on $\left(0,\frac{1}{e}\right)$ and increasing on $\left(\frac{1}{e},\infty\right)$. If $x_i,y_i,p_i>0\ \ (i=1,\ldots,n)$, then by (2.3) we have the inequality

$$\sum_{i=1}^{n}p_ix_i\ln x_i-\sum_{i=1}^{n}p_ix_i\ln y_i\geqslant\sum_{i=1}^{n}p_i\left(x_i-y_i\right) \tag{3.5}$$

or, equivalently

$$\prod_{i=1}^{n}\left(\frac{x_i}{y_i}\right)^{p_i x_i} \geqslant \exp\left[\sum_{i=1}^{n} p_i\left(x_i - y_i\right)\right]. \tag{3.6}$$

The equality holds in (3.6) iff $x_i = y_i \quad (i = 1, \ldots, n)$. If $(x_i - y_i)_{i=\overline{1,n}}$ is nondecreasing (nonincreasing), $(y_i)_{i=\overline{1,n}}$ is nondecreasing (nonincreasing) and $\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i y_i$, then, by Corollary 1, we have

$$\prod_{i=1}^{n} x_i^{x_i p_i} \geqslant \prod_{i=1}^{n} y_i^{y_i p_i}. \tag{3.7}$$

The equality holds in (3.7) iff $x_i = y_i \quad (i = 1, \ldots, n)$. If $x_i, y_i \geqslant \frac{1}{e}$, $(i = 1, \ldots, n)$, $(y_i)_{i=\overline{1,n}}$ is nondecreasing (nonincreasing), $(x_i - y_i)_{i=\overline{1,n}}$ is nondecreasing (nonincreasing) and $\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i y_i$, then (3.7) holds true.

If $x_i, y_i > 0 \quad (i = 1, \ldots, n)$, $(y_i)_{i=\overline{1,n}}$ is monotone nondecreasing (nonincreasing), $(x_i - y_i)_{i=\overline{1,n}}$ is monotone nondecreasing (nonincreasing) in mean by rapport of $(p_i)_{i=\overline{1,n}}$, and $\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i y_i$, then (3.7) also holds. If $x_i, y_i \geqslant \frac{1}{e}$ $(i = 1, \ldots, n)$, then the last equality may be replaced by the more general condition $\sum_{i=1}^{n} p_i x_i \geqslant \sum_{i=1}^{n} p_i y_i$ and the inequality (3.7) will still remain valid.

Similar results may be stated for $f : [0, \infty) \to \mathbb{R}$, $f(x) = x^r$, $r \geqslant 1$ or $f(x) = \left(\frac{x}{1-x}\right)^r$, $x \in \left(0, \frac{1}{2}\right]$, $r \geqslant 1$. We omit the details.

REMARK 4. The integral version and the version for isotonic linear functionals were considered in [15].

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