Mathematical
Inequalities
& Applications
Volume 7, Number 2 (2004), 217–225

GENERALIZATION OF INEQUALITIES OF HARDY–HILBERT TYPE

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(*communicated by I. Perić*)

Abstract. A generalizations of the well-known Hilbert's and Hardy-Hilbert's inequality are given both in discrete and integral case.

1. Introduction

In the recent years Hilbert's inequality attracted significant attention and some new inequalities of Hilbert type were stated. In [6] we obtained new, more general form of this inequality which includes some previously known results as a special case. In this paper we also state inequality of Hardy-Hilbert type equivalent to our two-dimensional result in [6] providing its special cases as well as discrete analogues.

Throughout the paper we suppose that all integrals and all series converge. We also assume nonnegativity of all real functions of the real variable and all real sequences, so we omit these type of conditions in all theorems and corollaries.

We start with reviewing some known results of this type.

THEOREM A. *(integral case) If $p > 1$ and $\frac{1}{p}+\frac{1}{q}=1$, then the following inequalities hold and are equivalent*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(x)dx\right)^{\frac{1}{q}}, \tag{1}$$

$$\int_0^\infty\left(\int_0^\infty \frac{f(x)}{x+y}dx\right)^p dy < \left(\frac{\pi}{\sin(\pi/p)}\right)^p\int_0^\infty f^p(x)dx, \tag{2}$$

where the constants $\frac{\pi}{\sin(\pi/p)}$ and $\left(\frac{\pi}{\sin(\pi/p)}\right)^p$ are the best possible.

THEOREM A. *(discrete case) If $p > 1$ and $\frac{1}{p}+\frac{1}{q}=1$, then the following inequalities hold and are equivalent*

$$\sum_{n=1}^\infty\sum_{m=1}^\infty \frac{a_nb_m}{n+m} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{n=1}^\infty a_n^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^\infty b_m^q\right)^{\frac{1}{q}} \tag{3}$$

Mathematics subject classification (2000): 26D15.

Key words and phrases: Hilbert's inequality, Hardy-Hilbert's inequality, Hölder's inequality, Beta function, Gamma function.

Paper MIA-07-24

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{a_n}{n+m}\right)^p < \left(\frac{\pi}{\sin(\pi/p)}\right)^p \sum_{n=1}^{\infty} a_n^p, \tag{4}$$

where the constants $\frac{\pi}{\sin(\pi/p)}$ *and* $\left(\frac{\pi}{\sin(\pi/p)}\right)^p$ *are the best possible.*

Inequalities (1) and (3) are known as Hilbert's inequalities (for Hilbert's and related inequalities see [2, Chapter V]). We shall call the inequalities (2) and (4) Hardy-Hilbert's inequalities, because of the similarity with Hardy's inequality as it was pointed out by A.Kufner ([8]).

Let's also mention recent results by Yang ([3], [4], see also [7]):

THEOREM B. *(integral case) If* $p > 1$ *and* $\frac{1}{p}+\frac{1}{q}=1$, $\lambda > 2-\min\{p,q\}$, *then the following inequalities hold and are equivalent*

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)dxdy}{(x+y)^{\lambda}} < B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)\left(\int_0^{\infty}x^{1-\lambda}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^{\infty}x^{1-\lambda}g^q(x)dx\right)^{\frac{1}{q}},$$

$$\int_0^{\infty} y^{(\lambda-1)(p-1)}\left(\int_0^{\infty}\frac{f(x)dx}{(x+y)^{\lambda}}\right)^p dy < B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)^p\int_0^{\infty}x^{1-\lambda}f^p(x)dx,$$

where $B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)$ *and* $B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)^p$ *are the best possible, while* B *is the beta-function.*

THEOREM B. *(discrete case) If* $p > 1$, $\frac{1}{p}+\frac{1}{q}=1$, *and* $2-\min\{p,q\} < \lambda \leqslant 2$, *then the following inequalities hold and are equivalent*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_n b_m}{(n+m)^{\lambda}} < B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)\left(\sum_{m=1}^{\infty}m^{1-\lambda}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}m^{1-\lambda}b_m^q\right)^{\frac{1}{q}},$$

$$\sum_{m=1}^{\infty}m^{(\lambda-1)(p-1)}\left(\sum_{n=1}^{\infty}\frac{a_n}{(n+m)^{\lambda}}\right)^p < B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)^p\sum_{n=1}^{\infty}n^{1-\lambda}a_n^p,$$

where $B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)$ *and* $B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)^p$ *are the best possible.*

