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REFINED GEOMETRIC INEQUALITIES BETWEEN TWO OR MORE TRIANGLES OBTAINED BY DEDUBLATION

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Abstract. We study a class of inequalities between two or more triangles which extend the known metric relations between the elements of a single triangle. The common idea is that any quadratic type inequality between the elements of one triangle can have a "dedublated form" when written between the elements of two (or more) triangles with the optimal inequality being possible only when the triangles are similar. For example, we extend the well known quadratic form inequalities of Gerretsen [2, page 8] and give the new, dedublated form inequalities for the relations, which, in the case of a single triangle, correspond to the distances between the important points of the triangle such as circumcentre, incentre, orthocentre and the centre of mass.

1. Introduction

Let $A_iB_iC_i$, $i=1,2$ be two triangles of sides $(a_i,b_i,c_i)_{i=1,2}$ and let S_i, $i=1,2$ denote their areas respectively. About 30 years ago Pedoe proved that

$$a_i^2\left(-a_2^2+b_2^2+c_2^2\right)+b_1^2\left(-b_2^2+c_2^2+a_2^2\right)+c_1^2\left(-c_2^2+a_2^2+b_2^2\right)\geqslant 16S_1S_2 \qquad (1)$$

with equality if and only if the two triangles $A_1B_1C_1$, $A_2B_2C_2$ are similar [1]. Pedoe found (1) in 1941 when he studied the conditions that allowed for an orthogonal projection of an arbitrary triangle to a triangle of a given shape to exist. However it was first discovred in 1891 by Neuberg and today inequality (1) is known as the Neuberg-Pedoe inequality [2, page 354]. Pedoe's result has attracted a large interest, for example up to last decade the review [2, Chapter XII, section 3] cites 38 references! Carlitz [3] gave an interesting algebraic proof of (1) based on a classic inequality of Aczel [4]. Among other results proved in that paper there is also the following:

PROPOSITION 1 [3]. *If triangles $A_iB_iC_i$, $i=1,2$ have circumcentres O_i, centroids G_i and their circumradii R_i, $i=1,2$, then the we have that*

$$R_1R_2-\frac{1}{9}(a_1a_2+b_1b_2+c_1c_2)\geqslant O_1G_1\cdot O_2G_2 \qquad (2)$$

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with equality if and only if the two triangles are similar.

(2) extends the well known result that

$$OG^2 = R^2 - \frac{1}{9}(a^2+b^2+c^2). \tag{3}$$

Klamkin later shown that the Neuberg-Pedoe's inequality can be derived via Cauchy-Schwartz theorem [5]. He also generalized inequality (2) to the case of two n-simplexes based on an idea from mechanics via the polar moment of inertia.

PROPOSITION 2 [5]. *If* $\left\{\left(V_i^k, w_i^k\right)\right\}_{i=1,2}^{k=1,\dots,n}$ *are two sets of weighted n-points (with positive weights* $\left\{w_i^k\right\}_{i=1,2}^{k=1,\dots,n}$*) forming two simplexes with circumenters O_i, centroids G_i and circumradii R_i, $i=1,2$ then we have*

$$R_1R_2 \geqslant O_1G_1 \cdot O_2G_2 + \frac{1}{2|W_1W_2|}\sum\sum\sqrt{w_1^k w_1^j w_2^k w_2^j A_1^{kj} A_2^{kj}} \tag{4}$$

where $W_i = \sum\limits_{k=1,n} w_k$.

