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AUXILIARY PRINCIPLE TECHNIQUE FOR MULTIVALUED MIXED QUASI–VARIATIONAL INEQUALITIES

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Abstract. In this paper, we suggest and analyze a class of predictor-corrector methods for solving mixed quasi variational inequalities by using the auxiliary principle technique. We prove that the convergence of these predictor-corrector type methods requires only the partial relaxed strongly monotonicity, which is a weaker condition than co-coercivity. Since the mixed quasi variational inequalities include (quasi) variational inequalities as special cases, our results continue to hold for these problems. Our results represent an improvement and refinement of the previously known results.

1. Introduction

Variational inequalities theory has emerged as a significant and important branch of applicable mathematics. This theory provides a general and unified treatment of equilibrium problems arising in economics, finance, transportation, elasticity, optimization, structural analysis and operations research, see [1-24]. Variational inequalities have been generalized and extended in several directions using novel and innovative techniques to tackle some complicated and complex problems. A useful and important generalization of variational inequalities is called the mixed quasi variational inequalities, which has several applications in fluid flow through porous media, elasticity and structural analysis involving the nonlinear bifunction. Clearly this technique combines both theoretical and algorithmic advances with a new domain of applications. As a result of interaction between different branches of applied and engineering sciences, we now have a variety of techniques for solving variational inequalities. Though the problems in each of these areas may look completely different, the resulting algorithms can be very closely related. Due to the presence of the nonlinear bifunction, projection methods, its variant forms and Wiener-Hopf equations can not be extended for the mixed quasi variational inequalities. This fact has provided some motivation to develop other techniques for solving these mixed quasi variational inequalities. If the bifunction is a proper, convex and lower semicontinuous function with respect to the first argument, then it has been shown in [18] that the mixed quasi variational inequalities are equivalent to the fixed-point and resolvent equations by using the resolvent operator technique.

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This alternative equivalent formulation has been used to suggest and analyze several iterative methods for solving mixed quasi variational inequalities and related optimization problems. It has been shown that the convergence analysis of these iterative methods requires the operator to be both strongly monotone and Lipschitz continuous. Secondly, it is very difficult to compute the resolvent of the operator except in some simple cases. These strict conditions rule out many applications to important problems. To overcome these difficulties, one uses the auxiliary principle technique, the origin of which can be traced back to Lions and Stampacchia [10]. This technique has been used by Glowinski, Lions and Tremolieres [6] to study the existence of a solution of a class of variational inequalities , known as mixed variational inequalities. In recent years, Noor [19-21] has used this technique to suggest various type of iterative methods for mixed variational inequalities. In this paper, we extend this technique for mixed quasi variational inequalities involving the nonlinear bifunction and suggest some predictor-corrector type methods. We also show that one-step, two-step and three-step splitting forward and backward methods for solving variational inequalities can be obtained as special cases from the proposed methods. In the implementation of theses methods, one does not have to find the projection or the resolvent of the operator, which is an other advantage of these proposed methods. We prove that the convergence of these predictor-corrector methods requires only the relaxed strongly monotonicity of the operator. It is worth mentioning that the relaxed strongly monotonicity implies co-coercivity, but not conversely. This shows that the relaxed strongly mononicity is a weaker condition than co-coercivity. Consequently our results improve the previously known results of Zhu and Marcotte [24] for solving classical variational inequalities. Since the mixed quasi variational inequalities include (quasi) variational inequalities as special cases, our results continue to hold for these problems. Our results can be viewed as improvement and extension of the results of Noor [19-21] for solving (mixed) variational inequalities and related complementarity problems

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$, respectively. Let $C(H)$ be the family of all non-empty compact subsets of H. Let $T : H \longrightarrow C(H)$ be a multivalued operator and $g : H \longrightarrow H$ be a single-valued operator. Let K be a nonempty, closed and convex set in H.

Given a nonlinear bifunction $\varphi(.,.) : H \times H \longrightarrow R \cup \{+\infty\}$, we consider the problem of finding $u \in H, \nu \in T(u)$, such that

$$\langle \nu, g(v) - g(u) \rangle + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geqslant 0, \qquad \forall v \in H. \tag{2.1}$$

Inequality of type (2.1) is called the *multivalued mixed quasi variational inequality*. It can be shown that a wide class of multivalued odd order and nonsymmetric free, obstacle, moving, equilibrium and optimization problems arising in pure and applied sciences can be studied via the multivalued mixed quasi variational inequalities (2.1), see, for example, Noor [18].