2. Integral case

In this section we extend some of our results from [6] by stating equivalent inequalities of Hardy-Hilbert type. We start with the main result:

THEOREM 1. *If* $\lambda > 0$, $p > 1$, $\frac{1}{p}+\frac{1}{q}=1$, *then the following inequalities hold and are equivalent*

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)dxdy}{(x+y)^{\lambda}} < C_1\left(\int_0^{\infty}x^{1-\lambda+p(A_1-A_2)}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^{\infty}x^{1-\lambda+q(A_2-A_1)}g^q(x)dx\right)^{\frac{1}{q}} \tag{5}$$

and

$$\int_0^\infty y^{(\lambda-1)(p-1)+p(A_1-A_2)}\left(\int_0^\infty \frac{f(x)dx}{(x+y)^\lambda}\right)^p dy < C_1^p \int_0^\infty x^{1-\lambda+p(A_1-A_2)} f^p(x)dx, \quad (6)$$

where $C_1 = \left(B(1-A_2p, \lambda-1+A_2p)\right)^{\frac{1}{p}}\left(B(1-A_1q, \lambda-1+A_1q)\right)^{\frac{1}{q}}$, $A_1 \in (\frac{1-\lambda}{q}, \frac{1}{q})$ *and* $A_2 \in (\frac{1-\lambda}{p}, \frac{1}{p})$.

Proof. Let's show that (5) and (6) are equivalent.

Suppose that inequality (5) is valid.

By putting

$$g(y) = y^{(\lambda-1)(p-1)+p(A_1-A_2)}\left(\int_0^\infty \frac{f(x)dx}{(x+y)^\lambda}\right)^{p-1},$$

taking into account $\frac{1}{p}+\frac{1}{q}=1$ and using (5) we have

$$\begin{aligned}
&\int_0^\infty y^{(\lambda-1)(p-1)+p(A_1-A_2)}\left(\int_0^\infty \frac{f(x)dx}{(x+y)^\lambda}\right)^p dy = \int_0^\infty\int_0^\infty \frac{f(x)g(y)dxdy}{(x+y)^\lambda}\\
&< C_1\left(\int_0^\infty x^{1-\lambda+p(A_1-A_2)} f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{1-\lambda+q(A_2-A_1)} g^q(y)dy\right)^{\frac{1}{q}}\\
&= C_1\left(\int_0^\infty x^{1-\lambda+p(A_1-A_2)} f^p(x)dx\right)^{\frac{1}{p}}\times\\
&\quad\times\left(\int_0^\infty y^{(\lambda-1)(p-1)+p(A_1-A_2)}\left(\int_0^\infty \frac{f(x)dx}{(x+y)^\lambda}\right)^p dy\right)^{\frac{1}{q}}
\end{aligned}$$

wherefrom we have (6).

Now, suppose that inequality (6) is valid. Applying Hölder's inequality and (6) we have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{f(x)g(y)dxdy}{(x+y)^\lambda}\\
&= \int_0^\infty\left(y^{-\frac{1-\lambda+q(A_2-A_1)}{q}}\int_0^\infty \frac{f(x)dx}{(x+y)^\lambda}\right) y^{\frac{1-\lambda+q(A_2-A_1)}{q}} g(y)dy\\
&\leqslant \left(\int_0^\infty y^{-\frac{p}{q}(1-\lambda+q(A_2-A_1))}\left(\int_0^\infty \frac{f(x)}{(x+y)^\lambda}dx\right)^p dy\right)^{\frac{1}{p}}\left(\int_0^\infty y^{1-\lambda+q(A_2-A_1)} g^q(y)dy\right)^{\frac{1}{q}}\\
&= \left(\int_0^\infty y^{(p-1)(\lambda-1)+p(A_1-A_2)}\left(\int_0^\infty \frac{f(x)}{(x+y)^\lambda}dx\right)^p dy\right)^{\frac{1}{p}}\left(\int_0^\infty y^{1-\lambda+q(A_2-A_1)} g^q(y)dy\right)^{\frac{1}{q}}
\end{aligned}$$

$$< C_1\left(\int_0^\infty x^{1-\lambda+p(A_1-A_2)}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{1-\lambda+q(A_2-A_1)}g^q(y)dy\right)^{\frac{1}{q}}$$

wherefrom we have (5).