Clearly the case $n=3$, $w_1^k = w_2^k = 1$, $k=\overline{1,3}$ reduces (4) to inequality (2) of Carlitz. For the triangle case relation (3) implies that

$$9R^2 \geqslant \left(a^2+b^2+c^2\right). \tag{5}$$

(5) also follows from the fact that $OH^2 = 9R^2-(a^2+b^2+c^2) \geqslant 0$ where H is the triangle's orthocenter. In this respect it is well known that a much finer inequality is valid, namely [2, page 50]

$$8R^2+4r^2 \geqslant a^2+b^2+c^2 \tag{6}$$

which holds in any triangle ABC. In fact it is known [2, pages 278-283] that in every triangle we have that $2IH^2 = 8R^2+4r^2-(a^2+b^2+c^2)$, where I is the incenter of triangle ABC. In view of the Euler inequality, $R \geqslant 2r$, valid in every triangle ABC we see that (6) is finer than (5). It can be shown that the coefficients in (6) are the best for an inequality of this form. In [8] we have recently considerably extended (6) to deal with the general power case as follows.

PROPOSITION 3 [8]. *If ABC is a triangle of sides a, b, c, and R, r are the radii of the circumscribed and inscribed circles, respectively, then the best inequality of the form*

$$u(n)R^n + v(n)r^n \geqslant a^n+b^n+c^n \tag{7}$$

for $n \geqslant 0$ is given by $u(n) = 2^{1+n}$ and $v(n) = 2^n\left(3^{1+\frac{n}{2}} - 2^{1+n}\right)$, for all $n \geqslant 0$.

2. The main results

In this paper we seek to introduce a general idea for obtaining inequalities of the form (2) and (6) between two or more triangles. This is based on the dedublation process encountered in the theory of quadratic forms in linear algebra. First we shall establish an inequality which, although similar in spirit to the Carlitz-Klamkin inequality (2,4), it is a much finer one. In the second part we shall present three new inequalities linking the distances between some fundamental points of two triangles: the orthocentres, centres of mass and the incenters. The underlying theme common to all the proofs is the use of the "Principle of Isosceles Triangle" property for triangle geometric inequalities which was first discussed in [6] and later developed in [7].

THEOREM 1. *Let $A_iB_iC_i$, $i=1,2$ be two triangles of sides $(a_i,b_i,c_i)_{i=1,2}$, areas $S_{1,2}$ and of radii $(R_i,r_i)_{i=1,2}$ where R_i, r_i denote the lengths of the radii of the circumscribed and inscribed circles for $A_iB_iC_i$, respectively. Then we have*

$$8R_1R_2+4r_1r_2 \geqslant a_1a_2+b_1b_2+c_1c_2 \geqslant 36r_1r_2 \tag{8}$$

The equality is possible only when the two triangles are similar and either equilateral or right angle ones.

REMARKS. Inequality (8) has appeared in somewhat modified form in [9]. We note that the inequality from the left hand side in (8) cannot result as an application of the Aczel result [4] as was the case with Carlitz inequality (2). This is because we have more than one term with the same sign on each side of the inequality from the left hand side at (8). Furthermore the polar moment of inertia idea is also not applicable. This is partly the reason what makes the case of inequality (8) more interesting than previous results. The current reference in the field [2, page 376] gives again result (2) without any other improvement.

Proof. We first discuss the inequality on the right hand side of (8). We shall use the following well known result [2].

LEMMA 1. *In every triangle we have with the above notation*

$$3\sqrt{3}R \geqslant a+b+c \geqslant 6\sqrt{3}r. \tag{9}$$

First we apply the arithmetic-geometric (AM-GM) inequality to obtain that

$$a_1a_2+b_1b_2+c_1c_2 \geqslant 3(a_1a_2b_1b_2c_1c_2)^{\frac{1}{3}} \tag{10}$$

Next we apply the well known relations which connect the radii and the lengths of the sides in a triangle [2], namely

$$abc = 4RS = 4Rpr \tag{11}$$

where S is the triangle's area and p is the semiperimeter for the triangle ABC. With (11) in (10) and taking into account (9) and the well known Euler inequality $R \geqslant 2r$ one easily gets from (10) that

$$3(a_1a_2b_1b_2c_1c_2)^{\frac{1}{3}} \geqslant 3\cdot\left(4^2R_1R_2p_1p_2r_1r_2\right)^{\frac{1}{3}} \geqslant 3\cdot 4\left(27(r_1r_2)^3\right)^{\frac{1}{3}} = 36r_1r_2 \tag{12}$$

which is the desired result on the right-hand side in inequality (8). For this part it is clear that equality is only possible when all the inequality signs become equalities, that is when both triangles are equilateral.

