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INEQUALITIES INVOLVING CERTAIN FAMILIES OF INTEGRAL AND CONVOLUTION OPERATORS

YONG CHAN KIM AND H. M. SRIVASTAVA

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Abstract. The main object of the present paper is to derive several strict inequalities associated with some families of integral and convolution operators which are defined for the class of normalized analytic functions in the open unit disk. A number of interesting consequences of these inequalities are also considered.

1. Introduction and Definitions

Let $\mathscr{A}$ denote the class of functions f *normalized* in the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Also, for functions f given by (1.1) and g given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.2}$$

we denote by $(f * g)(z)$ the *Hadamard product* or *convolution* of f and g, defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z). \tag{1.3}$$

Recently, Komatu [7] introduced a certain integral operator $\mathscr{L}_a^{\lambda}$ $(a > 0;\ \lambda \geqslant 0)$ defined by

$$\mathcal{L}_a^{\lambda} f(z) := \frac{a^{\lambda}}{\Gamma(\lambda)} \int_0^1 t^{a-2} \left(\log \frac{1}{t}\right)^{\lambda - 1} f(zt)\, dt \tag{1.4}$$

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$$(z \in \mathbb{U};\ a > 0;\ \lambda \geqslant 0;\ f \in \mathcal{A}).$$

Thus, if $f \in \mathcal{A}$ is of the form (1.1), it is easily seen from (1.4) that

$$\mathcal{L}_a^\lambda f(z) = z + \sum_{n=2}^{\infty} \left(\frac{a}{a+n-1} \right)^\lambda a_n z^n \qquad (a > 0;\ \lambda \geqslant 0). \tag{1.5}$$

The *special* case $a = 2$ of the integral operator $\mathcal{L}_a^\lambda$ is essentially the *multiplier transformation* (or *fractional integral* and *fractional derivative*), which was considered by Flett [3] (and, subsequently, by Jung *et al.* [5]). For the *general* integral operator $\mathcal{L}_a^\lambda$, it is not difficult to deduce from (1.5) that

$$z \left(\mathcal{L}_a^{\lambda+1} f(z) \right)' = a\mathcal{L}_a^\lambda f(z) - (a-1)\, \mathcal{L}_a^{\lambda+1} f(z) \qquad (a > 0;\ \lambda \geqslant 0). \tag{1.6}$$

Earlier in 1975, Ruscheweyh [9] introduced another linear operator

$$\mathcal{D}^\alpha : \mathcal{A} \to \mathcal{A}$$

defined by the Hadamard product (or convolution) as follows:

$$\mathcal{D}^\alpha f(z) := \frac{z}{(1-z)^{\alpha+1}} * f(z) \qquad (z \in \mathbb{U};\ \alpha > -1;\ f \in \mathcal{A}), \tag{1.7}$$

which, for $f \in \mathcal{A}$ of the form (1.1), implies that

$$\mathcal{D}^\alpha f(z) = z + \sum_{n=2}^{\infty} \frac{(\alpha)_n}{\alpha \cdot (n-1)!} a_n z^n \qquad (\alpha > -1), \tag{1.8}$$

where $(\mu)_\nu$ denotes the Pochhammer symbol (or the *shifted* factorial, since

$$(1)_n = n! \qquad \text{for} \qquad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$$

defined, in terms of the Gamma function, by

$$(\mu)_\nu := \frac{\Gamma(\mu+\nu)}{\Gamma(\mu)} = \begin{cases} 1 & (\nu = 0;\ \mu \in \mathbb{C} \setminus \{0\}) \\ \mu(\mu+1)\cdots(\mu+n-1) & (\nu = n \in \mathbb{N};\ \mu \in \mathbb{C}). \end{cases} \tag{1.9}$$

Clearly, we have

$$\mathcal{D}^0 f(z) = f(z), \qquad \mathcal{D}^1 f(z) = zf'(z), \tag{1.10}$$

and (in general)

$$\mathcal{D}^n f(z) = \frac{z\left(z^{n-1} f(z)\right)^{(n)}}{n!} \qquad (n \in \mathbb{N}_0). \tag{1.11}$$

Furthermore, it follows from (1.8) that (*cf.*, *e.g* ., [6, p. 513, Equation (3.4)]; see also [1])

$$z\left(\mathcal{D}^\alpha f(z)\right)' = (\alpha+1)\, \mathcal{D}^{\alpha+1} f(z) - \alpha \mathcal{D}^\alpha f(z) \qquad (\alpha > -1). \tag{1.12}$$

Next we recall the classes Φ *and* Ψ of complex-valued functions, which are given by Definition 1 and Definition 2 below.

