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THE PHASE–SPACE VIEW OF CONSERVATION LAWS

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Abstract. We discuss the phase-space of conservation laws in Lagrangian and Hamiltonian formalism from a very general point of view deriving all the geometrical properties of this reduced configuration space. Such a mathematical approach, based on an integration of the system dependent on the inequality between the number of dimensions in the configuration space and the number of conservation laws, is extremely useful in connection to the derivation of a General Conservation Principle (from which particular conservation laws can be derived). Properties and behaviours of general solutions are discussed in relation to the existence of first integrals of motion.

1. Introduction

Conservation laws and symmetries have always had a fundamental role both in physics and mathematics. For instance, their existence allows to "reduce" the problem of evolution of systems due to the first integrals of motion which can be found. Noether theorem is a preminent result in this sense, since it establishes a link between conservation laws and symmetries. Moreover, we think that conservation laws can play a deeper role in the definition of physical theories and, in particular, to define space-times which can be considered of *physical interest*.

The aim of this paper is to discuss the features of a configuration space and then of the related phase-space (*i.e.* the space which is the union of configuration space and momentum space) in relation to the conservation laws. More precisely, we will discuss the Lagrangian approach to conservation laws (in this case the phase-space is nothing else but the *tangent space* to the manifold of configuration space, technically speaking, the $2n$-dimensional space of configurations and generalized velocities), and the Hamiltonian approach where a phase-space can be properly defined. The result of such a study is the fact that a *General Conservation Law Theorem* can be deduced and thus the *structure* of the general solution of a dynamical system directly depends on the existence of symmetries in the various "directions" of configuration space.

The philosophy which underlies our approach is the fact that we believe that the violation of conservation laws (and then the symmetry breaking) it is artificially introduced in contemporary Physics. We believe that conservation laws are never

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violated and such a property allows the solution of a wide variety of phenomena such as the Einstein-Podolsky-Rosen paradox [1], topology changes [2], and the segregation of matter and antimatter in different universes [3]. Such results do not come from some *a priori* request of the theory, but they are derived from the existence of a *General Conservation Law* (in a 5-dimensional space-time) where no violation is allowed [4, 5].

In this stream of research, we analyze the mathematical properties of phase-space in the Lagrangian and Hamiltonian approaches to investigate the deep relations among symmetries, first integrals of motion, conservation laws and dimensionality of configuration space.

The paper is organized as follows. In Section II, we develop some general considerations on conservation laws and we set the mathematical machinery. Sections III and IV are devoted to Lagrangian and Hamiltonian approaches, respectively. Discussion and conclusions are drawn in Section V.

2. General considerations on conservation laws

General conservation laws can be discussed from a geometrical point of view by using the Lie derivative and mathematical tools such as functions, vectors, tensors and differential forms. As a general remark, we can say that every time the action of the Lie derivative on a geometric object gives as result zero, the object is conserved. Such a property is *covariant* by definition and specifies the number of dimensions and the nature of configuration space where the given (physical) system is defined. It is interesting to note that the existence of such conserved quantities always implies a *reduction* of dynamics, *i.e.* the degree of equations of motion is reduced due to the existence of first integrals [6, 7].

Before taking into account specific theories, let us remind some properties of the Lie derivative and how conservation laws are related to it. Let L_X be the Lie derivative

$$(L_X\omega)\xi = \frac{d}{dt}\omega(g^t_*\xi)\,, \tag{1}$$

where ω is a differential form of R^n defined on the vector field ξ, g^t_* is the differential of the phase flux $\{g_t\}$ given by the vector field X on a differential manifold $\mathcal{M}$. Let $\rho_t = \rho_{g-t}$ be the action of a one–parameter group able to act on functions, vectors and forms (in general tensors) on the vector spaces $C^\infty(\mathcal{M})$, $D(\mathcal{M})$, and $\Lambda(\mathcal{M})$ constructed starting from $\mathcal{M}$. If g_t takes the point $m \in \mathcal{M}$ in $g_t(m)$, then ρ_t takes back on m the vectors and the forms defined on $g_t(m)$; ρ_t is a *pull back*. Then the property

