ESTIMATES OF THE DIFFERENCE BETWEEN TWO WEIGHTED INTEGRAL MEANS VIA WEIGHTED MONTGOMERY IDENTITY

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Abstract. Some new generalizations of estimates of difference between two weighted integral means are given, by using Euler-type identities and weighted Montgomery identity

1. Introduction

In the recent paper J. Pečarić, I. Perić and A. Vukelić proved generalizations of the following inequalities using the Euler-type identities (see [10]):

THEOREM 1. *Assume (p, q) is a pair of conjugate exponents, that is $1 \leqslant p, q \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $|f'|^p : [a, b] \to \mathbb{R}$ be an R-integrable function. Then for $a \leqslant c < d \leqslant b$, we have*

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - \frac{1}{d-c} \int_c^d f(t)\, dt \right|$$

$$\leqslant \begin{cases} \frac{(c-a)^2 + (b-d)^2}{2(b-a-d+c)} \cdot \left\| f' \right\|_\infty, & f' \in L_\infty[a, b], \\[2ex] \left[\frac{(c-a)^{q+1} + (b-d)^{q+1}}{(q+1)(b-a)^{q-1}(b-a-d+c)} \right]^{\frac{1}{q}} \cdot \|f'\|_p, & f' \in L_p[a, b], \\[2ex] \frac{1}{2} \left(\frac{c-a+b-d}{b-a} + \left| \frac{c-a-b+d}{b-a} \right| \right) \cdot \|f'\|_1, & f' \in L_1[a, b]. \end{cases}$$

In the special case when we take $c = d = x$, assuming $\frac{1}{d-c} \int_c^d f(t)\, dt = f(x)$ as a limit case, the first inequality reduces to the Ostrowski inequality; the second reduces to the inequality proved by A. M. Fink in [9]; and the third to the inequality proved by Lj. Dedić, M. Matić and J. Pečarić in [8].

Similar results also were obtained by P. Cerone and S. S. Dragomir (see [7]) using the Montgomery identity:

THEOREM 2. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continous mapping as is also $u : [c, d] \to \mathbb{R}$ with $[c, d] \subseteq [a, b]$. The following inequalities are then valid. Namely*

$$\left| \int_c^d u(x) f(x)\, dx - \frac{1}{b-a} \int_a^b f(x)\, dx \int_c^d u(x)\, dx \right|$$

$$\leqslant \begin{cases} \dfrac{\|f'\|_\infty}{b-a} \left\{ \dfrac{\int_c^d u(x) dx}{2} \left[(c-a)^2 + (b-d)^2 \right] + \int_c^d |\phi(t)|\, dt \right\}, & f' \in L_\infty [a, b], \\[4mm] \dfrac{\|f'\|_p}{b-a} \left\{ \dfrac{\left| \int_c^d u(x) dx \right|^q}{q+1} \left[(c-a)^{q+1} + (b-d)^{q+1} \right] + \int_c^d |\phi(t)|^q\, dt \right\}^{\frac{1}{q}} & f' \in L_p [a, b], \\[4mm] \dfrac{\|f'\|_1}{b-a} \max \left\{ \left| \int_c^d u(x)\, dx \right| \theta, \sup_{t \in [c,d]} |\phi(t)| \right\}, & f' \in L_1 [a, b], \end{cases}$$

where

$$\phi(t) = \left| \begin{array}{cc} t - a & b - t \\ \int_c^t u(x)\, dx & \int_t^d u(x)\, dx \end{array} \right|$$

and

$$\theta = \max \{ c - a, b - d \} = \frac{b-a}{2} - \frac{d-c}{2} + \left| \frac{b+a}{2} - \frac{d+c}{2} \right|.$$

If we put $u \equiv 1$ in the previous theorem we get the same result as in the Theorem 1.

In this paper we generalize these results and other results from [10], [7] and [5]. We will estimate the difference between two weighted integral means, each having it's own weight, on two different intervals $[a, b]$ and $[c, d]$. This will be done for the functions whose derivatives f' are from L_p spaces, $1 \leqslant p \leqslant \infty$ and for both possible cases $[c, d] \subseteq [a, b]$ and $[a, b] \cap [c, d] = [c, b]$ (other two cases, when $[a, b] \cap [c, d] \neq \emptyset$ we simply get by change $a \leftrightarrow c$, $b \leftrightarrow d$). In [10] it was done for the case $[c, d] \subseteq [a, b]$ without weight function, and in [7] for the same case with only one weight function. In the Section 2 we use weighted Montgomery identity with two different methods. In the Section 3 we generalize the result for the functions whose n-th derivatives $f^{(n)}$ are from L_p spaces, $1 \leqslant p \leqslant \infty$. In Section 4 we estimate integral means via Euler type identities for case $[a, b] \cap [c, d] = [c, b]$ (generalization of results from [10]) and also we give the proof of the sharpness of given inequalities from [10].