We note that, if $T: H \longrightarrow H$ is a single-valued operator, then problem (2.1) is equivalent to finding $u \in H,\ g(u) \in K$ such that

$$\langle Tu, g(v) - g(u)\rangle + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geqslant 0, \quad \forall v \in H, \tag{2.2}$$

which is known as the general mixed quasi variational inequality.

If $\varphi(g(v), g(u)) = \varphi(g(v)), \forall u \in H,$ is the indicator function of a closed convex set K in $H,$ then problem (2.1) is equivalent to finding $u \in H, g(u) \in K, \nu \in T(u)$ such that

$$\langle \nu, g(v) - g(u)\rangle \geqslant 0, \quad \forall g(v) \in K. \tag{2.3}$$

The problem is called the multivalued variational inequality. For the applications and numerical methods, see [17,20].

If T is a single-valued operator, then problem (2.3) is equivalent to finding $u \in H, g(u) \in K$ such that

$$\langle T(u), g(v) - g(u)\rangle \geqslant 0, \quad \forall g(v) \in K, \tag{2.4}$$

which is called the *general variational inequality* introduced and studied by Noor [12] in 1988. It turned out that odd-order, nonsymmetric free, moving and equilibrium problems can be studied in the unified framework of the general variational inequalities (2.4), see [15-21].

We remark that, if $g \equiv I$, the identity operator, then problem (2.4) is equivalent to finding $u \in K$ such that

$$\langle T(u), v - u\rangle \geqslant 0, \qquad \forall v \in K, \tag{2.5}$$

which is called the classical variational inequality introduced and studied by Stampacchia [23] in 1964. For a number of applications, numerical methods and related formulations, the reader is referred to M. Aslam Noor, K. Inaayat Noor and Th. M. Rassias [22] (among all other references cited in the paper).

If $K^* = \{u \in H : \langle u, v\rangle \geqslant 0, \forall v \in K\}$ is a polar cone of a convex cone K in $H,$ then problem (2.4) is equivalent to finding $u \in H$ such that

$$g(u) \in K, \quad T(u) \in K^*, \quad \text{and} \quad \langle T(u), g(u)\rangle = 0, \tag{2.6}$$

which is known as the general complementarity problem. We note that if $g(u) = u - m(u)$, where m is a point-to-point mapping, then problem(2.6) is called the quasi(implicit) complementarity problem, see the references for the formulation and numerical methods.

It is clear that problems (2.2)-(2.6) are special cases of the multivalued variational inequality (2.1). In brief, for a suitable and appropriate choice of the operators T, g, and the space H, one can obtain a wide class of variational inequalities and complementarity problems. This clearly shows that problem (2.1) is quite general and unifying one. Furthermore, problem (2.1) has many important applications in various branches of pure and applied sciences; see the references.

We also need the following well known results and concepts.

LEMMA 2.1. $\forall u, v \in H,$ *we have*

$$2\langle u, v\rangle = ||u+v||^2 - ||u||^2 - ||v||^2 \tag{2.7}$$

DEFINITION 2.1. $\forall u_1, u_2, z \in H, w_1 \in T(u_1), w_2 \in T(u_2)$, the multivalued operator $T : H \to C(H)$ is said to be:

i *g-partially relaxed strongly monotone*, iff there exists a constant $\alpha > 0$, such that

$$\langle w_1 - w_2, g(z) - g(u_2)\rangle \geqslant -\alpha||g(u_1) - g(z)||^2$$

ii *g-co-coercive*, iff there exists a constant $\mu > 0$, such that

$$\langle w_1 - w_2, g(u_1) - g(u_2)\rangle \geqslant \mu||w_1 - w_2||^2.$$

iii *M-Lipschitz continuous*, iff there exists a constant $\delta > 0$, such that

$$M(T(u_1), T(u_2)) \leqslant \delta||u_1 - u_2||,$$

where $M(.,.)$ is the Hausdorff metric on $C(H)$.

We remark that, if $z = u_1$, then g-partially relaxed strongly monotonicity is exactly g-monotonicity of the operator T. For $g \equiv I$, the indentity operator, Definition 2.1 reduces to the definition of partially relaxed strongly monotonicity and co-coercivity of the operator. It has been shown in [20] that partially relaxed strongly monotonicity implies co-coercivity, but the converse is not true. This implies that the partially relaxed strong monotonicity is a weaker condition than co-coercivity.