$$\rho_{t+s} = \rho_t\rho_s \tag{2}$$

holds since

$$g_{t+s} = g_t \circ g_s\,. \tag{3}$$

On the functions $f, g \in C^\infty(\mathcal{M})$ we have

$$\rho_t(f\,g) = (\rho_t f)(\rho_t g)\,; \tag{4}$$

on the vectors $X, Y \in D(\mathcal{M})$,

$$\rho_t[X,Y] = [\rho_t X, \rho_t Y]\,; \tag{5}$$

on the forms $\omega, \mu \in \Lambda(\mathcal{M})$

$$\rho_t(\omega \wedge \mu) = (\rho_t \omega) \wedge (\rho_t \mu)\,. \tag{6}$$

L_X is the infinitesimal generator of the one–parameter group ρ_t, and, being a derivative on the algebras $C^\infty(\mathcal{M})$, $D(\mathcal{M})$, and $\Lambda(\mathcal{M})$, the following properties must hold

$$L_X(f\,g) = (L_X f)g + f\,(L_X g)\,, \tag{7}$$

$$L_X[Y,Z] = [L_X Y, Z] + [Y, L_X Z]\,, \tag{8}$$

$$L_X(\omega \wedge \mu) = (L_X \omega) \wedge \mu + \omega \wedge (L_X \mu)\,, \tag{9}$$

which are nothing else but the Leibniz rules for functions, vectors and differential forms, respectively. Furthermore,

$$L_X f = Xf\,, \tag{10}$$

$$L_X Y = adX(Y) = [X,Y]\,, \tag{11}$$

$$L_X d\omega = dL_X \omega\,, \tag{12}$$

where ad is the self–adjoint operator and d is the external derivative by which a $p-$form becomes a $(p+1)$–form. This is the mathematics we need in order to define the phase space of conservation laws depending on the geometrical (physical) object we shall take into account. As a general remark, we have to say that the dimensions of the space we are considering are those of configuration space. For example a point on a sphere is identified by two angles $\{\theta, \phi\}$ and a radius ρ so that its configuration space is 3-dimensional. Considering the generalized velocities $\{\dot\theta, \dot\phi, \dot\rho\}$ we obtain a 6-dimensional tangent space (*i.e.* a phase space).

3. The Lagrangian approach to conservation laws

The discussion can be specified by considering a Lagrangian $\mathcal{L}$ which is a function defined on the tangent space of configurations $T\mathcal{Q} \equiv \{q_i, \dot q_i\}$, that is

$$\mathcal{L} : T\mathcal{Q} \longrightarrow \Re\,. \tag{13}$$

In this case, the vector field X is

$$X = \alpha^i(q)\frac{\partial}{\partial q^i} + \dot\alpha^i(q)\frac{\partial}{\partial \dot q^i}\,, \tag{14}$$

where the dot denotes the derivative with respect to t, and because of Eq.(10), we have

$$L_X \mathcal{L} = X\mathcal{L} = \alpha^i(q)\frac{\partial \mathcal{L}}{\partial q^i} + \dot\alpha^i(q)\frac{\partial \mathcal{L}}{\partial \dot q^i}\,. \tag{15}$$

It is important to note that t is simply a parameter which specifies the evolution of the system. The condition

$$L_X\mathcal{L} = 0 \tag{16}$$

implies that the phase flux is conserved along X: this means that a constant of motion exists for $\mathcal{L}$ and a conservation law is associated to the vector X. In fact, by taking into account the Euler–Lagrange equations

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}^i} - \frac{\partial\mathcal{L}}{\partial q^i} = 0\,, \tag{17}$$

it is easy to show that

$$\frac{d}{dt}\left(\alpha^i\frac{\partial\mathcal{L}}{\partial\dot{q}^i}\right) = L_X\mathcal{L}\,. \tag{18}$$