We now discuss the inequality on the left-hand side in (1). Consider the function

$$f(a_i, b_i, c_i) = 8R_1R_2 + 4r_1r_2 - (a_1a_2 + b_1b_2 + c_1c_2) \tag{13}$$

given by the difference between the two terms in the inequality on the left-hand side in (8). We shall use the sines theorem and the well known relations for the inscribed radii [2]

$$r_i = 4R_i \sin(\alpha_i/2)\sin(\beta_i/2)\sin(\gamma_i/2), i = 1,2 \tag{14}$$

where α_i, β_i, γ_i are the triangle's angles. Thus (13) becomes (cyclical quantities)

$$f(\alpha_i, \beta_i, \gamma_i) = 4R_1R_2\left(2 + 16\prod \sin(\alpha_1/2)\sin(\alpha_2/2) - \sum \sin(\alpha_1)\sin(\alpha_2)\right). \tag{15}$$

The critical points for f are found from

$$\frac{\partial f}{\partial t} = 0 \text{ where } t \in \{\alpha_i, \beta_i, \gamma_i\},\ i = 1,2. \tag{16}$$

First notice that if at least one of the triangles is degenerate then the given inequality is trivial. For example if $\alpha_1 = 0$ then this follows from the fact that the two minus terms which are left at (15) are always (as products of sines) less than 2. Furthermore in such cases there is equality if and only if each of the triangles have two right angles, respectively.

If the triangles are both nondegenerate, then the conditions at (16) for $t \in \alpha_i, i = 1,2$ give that

$$(1 + \cos(\alpha_1))\cos(\alpha_2) = (1 + \cos(\alpha_2))\cos(\alpha_1) \tag{17}$$

Analogously one obtains similar relations from (16) at the critical points when considered for $t \in \beta_i, \gamma_i, i = 1,2$. (17) suggests considering the function

$$g(t) = \frac{\cos(t)}{1 + \cos(t)} \tag{18}$$

for $0 < t < \pi$. g is diferentiable in this range and its derivative is $g'(t) = -\frac{sin(t)}{(1+\cos(t))^2}$ which is strictly negative for all $0 < t < \pi$. Consequently g is strictly decreasing therefore relation (17) necessarily attracts that $\alpha_1 = \alpha_2$. Analogously $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$ at the critical, nondegenerate points.

In effect this has reduced the problem to the study of the behaviour of f at (15) when the two triangles are similar. However, in this case the left hand side inequality in (8) reduces to (6) which is true. In summary the function f in (13) is positive at any critical points of its domain and also along the boundary of the domain and consequently the left hand side inequality (8) holds for any given pairs of triangles as in the statement of the Theorem 1. Furthermore we have shown that the only case for equality is that when the two triangles are equilateral and, in the case of the left-hand side inequality at (8), the equality being possible also when the two triangles are both degenerate having two right angles. This ends the proof of the theorem.

REMARK. The inequality on the right-hand side in (8) can be strengthened as was given by Tsintsifas [10] in the form $a_1a_2+b_1b_2+c_1c_2 \geqslant 4\sqrt{3}\sqrt{S_1S_2} \geqslant 36r_1r_2$. Here equality is possible only when the two triangles are equilateral.

COROLLARY 1.1. *Using the same notation as in the theorem 1 one can show that for the case of n triangles we have*

$$2^{n+1}R_1R_2...R_n+2^n\left(3^{1+\frac{n}{2}}-2^{n+1}\right)r_1r_2...r_n \geqslant a_1a_2...a_n+b_1b_2...b_n+c_1c_2...c_n \geqslant 2^n3^{1+\frac{n}{2}}r_1r_2...r_n \tag{19}$$

where $n \geqslant 2$ *is an integer.*

Proof. One can apply the same method of proof as in Theorem 1 using the result of Proposition 3 and taking into account that the method of proof is not dependent on the number n, $n \geqslant 2$, of triangles.