DEFINITION 1. (*cf.* Aouf *et al.* [2]). Let Φ be the set of complex-valued functions

$$\phi(r,s,t) : \mathbb{C}^3 \to \mathbb{C}$$

satisfying each of the following conditions:

(i) $\phi(r,s,t)$ is continuous in a domain $\mathbb{D} \subset \mathbb{C}^3$;

(ii) $(0,0,0) \in \mathbb{D}$ and $|\phi(0,0,0)| < 1$;

(iii) $\left|\phi\left(e^{i\theta}, \frac{k+\alpha}{\alpha+1}\, e^{i\theta}, \frac{(\alpha+1)(\alpha+2k)\, e^{i\theta}+L}{(\alpha+1)(\alpha+2)}\right)\right| > 1$

when

$$\left(e^{i\theta}, \frac{k+\alpha}{\alpha+1}\, e^{i\theta}, \frac{(\alpha+1)(\alpha+2k)\, e^{i\theta}+L}{(\alpha+1)(\alpha+2)}\right) \in \mathbb{D}$$

with

$$\Re\left(e^{-i\theta} L\right) \geqslant k(k-1) \qquad (\theta \in \mathbb{R};\ k \geqslant 1).$$

DEFINITION 2. (*cf.* Aouf *et al.* [2]). Let Ψ be the set of complex-valued functions

$$\psi(r,s,t) : \mathbb{C}^3 \to \mathbb{C}$$

satisfying each of the following conditions:

(i) $\psi(r,s,t)$ is continuous in a domain $\mathbb{D} \subset \mathbb{C}^3$;

(ii) $(0,0,0) \in \mathbb{D}$ and $|\psi(0,0,0)| < 1$;

(iii) $\left|\psi\left(e^{i\theta}, \frac{k+a-1}{a}\, e^{i\theta}, \frac{\left[(k-1)(2a-1)+a^2\right] e^{i\theta}+L}{a^2}\right)\right| > 1$

when

$$\left(e^{i\theta}, \frac{k+a-1}{a}\, e^{i\theta}, \frac{\left[(k-1)(2a-1)+a^2\right] e^{i\theta}+L}{a^2}\right) \in \mathbb{D}$$

with

$$\Re\left(e^{-i\theta} L\right) \geqslant k(k-1) \qquad (\theta \in \mathbb{R};\ k \geqslant 1).$$

Based upon the function classes Φ and Ψ given by Definition 1 and Definition 2, respectively, we aim here at deriving several strict inequalities associated with the convolution (or *Ruscheweyh derivative*) operator $\mathcal{D}^{\alpha}$ *and* the integral operator $\mathcal{L}_a^{\lambda}$.

The following known result will be required in our present investigation.

LEMMA 1. *(cf., e.g.,* Miller and Mocanu [8]; see also Jack [4]*). Let the (nonconstant) function* $w(z)$ *be analytic in* $\mathbb{U}$ *with*

$$w(0) = 0 \qquad \text{and} \qquad w(z) \not\equiv 0 \qquad (z \in \mathbb{U}). \tag{1.13}$$

If $|w(z)|$ *attains its maximum value on the circle* $|z| = r < 1$ *at a point* $z_0 \in \mathbb{U}$*, then*

$$z_0\, w'(z_0) = k w(z_0) \tag{1.14}$$

and

$$\Re\left(1+\frac{z_0\, w''(z_0)}{w'(z_0)}\right) \geqslant k, \tag{1.15}$$

where k *is a real number and* $k \geqslant 1$.

The assertion (1.14) of Lemma 1 is popularly known as *Jack's Lemma*.

2. Inequalities Associated with the Convolution Operator $\mathcal{D}^\alpha$

By using (1.4) and Lemma 1, we prove

THEOREM 1. *Let* $\alpha > -1$. *Suppose also that* $f(z) \in \mathcal{A}$ *and* $\phi(r,s,t) \in \Phi$. *If*

$$\left(\mathcal{D}^\alpha f(z), \mathcal{D}^{\alpha+1} f(z), \mathcal{D}^{\alpha+2} f(z)\right) \in \mathbb{D} \subset \mathbb{C}^3 \tag{2.1}$$

and

$$\left|\phi\left(\mathcal{D}^\alpha f(z), \mathcal{D}^{\alpha+1} f(z), \mathcal{D}^{\alpha+2} f(z)\right)\right| < 1, \tag{2.2}$$

then

$$|\mathcal{D}^\alpha f(z)| < 1 \qquad (z \in \mathbb{U};\ \alpha > -1).$$

Proof. If we define

$$w(z) := \mathcal{D}^\alpha f(z) \qquad (\alpha > -1;\, f \in \mathcal{A}),$$

then [*cf.* Equation (1.13)]

$$w(z) \in \mathcal{A} \qquad \text{and} \qquad w(z) \not\equiv 0 \qquad (z \in \mathbb{U}).$$