If (16) holds, the relation

$$\Sigma_0 = \alpha^i\frac{\partial\mathcal{L}}{\partial\dot{q}^i} \tag{19}$$

identifies a constant of motion. Alternatively, using a generalized differential for the Lagrangian $\mathcal{L}$, the Cartan one–form,

$$\theta_{\mathcal{L}} \equiv \frac{\partial\mathcal{L}}{\partial\dot{q}^i}\,dq^i \tag{20}$$

and defining the inner derivative

$$i_X\theta_{\mathcal{L}} = \langle\,\theta_{\mathcal{L}}, X\rangle\ , \tag{21}$$

we get, as above,

$$i_X\theta_{\mathcal{L}} = \Sigma_0 \tag{22}$$

if, again, condition (16) holds. This representation identifies cyclic variables. Using a point transformation on vector field (14), it is possible to get

$$\tilde{X} = (i_X dQ^k)\,\frac{\partial}{\partial Q^k} + \left[\frac{d}{dt}(i_X dQ^k)\right]\frac{\partial}{\partial\dot{Q}^k}\,. \tag{23}$$

From now on, Lagrangians and vector fields transformed by the non–degenerate transformation

$$Q^i = Q^i(q)\,,\quad \dot{Q}^i(q) = \frac{\partial Q^i}{\partial q^j}\,\dot{q}^j \tag{24}$$

will be denoted by a tilde. However the Jacobian determinant

$$\mathcal{J} = \|\,\partial Q^i/\partial q^j\,\| \tag{25}$$

has to be non–zero. The Jacobian, and this fact is important for the implications discussed in [5], indicates whether a degenerate coordinate transformation can occur or not.

Now we have that when X is a symmetry for the Lagrangian $\mathcal{L}$ also $\tilde{X}$ is a symmetry for the Lagrangian $\tilde{\mathcal{L}}$ giving rise to a conserved quantity, thus it is always possible to choose a coordinate transformation so that

$$i_X dQ^1 = 1\,,\quad i_X dQ^i = 0\,,\quad i \neq 1\,, \tag{26}$$

and then

$$\tilde{X} = \frac{\partial}{\partial Q^1}\,, \qquad \frac{\partial \tilde{\mathcal{L}}}{\partial Q^1} = 0\,. \tag{27}$$

It is evident that Q^1 is a cyclic coordinate because dynamics can be reduced. Specifically, the "reduction" is connected to the existence of the second of (27). However, the change of coordinates is not unique and an opportune choice of coordinates is always important. Furthermore, it is possible that more symmetries are existent. In this case more cyclic variables must exist. For example, if X_1, X_2 are vector fields which induce conservation laws and they commute, $[X_1, X_2] = 0$, we obtain two cyclic coordinates by solving the system

$$\begin{gathered} i_{X_1} dQ^1 = 1\,, \quad i_{X_2} dQ^2 = 1\,, \\ i_{X_1} dQ^i = 0\,, \quad i \neq 1\,; \quad i_{X_2} dQ^i = 0\,, \quad i \neq 2\,. \end{gathered} \tag{28}$$

If they do not commute, this procedure does not work since commutation relations are preserved by diffeomorphisms. In this case

$$X_3 = [X_1, X_2] \tag{29}$$

is again a symmetry since

$$L_{X_3}\mathcal{L} = L_{X_1} L_{X_2}\mathcal{L} - L_{X_2} L_{X_1}\mathcal{L} = 0\,. \tag{30}$$

If X_3 is independent of X_1, X_2 we can go on until the vector fields close the Lie algebra. This means that the conservation laws are independent of the algebra of vector fields and therefore symmetries are conserved in every case. This point is extremely important from a physical point of view since it states that symmetry breakings are not requested by the theory [4].

A reduction procedure by cyclic coordinates can be achieved in three steps: $i)$ we choose a symmetry and obtain new coordinates as above and after this first reduction, we get a new Lagrangian $\tilde{\mathcal{L}}$ with a cyclic coordinate; $ii)$ we search for new symmetries in this new space and iterate the reduction technique until it is possible; $iii)$ the process stops if we select a pure kinetic Lagrangian where all coordinates are cyclic. This case is not common in theoretical physics but it is *the ideal case in which all conservation laws are enclosed in the same dynamics*. From our point of view, the *General Conservation Principle* must have such a form [4].

Going back to the development of dynamics, every symmetry selects a constant conjugate momentum since, by the Euler–Lagrange equations we get

$$\frac{\partial \tilde{\mathcal{L}}}{\partial Q^i} = 0 \Longleftrightarrow \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{Q}^i} = \Sigma_i\,. \tag{31}$$

Vice-versa, the existence of a constant conjugate momentum means that a cyclic variable has to exist. In other words, a symmetry exists.