COROLLARY 1.2. *With the notation of Corollary 1 we have that, for every set of* $m \geqslant 1$ *triangles, the double inequality*

$$\begin{aligned} 2^{s(m)+1}R_1^{q_1}R_2^{q_2}...R_m^{q_m}&+2^{s(m)}\left(3^{1+\frac{s(m)}{2}}-2^{s(m)+1}\right)r_1^{q_1}r_2^{q_2}...r_m^{q_m}\\ &\geqslant a_1^{q_1}a_2^{q_2}...a_m^{q_m}+b_1^{q_1}b_2^{q_2}...b_m^{q_m}+c_1^{q_1}c_2^{q_2}...c_m^{q_m}\\ &\geqslant 2^{s(m)}3^{1+\frac{s(m)}{2}}r_1^{q_1}r_2^{q_2}...r_m^{q_m} \end{aligned} \tag{20}$$

for all $q_1,q_2,...,q_m \geqslant 0, m \geqslant 1$, *integer and where* $s(m)=\sum_{k=1}^{m}q_k$.

Proof. We apply Corollary 1 via the well known result of Oppenheimer [11] which states that if $0 \leqslant \alpha \leqslant 1$ and a,b,c are the lengths of the sides of a triangle then $a^\alpha, b^\alpha, c^\alpha$ have the same property. In turn this fact can be iterated as many times as necessary taking into account that Corollary 1 is valid for any number of arbitrary triangles.

COROLLARY 1.3. *With the notation of Corollary 1 for every set of* $m \geqslant 1$ *triangles we have the double inequality*

$$\begin{aligned} 2^{s(m)+1}R_1^{q_1}R_2^{q_2}...R_m^{q_m}&+2^{s(m)}\left(3^{1+\frac{s(m)}{2}}-2^{s(m)+1}\right)r_1^{q_1}r_2^{q_2}...r_m^{q_m}\\ &\geqslant a_1^{q_1}...a_t^{q_t}m_{a_{t+1}}^{w_{t+1}}...m_{a_n}^{w_n}+b_1^{q_1}...b_t^{q_t}m_{b_{t+1}}^{w_{t+1}}...m_{b_n}^{w_n}+c_1^{q_1}...c_t^{q_t}m_{c_{t+1}}^{w_{t+1}}...m_{c_n}^{w_n}\\ &\geqslant 2^{s(m)}3^{1+\frac{s(m)}{2}}r_1^{q_1}...r_m^{q_m} \end{aligned} \tag{21}$$

for all $q_1,q_2,...,q_m \geqslant 0, m \geqslant 1$, *integer and where* $s(m)=\sum_{k=1}^{m}q_k$.

In (21) m_{a_k} represents the length of the median corresponding to the side a_k and R_k, r_k are the lengths of the radii of the circumscribed and inscribed circles in the triangle whose sides are $m_{a_k}, m_{b_k}, m_{c_k}$ for all $k=t+1,...,n$. For example, it is well known (and easy to check) that $R_k = \frac{m_{a_k} m_{b_k} m_{c_k}}{3Sk}, r_k = \frac{3S_k}{m_{a_k}+m_{b_k}+m_{c_k}}$ where S_k denotes the area of the triangle $A_kB_kC_k$ whose sides are $a_k, b_k, c_k, k = t+1, ..., n$ [2, p. 109].

Proof. We apply Corollary 2 together with the fact that if a_k, b_k, c_k are the lengths of the sides of a triangle then $m_{a_k}, m_{b_k}, m_{c_k}$ (the lengths of the medians in triangle $A_kB_kC_k$) have the same property for all $k = t+1, ..., n$.