2. Estimates of the difference between two weighted integral means via weighted Montgomery identity.

In this section we will give the estimates of the difference between two weighted integral means using the weighted Montgomery identity. The first method is by subtracting two weighted Montgomery identities (which is possible for both cases $a \leqslant c < d \leqslant b$ and $a \leqslant c < b \leqslant d$). The second is by integrating weighted Montgomery identity (only for the case $a \leqslant c < d \leqslant b$). This will be done for the functions whose derivatives f' are from L_p spaces, $1 \leqslant p \leqslant \infty$.

The weighted Montgomerty's identity for Riemann-Stieltjes integrals is (see [3]):

$$f(x) = \frac{1}{W(b) - W(a)} \int_a^b f(t) \, dW(t) + \int_a^b P(W(x), W(t)) \, df(t), \quad (1)$$

where $f : [a, b] \rightarrow \mathbb{R}$ is continuous function of bounded variation on $[a, b]$, $W : [a, b] \rightarrow \mathbb{R}$ is of bounded variation such that $W(a) \neq W(b)$, $x \in [a, b]$ and

$$P(W(x), W(t)) = \begin{cases} \frac{W(t) - W(a)}{W(b) - W(a)}, & a \leqslant t \leqslant x, \\ \\ \frac{W(t) - W(b)}{W(b) - W(a)}, & x < t \leqslant b, \end{cases}$$

is a weighted Peano kernel.

Let's take $W(t) = \int_a^t w(x) \, dx$, $t \in [a, b]$ (so $dW(t) = w(t) \, dt$), where $w : [a, b] \rightarrow [0, \infty\rangle$ is some nonnegative normalized weighted density function i.e. integrable function satisfying $\int_a^b w(t) \, dt = 1$. If $t \leqslant a$, $W(t) = 0$ and if $t \geqslant b$, $W(t) = 1$. The weighted Peano kernel in this case reduces to:

$$P(W(x), W(t)) = \begin{cases} W(t), & a \leqslant t \leqslant x, \\ \\ W(t) - 1, & x < t \leqslant b. \end{cases} \quad (2)$$

THEOREM 3. *Let* $f : [a, b] \cup [c, d] \rightarrow \mathbb{R}$ *be a continuous function of bounded variation on* $[a, b] \cup [c, d]$, $w : [a, b] \rightarrow [0, \infty\rangle$ *and* $u : [c, d] \rightarrow [0, \infty\rangle$ *some normalized weighted functions,*

$$W(t) = \begin{cases} 0, & t < a, \\ \int_a^t w(t) \, dt, & a \leqslant t \leqslant b, \\ 1, & t > b, \end{cases}$$

$$U(t) = \begin{cases} 0, & t < c, \\ \int_c^t u(t) \, dt & c \leqslant t \leqslant d, \\ 1, & t > d, \end{cases} \quad (3)$$

and $[a, b] \cap [c, d] \neq \emptyset$. *Then, for both cases* $[c, d] \subseteq [a, b]$ *and* $[a, b] \cap [c, d] = [c, b]$, *(and also for* $[a, b] \subseteq [c, d]$ *and* $[a, b] \cap [c, d] = [a, d]$*) the next formula is valid*

$$\int_a^b w(t) f(t) \, dt - \int_c^d u(t) f(t) \, dt = \int_{\min\{a,c\}}^{\max\{b,d\}} K(t) \, df(t) \quad (4)$$

where

$$K(t) = U(t) - W(t), \ t \in [\min\{a, c\}, \max\{b, d\}].$$

Proof. For $x \in [a, b] \cap [c, d]$, we subtract identities

$$f(x) = \int_a^b w(t) f(t) \, dt + \int_a^b P(W(x), W(t)) \, df(t),$$

and

$$f(x) = \int_c^d u(t) f(t)\, dt + \int_c^d P(U(x), U(t))\, df(t).$$

Then put

$$K(x,t) = P(U(x), U(t)) - P(W(x), W(t)).$$

$K(x,t)$ doesn't depend on x, so we put $K(t)$ instead:

$$K(t) = \begin{cases} -W(t), & t \in [a,c], \\ -W(t) + U(t), & t \in \langle c,d \rangle, \\ 1 - W(t), & t \in [d,b], \end{cases} \quad \text{if} \quad [c,d] \subseteq [a,b]$$

$$K(t) = \begin{cases} -W(t), & t \in [a,c], \\ -W(t) + U(t), & t \in \langle c,b \rangle, \\ U(t) - 1, & t \in [b,d]. \end{cases} \quad \text{if} \quad [a,b] \cap [c,d] = [c,b]$$

In another way, the equality $K(t) = U(t) - W(t) = \int_a^t (u(x) - w(x))\, dx,\ t \in [a, \max\{b,d\}]$ is the obvious consequence of integration by parts of the left side of the equality (4). $\quad \square$

The next corollary is the generalisation of the Theorem 2 (from [7]).

COROLLARY 1. *Assume (p,q) is a pair of conjugate exponents, that is $1 \leqslant p, q \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $|f'|^p : [a,b] \cup [c,d] \to \mathbb{R}$ be a R-integrable function. Then we have*

$$\left| \int_a^b w(t) f(t)\, dt - \int_c^d u(t) f(t)\, dt \right| \leqslant \left(\int_{\min\{a,c\}}^{\max\{b,d\}} |K(t)|^q\, dt \right)^{\frac{1}{q}} \left\| f' \right\|_p.$$

The constant $\left(\int_{\min\{a,c\}}^{\max\{b,d\}} |K(t)|^q\, dt \right)^{\frac{1}{q}}$ is sharp for $1 < p \leqslant \infty$ and the best possible for $p = 1$.

Proof. Use the identity (1) and apply the Hölder inequality to obtain

$$\left| \int_a^b w(t) f(t)\, dt - \int_c^d u(t) f(t)\, dt \right|$$

$$= \left| \int_{\min\{a,c\}}^{\max\{b,d\}} K(t) f'(t)\, dt \right| \leqslant \left(\int_{\min\{a,c\}}^{\max\{b,d\}} |K(t)|^q\, dt \right)^{\frac{1}{q}} \left\| f' \right\|_p.$$

For the proof of the optimality of the constant $\left(\int_{\min\{a,c\}}^{\max\{b,d\}} |K(t)|^q \, dt \right)^{\frac{1}{q}}$ we will find a function f such that

$$\left| \int_{\min\{a,c\}}^{\max\{b,d\}} K(t) f'(t) \, dt \right| = \left(\int_{\min\{a,c\}}^{\max\{b,d\}} |K(t)|^q \, dt \right)^{\frac{1}{q}} \left(\int_{\min\{a,c\}}^{\max\{b,d\}} |f'(t)|^p \, dt \right)^{\frac{1}{p}}.$$

For $1 < p < \infty$ take f to be such that

$$f'(t) = sgn\, K(t) \cdot |K(t)|^{\frac{1}{p-1}}.$$

For $p = \infty$ take

$$f'(t) = sgn\, K(t).$$

For $p = 1$ we shall prove that

$$\left| \int_{\min\{a,c\}}^{\max\{b,d\}} K(t) f'(t) \, dt \right| \leqslant \max_{t \in [a,b] \cup [c,d]} |K(t)| \left(\int_{\min\{a,c\}}^{\max\{b,d\}} |f'(t)| \, dt \right) \tag{5}$$

is the best possible inequality. Suppose that $|K(t)|$ attains its maximum at $t_0 \in [a,b] \cup [c,d]$. First we assume that $K(t_0) > 0$. For ε small enough define $f_\varepsilon(t)$ by

$$f_\varepsilon(t) = \begin{cases} 0, & \min\{a,c\} \leqslant t \leqslant t_0, \\ \frac{1}{\varepsilon}(t - t_0), & t_0 \leqslant t \leqslant t_0 + \varepsilon, \\ 1, & t_0 + \varepsilon \leqslant t \leqslant \max\{b,d\}. \end{cases}$$

Then, for ε small enough

$$\left| \int_{\min\{a,c\}}^{\max\{b,d\}} K(t) f'(t) \, dt \right| = \left| \int_{t_0}^{t_0 + \varepsilon} K(t) \frac{1}{\varepsilon} dt \right| = \frac{1}{\varepsilon} \int_{t_0}^{t_0 + \varepsilon} K(t) \, dt.$$