Further remarks on the form of the Lagrangian $\mathcal{L}$ are necessary at this point. We shall take into account time–independent, Lagrangians $\mathcal{L} = \mathcal{L}(q^i, \dot{q}^j)$ which is non-degenerate, *i.e.*

$$\frac{\partial \mathcal{L}}{\partial t} = 0\,, \quad \det H_{ij} \equiv \det \left\| \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial \dot{q}^j} \right\| \neq 0\,, \tag{32}$$

where H_{ij} is the Hessian. As it is usual in analytic mechanics, $\mathcal{L}$ can be set in the form

$$\mathcal{L} = T(q^i, \dot{q}^i) - V(q^i)\,, \tag{33}$$

where T is the kinetic energy, usually a positive–defined quadratic form in the $\dot{q}^j$ and $V(q^i)$ is a potential term. The energy function associated with $\mathcal{L}$ is

$$E_{\mathcal{L}} \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\,\dot{q}^i - \mathcal{L}(q^j, \dot{q}^j)\,. \tag{34}$$

Considering again the symmetry, the condition (16) and the vector field X in Eq.(14) give a homogeneous polynomial of second degree in the velocities, plus an inhomogeneous term in the q^i. Due to (16), such a polynomial has to be identically zero and then each coefficient must be independently zero. If n is the dimension of the configuration space, we get $\{1 + n(n+1)/2\}$ partial differential equations whose solutions assign the symmetry, as we shall see below. Such a symmetry is over–determined and, if a solution exists, it is expressed in terms of integration constants which can be read as boundary conditions, which was exactly what we were looking for.

4. The Hamiltonian approach to conservation laws

From the Lagrangian formalism, we can pass to the Hamiltonian one through the Legendre transformations

$$\mathcal{H} = \pi_j \dot{q}^j - \mathcal{L}(q^j, \dot{q}^j)\,, \quad \pi_j = \frac{\partial \mathcal{L}}{\partial \dot{q}^j}\,, \tag{35}$$

which define, respectively, the Hamiltonian function and the conjugate momenta.

In the Hamiltonian formalism, the conservation laws are obtained when

$$[\Sigma_j, \mathcal{H}] = 0\,, \quad 1 \leqslant j \leqslant m\,, \tag{36}$$

This is the relation for conserved momenta which is usually adopted in quantum mechanics and, in order to obtain a symmetry, the Hamiltonian has to satisfy the relations

$$L_\Gamma \mathcal{H} = 0\,, \tag{37}$$

where the vector Γ is defined by

$$\Gamma = \dot{q}^i \frac{\partial}{\partial q^i} + \ddot{q}^i \frac{\partial}{\partial \dot{q}^i}\,. \tag{38}$$

Let us now go to the specific formalism of quantum mechanics. By the Dirac canonical quantization procedure, we have

$$\pi_j \longrightarrow \hat{\pi}_j = -i\partial_j\,, \tag{39}$$

$$\mathcal{H} \longrightarrow \hat{\mathcal{H}}(q^j, -i\partial_{q^j})\,. \tag{40}$$

If $|\Psi\rangle$ is a *state* of the system (i.e. the wave function of a particle), dynamics is given by the Schrödinger eigenvalue equation

$$\hat{\mathcal{H}}|\Psi\rangle = E|\Psi\rangle\,, \tag{41}$$

where, obviously, the whole wave-function is given by $|\phi(t,x)\rangle = e^{iEt/\hbar}|\Psi\rangle$.