DEFINITION 2.2. $\forall u, v \in H,$ the bifunction $\varphi(u, v)$ is said to be *skew-symmetric,* if

$$\varphi(u,u) - \varphi(u,v) - \varphi(v,u) + \varphi(v,v) \geqslant 0.$$

Note that if the bifunction $\varphi(.,.)$ is bilinear, then $\varphi(.,.)$ is nonnegative.

3. Main Results

In this section, we suggest and analyze a new iterative method for solving the problem (2.1) by using the auxiliary principle technique of Glowinski, Lions and Tremolieres [6] as developed by Noor [19-21].

For a given $u \in H,$ consider the problem of finding a unique $w \in H, \eta \in T(w)$ satisfying the auxiliary variational inequality

$$\langle \rho\eta + g(w) - g(u), g(v) - g(w)\rangle + \varphi(g(v), g(w)) - \varphi(g(w), g(w)) \geqslant 0 \quad \forall v \in H, \tag{3.1}$$

where $\rho > 0$ is a constant.

We note that, if $w = u$, then clearly w is a solution of the multivalued variational inequality (2.1). This observation enables us to suggest the following predictor-corrector method for solving the multivalued mixed quasi variational inequalities (2.1).

ALGORITHM 3.1. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\begin{aligned} &\langle \rho\eta_n + g(u_{n+1}) - g(w_n), g(v) - g(u_{n+1})\rangle + \rho\varphi(g(v), g(u_{n+1})) \\ &-\rho\varphi(g(u_{n+1}), g(u_{n+1})) \geqslant 0, \quad \forall v \in H \end{aligned} \tag{3.2}$$

$$\eta_n \in T(w_n) : ||\eta_{n+1} - \eta_n|| \leqslant M(T(w_{n+1}), T(w_n)) \tag{3.3}$$

$$\begin{aligned} &\langle \beta\xi_n + g(w_n) - g(y_n), g(v) - g(w_n)\rangle + \beta\varphi(g(v), g(w_n)) \\ &-\beta\varphi(g(w_n), g(w_n)) \geqslant \quad 0, \quad \forall v \in H \end{aligned} \tag{3.4}$$

$$\xi_n \in T(y_n) : ||\xi_{n+1} - \xi_n|| \leqslant M(T(y_{n+1}), T(y_n)) \tag{3.5}$$

and

$$\begin{aligned} &\langle \mu\nu_n + g(y_n) - g(u_n), g(v) - g(y_n)\rangle + \mu\varphi(g(v), g(y_n)) \\ &-\mu\varphi(g(y_n), g(y_n)) \geqslant 0, \forall v \in H \end{aligned} \tag{3.6}$$

$$\nu_n \in T(u_n) : ||\nu_{n+1} - \nu_n|| \leqslant M(T(u_{n+1}), T(u_n)), \quad n = 0, 1, 2, \ldots \tag{3.7}$$

where $\rho > 0$, $\mu > 0$ and $\beta > 0$ are constants.

Note that, if $\varphi(v, u) = \varphi(v), \forall u \in H,$ is the indicator function of a closed convex set K in $H,$ then Algorithm 3.1 reduces to the following predictor-corrector method for solving the variational inequalities (2.3), which is due to Noor [20].

ALGORITHM 3.2. For a given $u_0 \in H$, compute u_{n+1} by the iterative schemes

$$\begin{aligned} &\langle \rho\eta_n + u_{n+1} - w_n, g(v) - g(u_{n+1})\rangle \geqslant 0, \quad \forall g(v) \in K, \\ &\eta_n \in T(w_n) : ||\eta_{n+1} - \eta_n|| \leqslant M(T(w_{n+1}), T(w_n)) \end{aligned}$$

$$\begin{aligned} &\langle \beta\xi_n + w_n - y_n, g(vI - g(w_n)\rangle \geqslant 0, \quad \forall g(v) \in K \\ &\xi_n \in T(y_n) : ||\xi_{n+1} - \xi_n|| \leqslant M(T(y_{n+1}), T(y_n)) \end{aligned}$$

$$\begin{aligned} &\langle \mu\nu_n + y_n - u_n, g(v) - g(y_n)\rangle \geqslant 0, \quad \forall g(v) \in K. \\ &\nu_n \in T(u_n) : ||\nu_{n+1} - \nu_n|| \leqslant M(T(u_{n+1}), T(u_n)), \quad n = 0, 1, 2 \ldots \end{aligned}$$