At the end, we point out that we proved inequality (5) which appears in [6]. Hence, the inequality (6) holds, too. □

Now we can obtain Theorem B as a special case of Theorem 1 by putting $A_1 = A_2 = \frac{2-\lambda}{pq}$.

Further, for $\lambda > 0$, $A_1 = \frac{2-\lambda}{2q} \in (\frac{1-\lambda}{q}, \frac{1}{q})$ and $A_2 = \frac{2-\lambda}{2p} \in (\frac{1-\lambda}{p}, \frac{1}{p})$, we obtain

COROLLARY 1. *If $\lambda > 0$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ then the following inequalities hold and are equivalent*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)dxdy}{(x+y)^\lambda}$$
$$< B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)\left(\int_0^\infty x^{p-1-\frac{p\lambda}{2}}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty x^{q-1-\frac{q\lambda}{2}}g^q(x)dx\right)^{\frac{1}{q}},$$
$$\int_0^\infty y^{\frac{p\lambda}{2}-1}\left(\int_0^\infty \frac{f(x)dx}{(x+y)^\lambda}\right)^p dy < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)^p \int_0^\infty x^{p-1-\frac{p\lambda}{2}}f^p(x)dx.$$

REMARK. The first inequality in Corollary 1 is proven in [6].

Now we put $A_1 = \frac{1-b}{q} - \frac{1}{pq}$, and $A_2 = \frac{1-c}{p} - \frac{1}{pq}$ in Theorem 1. Taking the conditions $A_1 \in (\frac{1-\lambda}{q}, \frac{1}{q})$ and $A_2 \in (\frac{1-\lambda}{p}, \frac{1}{p})$ into account, we obtain conditions $0 < b + \frac{1}{p} < \lambda$ and $0 < c + \frac{1}{q} < \lambda$ and the following result is obtained:

COROLLARY 2. *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < b + \frac{1}{p} < \lambda$ and $0 < c + \frac{1}{q} < \lambda$, then the following inequalities hold and are equivalent*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)dxdy}{(x+y)^\lambda}$$
$$< C_2\left(\int_0^\infty x^{(p-1)(1-b)+c-\lambda}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty x^{(q-1)(1-c)+b-\lambda}g^q(x)dx\right)^{\frac{1}{q}},$$
$$\int_0^\infty y^{(\lambda-b)(p-1)+c-1}\left(\int_0^\infty \frac{f(x)dx}{(x+y)^\lambda}\right)^p dy < C_2^p \int_0^\infty x^{(p-1)(1-b)+c-\lambda}f^p(x)dx,$$

where $C_2 = \left(B(c+\frac{1}{q}, \lambda - c - \frac{1}{q})\right)^{\frac{1}{p}}\left(B(b+\frac{1}{p}, \lambda - b - \frac{1}{p})\right)^{\frac{1}{q}}$.

REMARK. The first inequality in Corollary 2 is proven in [6].

Now, for $\lambda = b + c + 1$ we obtain

COROLLARY 3. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $b + \frac{1}{p} > 0$, $c + \frac{1}{q} > 0$, *then the following inequalities hold and are equivalent*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)dxdy}{(x+y)^{b+c+1}} < C_3 \left(\int_0^\infty x^{p(1-b)-2} f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty x^{q(1-c)-2} g^q(x)dx \right)^{\frac{1}{q}},$$

$$\int_0^\infty y^{p(c+1)-2} \left(\int_0^\infty \frac{f(x)dx}{(x+y)^{b+c+1}} \right)^p dy < C_3^p \int_0^\infty x^{p(1-b)-2} f^p(x)dx,$$

where $C_3 = B\big(b + \frac{1}{p}, c + \frac{1}{q}\big)$.

REMARK. The first inequality in Corollary 3 is proven in [6] and it is connected with the result of Peachey in [5].