If a symmetry exists, the reduction procedure outlined above can be applied and then, from (31) and (35), we get

$$\begin{array}{rcl} \pi_1 \equiv \dfrac{\partial\mathcal{L}}{\partial\dot{Q}^1} = i_{X_1}\theta_{\mathcal{L}} & = & \Sigma_1\,, \\ \pi_2 \equiv \dfrac{\partial\mathcal{L}}{\partial\dot{Q}^2} = i_{X_2}\theta_{\mathcal{L}} & = & \Sigma_2\,, \\ \dots \quad \dots & & \dots\,, \end{array} \tag{42}$$

depending on the number of symmetry vectors. After Dirac quantization, we get

$$\begin{array}{rcl} -i\partial_1|\Psi\rangle & = & \Sigma_1|\Psi\rangle\,, \\ -i\partial_2|\Psi\rangle & = & \Sigma_2|\Psi\rangle\,, \\ \dots & & \dots\,, \end{array} \tag{43}$$

which are nothing else but translations along the Q^j axis singled out by the corresponding symmetry. Eqs. (43) can be immediately integrated and, being Σ_j real constants, we obtain oscillatory behaviours for $|\Psi\rangle$ in the directions of symmetries, i.e.

$$|\Psi\rangle = \sum_{j=1}^{m} e^{i\Sigma_j Q^j}|\chi(Q^l)\rangle\,, \quad m < l \leqslant n\,, \tag{44}$$

where m is the number of symmetries, l are the directions where symmetries do not exist and n is the number of dimensions of configuration space.

Vice-versa, dynamics given by (41) can be reduced by (43) if, and only if, it is possible to define constant conjugate momenta as in (42), *i.e.* oscillatory behaviours of a subset of solutions $|\Psi\rangle$ exist as a consequence of the fact that symmetries are present in the dynamics.

The m symmetries give first integrals of motion and then the possibility to select classical trajectories for particles. In one and two–dimensional configuration spaces, the existence of a symmetry allows the complete solution of the problem. Therefore, if $m = n$, the problem is completely solvable and a symmetry exists for every variable of configuration space. In conclusion, we can set out the following theorem, coming out from the above demonstration:

General Conservation Law Theorem: *The reduction procedure of dynamics, connected to the existence of symmetries, allows to select a subset of the general solution of equations of motion, in both Lagrangian and Hamiltonian approaches, where oscillatory behaviours are found. This fact gives conserved momenta and trajectories which are solutions of equations of motion. Vice-versa, if a subset of the general solution of equations of motion has an oscillatory behaviour, due to the equations of conjugate momenta, conserved momenta have to exist and symmetries are present. In other words, symmetries select exact solutions and reduce dynamics. In these cases, the general solution of a dynamical system can be split in a combination of functions each of them depending on a given variable. As a corollary, a Lagrangian (or a Hamiltonian)* *where only kinetic terms are present always gives rise to a full integrable dynamics. The phase-space where such an object is defined is the Phase-Space of General Conservation Laws.*

5. Discussion and Conclusions

The above statement deserves some discussion. As a first remark the general solution (44) can be interpreted as a superposition of particular solutions (the components in different directions) which result the more *solved* (*i.e.* separated in every direction of configuration space) if the more symmetries exist. Starting from such a consideration, as a consequence, we can establish a sort of *degree of solvability* among the components of a given physical system, connected to the number of symmetries: *i*) a system is *completely* solved and separated if a symmetry exists for *every* direction of configuration space (in this case, the system is fully integrable and the relations among its parts can be exactly obtained); *ii*) a system is *partially* solved and separated if a symmetry exists for *some* directions of configuration space (in this case, it is not always possible to get a general solution); *iii*) a system is *not* separated at all and *no* symmetry exists (a necessary and sufficient condition to get the general solution does not exist). In other words, we could also obtain the general solution in the last case, but not by a straightforward process of separation of variables induced by the reduction procedure.

A further remark deserves the fact that the eigen-functions of a given operator (in our case the Hamiltonian $\hat{\mathcal{H}}$) define a Hilbert space. The above result works also in this case, so that we can define, for a quantum system whose eigen-functions are given by a set of commuting Hermitian operators (e.g. the Hamiltonian, the linear and angular momenta, the spin and so on), a *Hilbert Space of General Conservation Laws*. The number of dimensions of such a space is given by the components of superposition (44) while the number of symmetries is given by the oscillatory components. Vice-versa, the oscillatory components are *always* related to the number of symmetries in the corresponding Hilbert space.

In conclusion, we can say that the existence of symmetries, implying conservation laws, determines also the structure of the configuration space (a vector space in the case of Hilbert) where the physical system is set. Furthermore, the degree of solvability and separability of a system is deeply related to the existence of conservation laws and, vice-versa, all physical quantities are conserved in a completely separated system.

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