In view of (1.12), we thus obtain

$$\mathcal{D}^{\alpha+1} f(z) = \frac{1}{\alpha+1}\left[\alpha w(z) + z w'(z)\right]$$

and

$$\mathcal{D}^{\alpha+2} f(z) = \frac{1}{\alpha+2}\left(\alpha w(z) + 2z w'(z) + \frac{z^2\, w''(z)}{\alpha+1}\right).$$

Assume that $z_0 = r_0\ e^{i\theta}$ $(r_0 < 1;\ \theta \in \mathbb{R})$ and

$$|w(z_0)| = \max_{|z| \leqslant r_0} |w(z)| = 1.$$

Then, writing $w(z_0) = e^{i\theta}$ and using (1.14), we have

$$\mathcal{D}^\alpha f(z_0) = w(z_0) = e^{i\theta},$$

$$\mathcal{D}^{\alpha+1} f(z_0) = \frac{\alpha+k}{\alpha+1}\, w(z_0) = \frac{\alpha+k}{\alpha+1}\, e^{i\theta},$$

and

$$\mathcal{D}^{\alpha+2} f(z_0) = \frac{1}{\alpha+2}\left((\alpha+2k)\, w(z_0) + \frac{z_0^2\, w''(z_0)}{\alpha+1}\right) = \frac{(\alpha+1)(\alpha+2k)\, e^{i\theta} + L}{(\alpha+1)(\alpha+2)},$$

where

$$L = z_0^2\, w''(z_0) \qquad \text{and} \qquad k \geqslant 1.$$

Furthermore, by applying (1.15), we see that

$$\Re\left(\frac{z_0\, w''(z_0)}{w'(z_0)}\right) = \Re\left(\frac{z_0^2\, w''(z_0)}{ke^{i\theta}}\right) \geqslant k-1$$

or

$$\Re\left(e^{-i\theta} L\right) \geqslant k(k-1) \qquad (\theta \in \mathbb{R};\ k \geqslant 1).$$

Since $\phi(r,s,t) \in \Phi,$ we also have

$$\begin{aligned}&\left|\phi\left(\mathcal{D}^{\alpha} f(z), \mathcal{D}^{\alpha+1} f(z), \mathcal{D}^{\alpha+2} f(z)\right)\right| \\ &\qquad = \left|\phi\left(e^{i\theta}, \frac{k+\alpha}{\alpha+1}\, e^{i\theta}, \frac{(\alpha+1)(\alpha+2k)\, e^{i\theta}+L}{(\alpha+1)(\alpha+2)}\right)\right| > 1,\end{aligned}$$

which contradicts the hypothesis (2.2) of Theorem 1. This implies that

$$|w(z)| = |\mathcal{D}^{\alpha} f(z)| < 1 \qquad (z \in \mathbb{U};\ \alpha > -1),$$

which completes the proof of Theorem 1. □

Throughout this paper, we define a projection map

$$p : \mathbb{C}^3 \to \mathbb{C}$$

by

$$p(r,s,t) = s.$$

Then the function $p \in \Phi$, so that from Theorem 1 we can deduce

COROLLARY 1. *Let the domain* $\mathbb{D}$ *be given as in Definition* 1. *Also let* $\alpha > -1$ *and suppose that* $f(z) \in \mathcal{A}$ *satisfies the following inclusion relation:*

$$\left(\mathcal{D}^{\alpha} f(z), \mathcal{D}^{\alpha+1} f(z), \mathcal{D}^{\alpha+2} f(z)\right) \in \mathbb{D} \subset \mathbb{C}^3. \tag{2.3}$$

If

$$\left|\mathcal{D}^{\alpha+1} f(z)\right| < 1 \qquad (z \in \mathbb{U}),$$

then

$$|\mathcal{D}^{\alpha} f(z)| < 1 \qquad (z \in \mathbb{U};\ \alpha > -1).$$

3. Inequalities Associated with the Integral Operator $\mathcal{L}_a^{\lambda}$

By making use of (1.6) instead of (1.12), we now prove

THEOREM 2. *Let* $a > 0$ *and* $\lambda \geqslant 1$. *Suppose also that* $f(z) \in \mathcal{A}$ *and* $\psi(r,s,t) \in \Psi$. *If*

$$\left(\mathcal{L}_a^{\lambda} f(z), \mathcal{L}_a^{\lambda-1} f(z), \mathcal{L}_a^{\lambda-2} f(z)\right) \in \mathbb{D} \subset \mathbb{C}^3 \tag{3.1}$$

and

$$\left|\psi\left(\mathcal{L}_a^{\lambda} f(z), \mathcal{L}_a^{\lambda-1} f(z), \mathcal{L}_a^{\lambda-2} f(z)\right)\right| < 1, \tag{3.2}$$

then

$$\left|\mathcal{L}_a^{\lambda} f(z)\right| < 1 \qquad (z \in \mathbb{U};\ a > 0;\ \lambda \geqslant 2).$$