Another way of generalization of Theorem 1 given in [6] is obtained by putting $x = u^\alpha$ and $y = v^\beta$. We repeat this result and state equivalent inequality of Hardy-Hilbert type:

COROLLARY 4. *If* $\lambda > 0$, $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then the following inequalities hold and are equivalent*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)dxdy}{(x^\alpha + y^\beta)^\lambda} < C_4 \left(\int_0^\infty x^{\alpha(2-\lambda-p+p(A_1-A_2))+p-1} f^p(x)dx \right)^{\frac{1}{p}} \times \left(\int_0^\infty x^{\beta(2-\lambda-q+q(A_2-A_1))+q-1} g^q(x)dx \right)^{\frac{1}{q}},$$

$$\int_0^\infty y^{\beta((\lambda-1)(p-1)+p(A_1-A_2)+1)-1} \left(\int_0^\infty \frac{f(x)dx}{(x^\alpha + y^\beta)^\lambda} \right)^p dy < C_4^p \int_0^\infty x^{\alpha(2-\lambda-p+p(A_1-A_2))+p-1} f^p(x)dx,$$

where $C_4 = \alpha^{-\frac{1}{q}} \beta^{-\frac{1}{p}} \Big(B(1 - A_2 p, \lambda - 1 + A_2 p) \Big)^{\frac{1}{p}} \Big(B(1 - A_1 q, \lambda - 1 + A_1 q) \Big)^{\frac{1}{q}}$, *while* $A_1 \in (\frac{1-\lambda}{q}, \frac{1}{q})$ *and* $A_2 \in (\frac{1-\lambda}{p}, \frac{1}{p})$.

3. Discrete case

Now we state analogue results for discrete case.

THEOREM 2. *If* $\lambda > 0$, $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then the following inequalities* *hold and are equivalent*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_n b_m}{(n+m)^{\lambda}} < C_1\left(\sum_{n=1}^{\infty} n^{1-\lambda+p(A_1-A_2)}a_n^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty} m^{1-\lambda+q(A_2-A_1)}b_m^q\right)^{\frac{1}{q}} \tag{7}$$

and

$$\sum_{m=1}^{\infty} m^{(\lambda-1)(p-1)+p(A_1-A_2)}\left(\sum_{n=1}^{\infty}\frac{a_n}{(n+m)^{\lambda}}\right)^p < C_1^p\sum_{n=1}^{\infty} n^{1-\lambda+p(A_1-A_2)}a_n^p, \tag{8}$$

where $C_1 = \left(B(1-A_2p,\lambda-1+A_2p)\right)^{\frac{1}{p}}\left(B(1-A_1q,\lambda-1+A_1q)\right)^{\frac{1}{q}}$, $A_1 \in (\frac{1-\lambda}{q},\frac{1}{q})$, $A_2 \in (\frac{1-\lambda}{p},\frac{1}{p})$ *and* $A_1,\ A_2 \geqslant 0$.

Proof. First, we prove inequality (7).

Applying Hölder's inequality we obtain

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_n b_m}{(n+m)^{\lambda}} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_n\frac{n^{A_1}}{m^{A_2}}b_m\frac{m^{A_2}}{n^{A_1}}}{(n+m)^{\lambda}}$$
$$\leqslant \left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{n^{A_1}}{m^{A_2}}\right)^p\frac{a_n^p}{(n+m)^{\lambda}}\right)^{\frac{1}{p}}\cdot\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{m^{A_2}}{n^{A_1}}\right)^q\frac{b_m^q}{(n+m)^{\lambda}}\right)^{\frac{1}{q}}.$$

Now, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{n^{A_1}}{m^{A_2}}\right)^p\frac{a_n^p}{(n+m)^{\lambda}} = \sum_{n=1}^{\infty}a_n^p n^{(A_1-A_2)p}\sum_{m=1}^{\infty}\left(\frac{n}{m}\right)^{A_2p}\frac{1}{(n+m)^{\lambda}}$$
$$< \sum_{n=1}^{\infty}a_n^p n^{(A_1-A_2)p}\int_0^{\infty}\left(\frac{n}{x}\right)^{A_2p}\frac{dx}{(n+x)^{\lambda}}$$
$$= \sum_{n=1}^{\infty}a_n^p n^{1-\lambda+(A_1-A_2)p}\int_0^{\infty}\frac{t^{-A_2p}}{(1+t)^{\lambda}}dt$$
$$= \frac{\Gamma(1-A_2p)\Gamma(\lambda-1+A_2p)}{\Gamma(\lambda)}\sum_{n=1}^{\infty}a_n^p n^{1-\lambda+(A_1-A_2)p},$$