Using the technique of the projection, Algorithm 3.2 can be written as

ALGORITHM 3.3. For a given $u_0 \in H$, compute u_{n+1} such that $\eta_n \in T(w_n), \xi_n \in T(y_n), \nu_n \in T(u_n)$ by the iterative schemes

$$\begin{aligned} g(u_{n+1}) &= P_K[g(w_n) - \rho\eta_n], \\ g(w_n) &= P_K[g(y_n) - \beta\xi_n], \\ g(y_n) &= P_K[g(u_n) - \mu\nu_n], \quad n = 0, 1, 2 \ldots, \end{aligned}$$

Algorithm 3.3 is a three-step forward-backward splitting method for solving multivalued general variational inequalities (2.3). For the convergence analysis of Algorithm 3.3, see Noor [20].

If the operator T is single-valued, then Algorithm 3.3 collapses to the following predictor-corrector method for solving general variational inequalities (2.4).

ALGORITHM 3.4. For a given $u_0 \in H$, compute u_{n+1} by the iterative schemes

$$\begin{aligned}
&\langle \rho T(w_n) + g(u_{n+1}) - g(w_n), g(v) - g(u_{n+1})\rangle \geqslant 0, \quad \forall g(v) \in K \\
&\langle \beta T(y_n) + g(w_n) - g(y_n), g(v) - g(w_n)\rangle \geqslant 0, \quad \forall g(v) \in K \\
&\langle \mu T(u_n) + g(y_n) - g(u_n), g(v) - g(y_n)\rangle \geqslant 0, \quad \forall g(v) \in K
\end{aligned}$$

We remark that Algorithm 3.4 can be written in the equivalent form as

ALGORITHM 3.5. For a given $u_0 \in H$, compute u_{n+1} by the iterative schemes

$$\begin{aligned}
g(y_n) &= P_K[g(u_n) - \mu T u_n] \\
g(w_n) &= P_K[g(y_n) - \beta T(y_n)] \\
g(u_{n+1}) &= P_K[g(w_n) - \rho T(w_n)], \quad n = 0, 1, 2 \ldots
\end{aligned}$$

which can be written in the following form, if g is invertible,

$$g(u_{n+1}) = P_K[I - \rho T g^{-1}]P_K[I - \beta T g^{-1}]P_K[I - \mu T g^{-1}]g(u_n), \quad n = 0, 1, 2 \ldots$$

which is a three-step forward-backward splitting algorithms.

Algorithm 3.5 is similar to the so-called θ-scheme of Glowinski and Le Tallec [7], which they suggested by using the Lagrangian multiplier method. It has been shown in [7] that three-step schemes are numerically efficient. The convergence analysis of Algorithm 3.5 has been considered by Noor [20].

For a suitable choice of the operators and the space H, one can obtain various new and known methods for solving variational inequalities and complementarity problems.

For the convergence analysis of Algorithm 3.1, we need the following result. The analysis is in the spirit of Noor [19-21]. For the sake of completeness and to convey an idea of the techniques involved, we give its proof.

LEMMA 3.1. *Let $u \in H$ be the exact solution of (2.1) and u_{n+1} be the approximate solution obtained from Algorithm 3.1. If the operator $T : H \longrightarrow C(H)$ is a $g-$partially relaxed strongly monotone operator with constant $\alpha > 0$ and the bifunction $\varphi(.,.)$ is skew-symmetric, then*

$$||g(u_{n+1}) - g(u)||^2 \leqslant ||g(u_n) - g(u)||^2 - (1 - 2\rho\alpha)||g(u_{n+1}) - g(u_n)||^2. \quad (3.8)$$

Proof. Let $u \in H, \nu \in T(u)$ be solution of (1). Then

$$\begin{aligned}
&\langle \rho \nu, g(v) - g(u)\rangle + \rho\varphi(g(v), g(u)) - \rho\varphi(g(u), g(u)) \geqslant 0, \quad \forall v \in H \quad (3.9) \\
&\langle \beta \nu, g(v) - g(u)\rangle + \beta\varphi(g(v), g(u)) + \beta\varphi(g(u), g(u)) \geqslant 0, \quad \forall v \in H \quad (3.10) \\
&\langle \mu \nu, g(v) - g(u)\rangle + \mu\varphi(g(v), g(u))\varphi(g(u), g(u)) \geqslant 0, \quad \forall v \in H, \quad (3.11)
\end{aligned}$$

where $\rho > 0$, $\beta > 0$ and $\mu > 0$ are constants.