APPLICATIONS. 1. Various interesting inequalities are obtained from the result of theorem 1. The case when all triangles are similar reduces exactly to inequality (7) above. For $n = 1, 2$ our result recovers known inequalities published by W.J. Blundon in the mid 60's, see [12-13] for details.

2. Corollary 2 considerably extends the results of G. Tsintsifas and W. Janous quoted in [2, page 376].

3. Take $m = 2$, $q_1 = q_2 = \frac{1}{2}$ in (20). This gives the inequality

$$4\sqrt{R_1R_2} + (6\sqrt{3} - 8)\sqrt{r_1r_2} \geqslant \sqrt{a_1a_2} + \sqrt{b_1b_2} + \sqrt{c_1c_2} \geqslant 6\sqrt{3}\sqrt{r_1r_2} \tag{22}$$

which generalizes the well known Blundon inequality [2, page 10]

$$2R + (3\sqrt{3} - 4)r \geqslant p \geqslant 3\sqrt{3}r \tag{23}$$

valid in every triangle.

4. By applying the Holder's inequality on the left hand side in (19) one gets that

$$\left(\prod_{i=1}^{n} (u(n)R_i^n + v(n)r_i^n)\right)^{\frac{1}{n}} \geqslant u(n)\prod_{i=1}^{n} R_i + v(n)\prod_{i=1}^{n} r_i \tag{24}$$

for all $\alpha \geqslant 0$. Also in view of the Euler inequality mentioned earlier one can easily see that

$$3^{1+\frac{n\alpha}{2}}(R_1...R_n)^{\alpha} \geqslant \left(\prod_{i=1}^{n} (u(n\alpha)R_i^{n\alpha} + v(n\alpha)r_i^{n\alpha})\right)^{\frac{1}{n}} \tag{25}$$

Indeed this follows from the fact $3^{1+\frac{n\alpha}{2}} = \left(\prod_{i=1}^{n} (u(n\alpha) + 2^{-n\alpha}v(n\alpha))\right)^{\frac{1}{n}}$ and the Euler inequality $R \geqslant 2r$. Therefore (8) coupled with (20-21) gives the extension to the case of n triangles of the classical, old result at (5).

More applications can be obtained by considering various dual transformations applied to the initial triangle. Some expressions of certain elements of the initial triangle can also serve to play the role of the lengths of the sides of a new triangle. We applied this idea already at Corollary 3 but other interesting examples can be given. For example, $\frac{a}{1+a}, \frac{b}{1+b}, \frac{c}{1+c}$ form a triangle whenever a, b, c are the sides of the given triangle [2, page 18]. A detailed list of such properties is given in [2, pages 18-25].

3. Further extensions

In this section we give a few inequalities between some important points of a triangle being similar in spirit to the Carlitz-Klamkin inequalities (2,4). These inequalities are obtained using the dedublation idea from the theory of quadratic forms and extend the situation from a single triangle to the case of a pair of triangles.

The well known Euler relation which gives the distance between the circumcenter O and the incenter I in any triangle reads [2, page 279]

$$OI^2 = R^2 - 2Rr. \tag{26}$$

We shall prove the following extension, similar in spirit to (2,4).

THEOREM 2. *Let $A_iB_iC_i$, $i = 1,2$ be two triangles of radii $(R_i, r_i)_{i=1,2}$ where R_i, r_i denote the lengths of the radii of the circumscribed and inscribed circles for $A_iB_iC_i$, respectively. Let O_i, I_i be their corresponding circumcenters and incenters, respectively. Then we have the inequality*

$$O_1I_1 \cdot O_2I_2 \leqslant \sqrt{R_1R_2}\left(\sqrt{R_1R_2} - 2\sqrt{r_1r_2}\right). \tag{27}$$

The equality is possible only when the two triangles are similar.