and analogously

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{m^{A_2}}{n^{A_1}}\right)^q\frac{b_m^q}{(n+m)^{\lambda}} < \sum_{m=1}^{\infty}b_m^q m^{(A_2-A_1)q}\int_0^{\infty}\left(\frac{m}{x}\right)^{A_1q}\frac{dx}{(m+x)^{\lambda}}$$
$$= \sum_{m=1}^{\infty}b_m^q m^{1-\lambda+(A_2-A_1)q}\int_0^{\infty}\frac{t^{-A_1q}}{(1+t)^{\lambda}}dt$$
$$= \frac{\Gamma(1-A_1q)\Gamma(\lambda-1+A_1q)}{\Gamma(\lambda)}\sum_{m=1}^{\infty}b_m^q m^{1-\lambda+(A_2-A_1)q},$$

since, for $A_2 \geqslant 0$, the function $f(x) = \left(\frac{n}{x}\right)^{A_2 p} \frac{1}{(n+x)^\lambda}$ is decreasing and for $A_1 \geqslant 0$, the function $g(x) = \left(\frac{m}{x}\right)^{A_1 p} \frac{1}{(m+x)^\lambda}$ is also decreasing.

Taking into account that the gamma-function is defined for positive reals, we obtain the conditions $A_1 \in \left(\frac{1-\lambda}{q}, \frac{1}{q}\right)$, $A_2 \in \left(\frac{1-\lambda}{p}, \frac{1}{p}\right)$.

Let's show that (7) and (8) are equivalent.

First, assume that inequality (7) is valid.

By putting

$$b_m = m^{(\lambda-1)(p-1)+p(A_1-A_2)} \left(\sum_{n=1}^{\infty} \frac{a_n}{(n+m)^\lambda} \right)^{p-1}$$

taking into account $\frac{1}{p} + \frac{1}{q} = 1$ and using (7) we have

$$\begin{aligned}
\sum_{m=1}^{\infty} m^{(\lambda-1)(p-1)+p(A_1-A_2)} \left(\sum_{n=1}^{\infty} \frac{a_n}{(n+m)^\lambda} \right)^p &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n b_m}{(n+m)^\lambda} \\
< C_1 \left(\sum_{n=1}^{\infty} n^{1-\lambda+p(A_1-A_2)} a_n^p \right)^{\frac{1}{p}} &\left(\sum_{m=1}^{\infty} m^{1-\lambda+q(A_2-A_1)} b_m^q \right)^{\frac{1}{q}} \\
= C_1 \left(\sum_{n=1}^{\infty} n^{1-\lambda+p(A_1-A_2)} a_n^p \right)^{\frac{1}{p}} &\times \\
\times \left(\sum_{m=1}^{\infty} m^{(\lambda-1)(p-1)+p(A_1-A_2)} \right. &\left. \left(\sum_{n=1}^{\infty} \frac{a_n}{(n+m)^\lambda} \right)^p \right)^{\frac{1}{q}}
\end{aligned}$$

wherefrom we have (8).

Now we assume that inequality (8) is valid. Applying Hölder's inequality and (8) we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n b_m}{(n+m)^\lambda} \\
&\quad = \sum_{m=1}^{\infty} \left(m^{-\frac{1-\lambda+q(A_2-A_1)}{q}} \sum_{n=1}^{\infty} \frac{a_n}{(n+m)^\lambda} \right) m^{\frac{1-\lambda+q(A_2-A_1)}{q}} b_m \\
&\quad \leqslant \left(\sum_{m=1}^{\infty} m^{-\frac{p}{q}((1-\lambda)+q(A_2-A_1))} \left(\sum_{n=1}^{\infty} \frac{a_n}{(n+m)^\lambda} \right)^p \right)^{\frac{1}{p}} \left(\sum_{m=1}^{\infty} m^{1-\lambda+q(A_2-A_1)} b_m^q \right)^{\frac{1}{q}} \\
&\quad = \left(\sum_{m=1}^{\infty} m^{(p-1)(\lambda-1)+p(A_1-A_2)} \left(\sum_{n=1}^{\infty} \frac{a_n}{(n+m)^\lambda} \right)^p \right)^{\frac{1}{p}} \left(\sum_{m=1}^{\infty} m^{1-\lambda+q(A_2-A_1)} b_m^q \right)^{\frac{1}{q}} \\
&\quad < C_1 \left(\sum_{n=1}^{\infty} n^{1-\lambda+p(A_1-A_2)} a_n^p \right)^{\frac{1}{p}} \left(\sum_{m=1}^{\infty} m^{1-\lambda+q(A_2-A_1)} b_m^q \right)^{\frac{1}{q}}
\end{aligned}$$