Now taking $v = u_{n+1}$ in (3.9) and $v = u$ in (3.2), we have

$$\langle \rho \nu, g(u_{n+1}) - g(u)\rangle + \rho\varphi(g(u_{n+1}), g(u)) - \rho\varphi(g(u), g(u)) \geqslant 0 \tag{3.12}$$

and

$$\begin{aligned}&\langle \rho\eta_n + g(u_{n+1}) - g(w_n), g(u) - g(u_{n+1})\rangle \\ &+\rho\varphi(g(u), g(u_{n+1}) - \rho\varphi(g(u_{n+1}), g(u_{n+1})) \geqslant 0.\end{aligned} \tag{3.13}$$

Adding (3.12) and (3.13), we have

$$\begin{aligned}\langle g(u_{n+1}) - g(w_n), g(u) - g(u_{n+1})\rangle &\geqslant \rho\langle \eta_n - \nu, g(u_{n+1}) - g(u)\rangle + \rho\{\varphi(g(u), g(u)) \\ &\quad -\varphi(g(u), g(u_{n+1})) - \varphi(g(u_{n+1}), g(u)) \\ &\quad +\varphi(g(u_{n+1}), g(u_{n+1}))\} \\ &\geqslant -\alpha\rho||g(u_{n+1}) - g(w_n)||^2,\end{aligned} \tag{3.14}$$

where we have used the fact that T is g-partially relaxed strongly monotone with constant $\alpha > 0$. and the skew-symmetry of the bifunction $\varphi(.,.)$.

Setting $u = g(u) - g(u_{n+1})$ and $v = g(u_{n+1}) - g(w_n)$ in (2.7), we obtain

$$\begin{aligned}2\langle g(u_{n+1}) - g(w_n), g(u) - g(u_{n+1})\rangle &= ||g(u) - g(w_n)||^2 - ||g(u) - g(u_{n+1})||^2 \\ &\quad -||g(u_{n+1}) - g(w_n)||^2.\end{aligned} \tag{3.15}$$

Combining (3.14) and (3.15), we have

$$||g(u_{n+1}) - g(u)||^2 \leqslant ||g(w_n) - g(u)||^2 - (1 - 2\alpha\rho)||g(u_{n+1}) - g(w_n)||^2. \tag{3.16}$$

Taking $v = u$ in (3.4) and $v = w_n$ in (3.10), we have

$$\langle \beta\nu, g(w_n) - g(u)\rangle + \beta\varphi(g(w_n), g(u)) - \beta\varphi(g(u), g(u)) \geqslant 0 \tag{3.17}$$

and

$$\begin{aligned}&\langle \beta\xi_n + g(w_n) - g(y_n), g(u) - g(w_n)\rangle \\ &+\beta\varphi(g(u), g(w_n)) - \beta\varphi(g(w_n), g(w_n)) \geqslant 0.\end{aligned} \tag{3.18}$$

Adding (3.17) and (3.18) and rearranging the terms, we have

$$\begin{aligned}\langle g(w_n) - g(y_n), g(u) - g(w_n)\rangle &\geqslant \beta\langle \xi_n - \nu, g(w_n) - g(u)\rangle\, \beta\{\varphi(g(u), g(u)) \\ &\quad -\varphi(g(u), g(w_n)) - \varphi(g(w_n), g(u)) \\ &\quad +\varphi(g(w_n), g(w_n))\} \\ &\geqslant -\beta\alpha||g(y_n) - g(w_n)||^2,\end{aligned} \tag{3.19}$$

since T is a g-partially relaxed strongly monotone operator with constant $\alpha > 0$ and $\varphi(.,.)$ is skew-symmetric.

Now taking $v = g(w_n) - g(y_n)$ and $u = g(u) - g(w_n)$ in (2.7), (3.19) can be written as

$$\begin{aligned}||g(u) - g(w_n)||^2 &\leqslant ||g(u) - g(y_n)||^2 - (1 - 2\beta\alpha)||g(y_n) - g(w_n)||^2 \\ &\leqslant ||g(u) - g(y_n)||^2, \quad \text{for } 0 < \beta < 1/2\alpha.\end{aligned} \tag{3.20}$$

Similarly, by taking $v = u$ in (3.6) and $v = u_{n+1}$ in (3.11) and using the g-partially relaxed strongly monotonicity of the operator T and the skew-symmetry of $\varphi(.,.)$, we have

$$\langle g(y_n) - g(u_n), g(u) - g(y_n)\rangle \geqslant -\mu\alpha||g(y_n) - g(u_n)||^2. \tag{3.21}$$