Proof. (27) follows easily from the simple inequality

$$\left(R_1^2 - 2R_1r_1\right)\left(R_2^2 - 2R_2r_2\right) \leqslant \left(R_1R_2 - 2\sqrt{R_1R_2r_1r_2}\right)^2 \tag{28}$$

which is equivalent to the obvious relation $R_1r_2 + R_2r_1 \geqslant 2\sqrt{R_1R_2r_1r_2}$.

COROLLARY 2.1. *When n is an even positive integer the result of theorem 2 can be generalized by induction as follows*

$$O_1I_1 \cdot O_2I_2 \cdot \ldots \cdot O_nI_n \leqslant \sqrt{R_1R_2...R_n}\left(\sqrt{R_1R_2...R_n} - 2^{\frac{n}{2}}\sqrt{r_1r_2...r_n}\right). \tag{29}$$

In every triangle the distance between the centroid G and the incentre I is given by [2, page 280]

$$9GI^2 = p^2 + 5r^2 - 16Rr. \tag{30}$$

(30) can also be extended to the dedublated form inequality for a pair of triangles as follows.

THEOREM 3. *Let $A_iB_iC_i$, $i = 1,2$ be two triangles of radii $(R_i, r_i)_{i=1,2}$ and of semiperimeters p_i, where R_i, r_i denote the lengths of the radii of the circumscribed and inscribed circles for $A_iB_iC_i$, respectively. Let G_i, I_i, $i = 1,2$, be their corresponding centroids and incenters, respectively. Then we have the inequality*

$$9G_1I_1 \cdot G_2I_2 \leqslant p_1p_2 + 5r_1r_2 - 16\sqrt{R_1R_2r_1r_2}. \tag{31}$$

The equality is possible only when the two triangles are similar.

Proof. We have to show that

$$\left(p_1^2 + 5r_1^2 - 16R_1r_1\right)\left(p_2^2 + 5r_2^2 - 16R_2r_2\right) \leqslant \left(p_1p_2 + 5r_1r_2 - 16\sqrt{R_1R_2r_1r_2}\right)^2. \tag{32}$$

First notice that one cannot apply Aczel's inequality [4] in (32) because the first two parenthesis at (32) are positive rather than negative which is what one would need for the inequality to be applied. Therefore we will have to do the full calculations! Let us introduce $x, y, z > 0$ defined by

$$p_2 = xp_1, r_2 = yr_1, R_2 = zR_1. \tag{33}$$

Consider

$$\begin{aligned} f(x,y,z) = &\left(p_1^2 + 5r_1^2 - 16R_1r_1\right)\left(x^2p_1^2 + 5y^2r_1^2 - 16yzR_1r_1\right) \\ &- \left(xp_1p_2 + 5yr_1r_2 - 16\sqrt{yzR_1R_2r_1r_2}\right)^2 \end{aligned} \tag{34}$$

Then f is differentiable and its partial derivatives are

$$f_x = 2p_1^2r_1\left(5r_1(x-y) - 16R_1(x-\sqrt{yz})\right) \tag{35.1}$$

$$f_y = \frac{2r_1}{\sqrt{zy}}\left(5p_1^2r_1(\sqrt{yz}-x) + 8zp_1^2R_1(x-\sqrt{yz}) + 40\sqrt{zy}R_1r_1^2(3\sqrt{yz}-2y-z)\right) \tag{35.2}$$

$$f_z = 16\sqrt{\frac{y}{z}}R_1r_1\left((x-\sqrt{yz})r_1^2\right) \tag{35.3}$$

Using relations (35) it is easy to see that the critical points (x_0, y_0, z_0) of f satisfy

$$x_0 = y_0 = z_0. \tag{36}$$

We shall end the proof by showing that f attains its maximum at any point (36). Due to the homogeneity in x, y, z in the structure of f it will be sufficient to do this for the case when the two triangles have the same circumradius, i.e. $z = 1$. Let us denote $\alpha = \frac{x^2}{y}, \beta = y$ and consider

$$\begin{aligned} f_1(\alpha,\beta) = &\left(p_1^2 + 5r_1^2 - 16R_1r_1\right)\left(\alpha^2p_1^2 + 5\beta^2r_1^2 - 16R_1r_1\right) \\ &- \left(\alpha p_1 + 5\beta r_1 - 16\sqrt{R_1r_1}\right)^2 \end{aligned} \tag{37}$$