wherefrom we have (7). □

Now we can obtain Theorem B as a special case of Theorem 2 by putting $A_1 = A_2 = \frac{2-\lambda}{pq}$.

Further, for $A_1 = \frac{2-\lambda}{2q}$ and $A_2 = \frac{2-\lambda}{2p}$ we obtain

COROLLARY 5. *If* $0 < \lambda \leqslant 2$, $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then the following inequalities hold and are equivalent*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_n b_m}{(n+m)^{\lambda}} < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)\left(\sum_{n=1}^{\infty} n^{p-1-\frac{p\lambda}{2}} a_n^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty} m^{q-1-\frac{q\lambda}{2}} b_m^q\right)^{\frac{1}{q}},$$

$$\sum_{m=1}^{\infty} m^{\frac{p\lambda}{2}-1}\left(\sum_{n=1}^{\infty} \frac{a_n}{(n+m)^{\lambda}}\right)^p < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)^p \sum_{n=1}^{\infty} n^{p-1-\frac{p\lambda}{2}} a_n^p.$$

On the other hand, if we put $A_1 = \frac{1-b}{q} - \frac{1}{pq} \geqslant 0$, and $A_2 = \frac{1-c}{p} - \frac{1}{pq} \geqslant 0$, we obtain the following result

COROLLARY 6. *If* $\lambda > 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < b + \frac{1}{p} < \min\{\lambda, 1\}$ *and* $0 < c + \frac{1}{q} < \min\{\lambda, 1\}$, *then the following inequalities hold and are equivalent*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_n b_m}{(n+m)^{\lambda}} < C_2\left(\sum_{n=1}^{\infty} n^{(p-1)(1-b)+c-\lambda} a_n^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty} m^{(q-1)(1-c)+b-\lambda} b_m^q\right)^{\frac{1}{q}},$$

$$\sum_{m=1}^{\infty} m^{(\lambda-b)(p-1)+c-1}\left(\sum_{n=1}^{\infty} \frac{a_n}{(n+m)^{\lambda}}\right)^p < C_2^p \sum_{n=1}^{\infty} n^{(p-1)(1-b)+c-\lambda} a_n^p,$$

where $C_2 = \left(B(c+\frac{1}{q}, \lambda - c - \frac{1}{q})\right)^{\frac{1}{p}}\left(B(b+\frac{1}{p}, \lambda - b - \frac{1}{p})\right)^{\frac{1}{q}}$

REMARK. The conditions $A_1 \geqslant 0$ and $A_2 \geqslant 0$ imply $b + \frac{1}{p} \leqslant 1$ and $c + \frac{1}{q} \leqslant 1$.

Now if we put $\lambda = b + c + 1$ we obtain

COROLLARY 7. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $b + \frac{1}{p} > 0$ *and* $c + \frac{1}{q} > 0$, *then the following inequalities hold and are equivalent*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_n b_m}{(n+m)^{b+c+1}} < C_3\left(\sum_{n=1}^{\infty} n^{p(1-b)-2} a_n^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty} m^{q(1-c)-2} b_m^q\right)^{\frac{1}{q}},$$

$$\sum_{m=1}^{\infty} m^{p(c+1)-2}\left(\sum_{n=1}^{\infty} \frac{a_n}{(n+m)^{b+c+1}}\right)^p < C_3^p \sum_{n=1}^{\infty} n^{p(1-b)-2} a_n^p,$$

where $C_3 = B\big(b + \frac{1}{p}, c + \frac{1}{q}\big)$.

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(Received October 10, 2003)

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