Letting $v = y_n - u_n$, and $u = u - y_n$ in (2.7), and combining the resultant with (3.21), we have

$$\begin{aligned} ||g(y_n) - g(u)||^2 &\leqslant ||g(u) - g(u_n)||^2 - (1 - 2\mu\alpha)||g(y_n) - g(u_n)||^2 \\ &\leqslant ||g(u) - g(u_n)||^2, \quad \text{for } 0 < \mu < \tfrac{1}{2\alpha}. \end{aligned} \tag{3.22}$$

Now

$$\begin{aligned} ||g(u_{n+1}) - g(w_n)||^2 &= ||g(u_{n+1}) - g(u_n) + g(u_n) - g(w_n)||^2 \\ &= ||g(u_{n+1}) - g(u_n)||^2 + ||g(u_n) - g(w_n)||^2 \\ &+ 2\langle g(u_{n+1}) - g(u_n), g(u_n) - g(w_n)\rangle . \end{aligned} \tag{3.23}$$

Combining (3.16), (3.20), (3.22) and (3.23), we obtain

$$||g(u_{n+1}) - g(u)||^2 \leqslant ||g(u_n) - g(u)||^2 - (1 - 2\beta\alpha)||g(u_{n+1}) - g(u_n)||^2,$$

the required result (3.8). □

THEOREM 3.1. *Let H be a finite dimensional space. Let $g : H \longrightarrow H$ be injective and $0 < \rho < \frac{1}{2\alpha}$. Let $T : H \longrightarrow C(H)$ be M-Lipschitz continuous operator. Then the sequence $\{u_n\}_1^\infty$ given by Algorithm 3.1 converges to a solution u of (2.1).*

Proof. Let $u \in H$ be a solution of (2.1). Since $0 < \rho < \frac{1}{2\alpha}$. From (3.8), it follows that the sequence $\{||g(u) - g(u_n)||\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Furthermore, we have

$$\sum_{n=0}^{\infty}(1 - 2\alpha\rho)||g(u_{n+1}) - g(u_n)||^2 \leqslant ||g(u_0) - g(u)||^2,$$

which implies that

$$\lim_{n\longrightarrow\infty} ||g(u_{n+1}) - g(u_n)|| = 0. \tag{3.24}$$

Let $\hat{u}$ be the limlit point of $\{u_n\}_1^\infty$; a subsequence $\{u_{n_j}\}_1^\infty$ of $\{u_n\}_1^\infty$ converges to $\hat{u} \in H$. Replacing w_n and y_n by u_{n_j} in (3.2), (3.4) and (3.6), taking the limit $n_j \longrightarrow \infty$ and using (3.24), we have

$$\langle \hat{v}, g(v) - g(\hat{u})\rangle + \varphi(g(v), g(\hat{u})) - \varphi(g(\hat{u}), g(\hat{u})) \geqslant 0, \quad \forall v \in H,$$

which implies that $\hat{u}$ solves the multivalued variational inequality (2.1) and

$$\|g(u_{n+1}) - g(u)\|^2 \leqslant \|g(u_n) - g(u)\|^2.$$

Thus, it follows from the above inequality that $\{u_n\}_1^\infty$ has exactly one limit point $\hat{u}$ and

$$\lim_{n\to\infty} g(u_n) = g(\hat{u}).$$

Since g is injective, thus

$$\lim_{n\to\infty} (u_n) = \hat{u}.$$

It remains to show that $v \in T(u)$. From (3.3) and using the M-Lipschitz continuity of T, we have

$$||v_n - v|| \leqslant M(T(u_n), T(u)) \leqslant \delta ||u_n - u||,$$

which implies that $v_n \longrightarrow v$ as $n \longrightarrow \infty$. Now consider

$$\begin{aligned} d(v, T(u)) &\leqslant ||v - v_n|| + d(v, T(u)) \\ &\leqslant ||v - v_n|| + M(T(u_n), T(u)) \\ &\leqslant ||v - v_n|| + \delta ||u_n - u|| \quad \longrightarrow 0 \text{ as } n \longrightarrow \infty \end{aligned}$$

where $d(v, T(u)) = inf\{||v-z|| : z \in T(u)\}$. and $\delta > 0$ is the M-Lipschitz continuity constant. From the above inequality, it follows that $d(v, T(u)) = 0$. This implies that $v \in T(u)$, since $T(u) \in C(H)$. This completes the proof. □

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