A simple calculation gives that its partial derivatives with respect to α, β are

$$f_{1,\alpha} = 2p_1^2r_1\left(5(\alpha-\beta)r_1 + 16(1-\alpha)R_1\right) \tag{38.1}$$

$$f_{1,\beta} = 10r_1^2\left(5(\beta-\alpha)p_1^2 + 16(1-\beta)R_1r_1\right) \tag{38.2}$$

Clearly $\alpha = \beta = 1$ is the only critical point. For $\alpha = 1$ $f_{1,\alpha}$ is strictly negative for $0 < \beta < 1$ and strictly positive for $\beta > 1$. Similarly at $\beta = 1$ $f_{1,\beta}$ is strictly negative for $0 < \alpha < 1$ and strictly positive for $\alpha > 1$. Therefore f_1 attains its maximum at $\alpha = \beta = 1$. Furthermore $f_1(1,1) = 0$ therefore $f_1(\alpha,\beta) \leqslant 0$ for all $\alpha, \beta > 0$. This shows that f attains its maximum at the critical points (36). In this case we have $f(x_0, x_0, x_0) = 0$ and hence

$$f(x,y,z) \leqslant 0 \text{ for all } x,y,z > 0. \tag{39}$$

Equality in (39) can happen only at the critical point (36) in which case the two triangles are similar and the proof is finished.

In every triangle the distance between the orthocentre H and the incentre I is given by [2, page 280]

$$HI^2 = 4R^2 + 4Rr + 3r^2 - p^2. \tag{40}$$

We shall extend (40) to the dedublated form inequality for a pair of triangles.

THEOREM 4. *Let $A_iB_iC_i$, $i = 1,2$ be two triangles of radii $(R_i, r_i)_{i=1,2}$ and of semiperimeters p_i, where R_i, r_i denote the lengths of the radii of the circumscribed and inscribed circles for $A_iB_iC_i$, respectively. Let $H_i, I_i, i = 1,2$, be their corresponding orthocenters and incenters, respectively. Then we have the inequality*

$$H_1I_1 \cdot H_2I_2 \leqslant 4R_1R_2 + 4\sqrt{R_1R_2r_1r_2} + 3r_1r_2 - p_1p_2 \tag{41}$$

The equality is possible only when the two triangles are similar.

Proof. This is similar to the proof of Theorem 3.

By combining the results of theorems 3 and 4 we get

THEOREM 5. *Under the hypothesis of theorems 3 and 4 we have the double inequality*

$$16\sqrt{R_1R_2r_1r_2} - 5r_1r_2 \leqslant p_1p_2 \leqslant 4R_1R_2 + 4R_1R_2 + 4\sqrt{R_1R_2r_1r_2} + 3r_1r_2. \tag{42}$$

Similar results can be established in connection to other distances between important points of the triangle. For example the following inequality makes the connection between the result of theorem 1 and the results in section 3.

THEOREM 6. *Let $A_iB_iC_i$, $i = 1,2$ be two triangles of radii $(R_i, r_i)_{i=1,2}$ and of semiperimeters p_i, where R_i, r_i denote the lengths of the radii of the circumscribed and inscribed circles for $A_iB_iC_i$, respectively. Let $H_i, O_i, i = 1,2$, be their corresponding orthocenters and circumcenters, respectively. Then we have the inequality*

$$H_1O_1 \cdot H_2O_2 \leqslant 9R_1R_2 + 8\sqrt{R_1R_2r_1r_2} + 2r_1r_2 - 2p_1p_2. \tag{43}$$

Proof. This is similar to the proof of Theorem 3.

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