Proof. If we set

$$w(z) := \mathcal{L}_a^{\lambda} f(z) \qquad (a > 1,\ \lambda \geqslant 2),$$

then [*cf.* Equation (1.13)]

$$w(z) \in \mathcal{A} \qquad \text{and} \qquad w(z) \not\equiv 0 \qquad (z \in \mathbb{U}).$$

By means of (1.6), we also have

$$\mathcal{L}_a^{\lambda-1} f(z) = \frac{1}{a}\left[(a-1)\,w(z) + zw'(z)\right]$$

and

$$\mathcal{L}_a^{\lambda-2} f(z) = \frac{1}{a^2}\left[(a-1)^2\ w(z) + (2a-1)\,zw'(z) + z^2\ w''(z)\right].$$

Suppose that $z_0 = r_0\ e^{i\theta}$ $(r_0 < 1;\ \theta \in \mathbb{R})$ and

$$|w(z_0)| = \max_{|z| \leqslant r_0} |w(z)| = 1.$$

Then, letting $w(z_0) = e^{i\theta}$ and using (1.14), we obtain

$$\mathcal{L}_a^{\lambda} f(z_0) = w(z_0) = e^{i\theta},$$

$$\mathcal{L}_a^{\lambda-1} f(z_0) = \frac{k+a-1}{a}\ e^{i\theta},$$

and

$$\mathcal{L}_a^{\lambda-2} f(z_0) = \frac{1}{a^2}\left(\left[(2a-1)(k-1) + a^2\right] e^{i\theta} + L\right),$$

where

$$L = z_0^2\ w'(z_0) \qquad \text{and} \qquad k \geqslant 1.$$

Moreover, we find from (1.15) that

$$\Re\left(\frac{z_0\ w''(z_0)}{w'(z_0)}\right) = \Re\left(\frac{z_0^2\ w''(z_0)}{ke^{i\theta}}\right) \geqslant k-1$$

or

$$\Re\left(e^{-i\theta}\ L\right) \geqslant k(k-1) \qquad (\theta \in \mathbb{R};\ k \geqslant 1).$$

Since $\psi(r,s,t) \in \Psi$, we also have

$$\left|\psi\left(\mathcal{L}_a^{\lambda} f(z), \mathcal{L}_a^{\lambda-1} f(z), \mathcal{L}_a^{\lambda-2} f(z)\right)\right|$$
$$= \left|\psi\left(e^{i\theta}, \frac{k+a-1}{a}\ e^{i\theta}, \frac{\left[(2a-1)(k-1)+a^2\right] e^{i\theta} + L}{a^2}\right)\right| > 1,$$

which contradicts the hypothesis (3.2) of Theorem 2. Therefore, we conclude that

$$|w(z)| = \left|\mathcal{L}_a^\lambda f(z)\right| < 1 \qquad (z \in \mathbb{U};\ a > 0;\ \lambda \geqslant 2),$$

which completes the proof of Theorem 2. □

REMARK. By setting $a = 2$ in Theorem 2, we obtain a result due to Aouf *et al.* [2, Theorem 1].

Since the aforementioned function $p \in \Psi$, by using Theorem 2, we finally obtain

COROLLARY 2. *Let the domain* $\mathbb{D}$ *be given as in Definition* 2*. Also let* $a > 0$ *and* $\lambda \geqslant 2$*, and suppose that the function* $f(z) \in \mathcal{A}$ *satisfies the following inclusion relation:*

$$\left(\mathcal{L}_a^\lambda f(z), \mathcal{L}_a^{\lambda-1} f(z), \mathcal{L}_a^{\lambda-2} f(z)\right) \in \mathbb{D} \subset \mathbb{C}^3. \tag{3.3}$$

If

$$\left|\mathcal{L}_a^{\lambda-1} f(z)\right| < 1 \qquad (z \in \mathbb{U}),$$

then

$$\left|\mathcal{L}_a^{\lambda+j} f(z)\right| < 1 \qquad (j = 0, 1, 2, \ldots;\ z \in \mathbb{U};\ a > 0;\ \lambda \geqslant 2).$$

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Yong Chan Kim
Department of Mathematics
College of Education
Yeungnam University
214-1 Daedong, Gyongsan 712-749
Korea
e-mail: kimyc@ynucc.yeungnam.ac.kr

H. M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3P4
Canada
e-mail: harimsri@math.uvic.ca