Now, from inequality (5) we have

$$\frac{1}{\varepsilon} \int_{t_0}^{t_0 + \varepsilon} K(t) \, dt \leqslant K(t_0) \int_{t_0}^{t_0 + \varepsilon} \frac{1}{\varepsilon} dt = K(t_0).$$

Since,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t_0}^{t_0 + \varepsilon} K(t) \, dt = K(t_0)$$

the statement follows. In case $K(t_0) < 0$, we take

$$f_\varepsilon(t) = \begin{cases} 1, & \min\{a,c\} \leqslant t \leqslant t_0, \\ -\frac{1}{\varepsilon}(t - t_0 - \varepsilon), & t_0 \leqslant t \leqslant t_0 + \varepsilon, \\ 0, & t_0 + \varepsilon \leqslant t \leqslant \max\{b,d\}. \end{cases}$$

and the rest of proof is the same as above. $\square$

REMARK 1. In the special case, if we take $w(t) = \frac{1}{b-a}$, $t \in [a,b]$ and $u(t) = \frac{1}{d-c}$, $t \in [c,d]$ the inequality from the Corollary 1, for $a \leqslant c < d \leqslant b$ reduces to the inequality from the Theorem 1.

COROLLARY 2. *Let* $f : [a,d] \to \mathbb{R}$ *be such that* f' *exists and is bounded on* $[a,d]$. *Then for* $a \leqslant c < b \leqslant d$ *we have inequality*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - \frac{1}{d-c} \int_c^d f(t)\,dt \right| \leqslant \left[\frac{c-a+d-b}{2} \right] \cdot \left\| f' \right\|_\infty. \tag{6}$$

Proof. If we put $p = \infty$, $q = 1$, and $w(t) = \frac{1}{b-a}$, $t \in [a,b]$ and $u(t) = \frac{1}{d-c}$, $t \in [c,d]$ in the inequality from the Corollary1 we have

$$\int_a^d |K(t)|\,dt = \int_a^c |-W(t)|\,dt + \int_c^b |-W(t) + U(t)|\,dt + \int_b^d |U(t) - 1|\,dt.$$

Then

$$I_1 = \int_a^c |-W(t)|\,dt = \int_a^c W(t)\,dt = \int_a^c \frac{t-a}{b-a}\,dt = \frac{(c-a)^2}{2(b-a)},$$

$$I_3 = \int_b^d |U(t) - 1|\,dt = \int_b^d (1 - U(t))\,dt = \int_b^d \frac{d-t}{d-c}\,dt = \frac{(d-b)^2}{2(d-c)},$$

and

$$I_2 = \int_c^b |-W(t) + U(t)|\,dt = \int_c^b \left| \frac{a-t}{b-a} + \frac{t-c}{d-c} \right| dt$$

$$= \frac{1}{(b-a)(d-c)} \int_c^b |(b-a+c-d)t - bc + ad|\,dt.$$

Since for $f(t) = (b-a+c-d)t - bc + ad$ we have $f(c) = (c-a)(c-d) \leqslant 0$ and $f(b) = (b-a)(b-d) \leqslant 0$, we have

$$I_2 = \frac{1}{(b-a)(d-c)} \int_c^b ((a-b+d-c)t + bc - ad)\,dt$$

$$= \frac{1}{(b-a)(d-c)} \cdot \frac{((a-b+d-c)t + bc - ad)^2}{2} \cdot \frac{1}{a-b+d-c} \Bigg|_c^b$$

$$= \frac{[(b-a)(d-b)]^2 - [(c-a)(d-c)]^2}{2(b-a)(d-c)(a-b+d-c)}$$

$$= \left[\frac{(b-a)(d-b)^2}{2(d-c)} - \frac{(d-c)(c-a)^2}{2(b-a)} \right] \frac{1}{a-b+d-c}.$$

Consequently

$$\int_a^d |K(x,t)|\,dt = I_1 + I_2 + I_3 = \frac{(d-b)^2 - (c-a)^2}{2(a-b+d-c)} = \frac{c-a+d-b}{2}. \quad \square$$

COROLLARY 3. *Let* $f : [a,d] \to \mathbb{R}$ *be such that* f' *is an R-integrable on* $[a,d]$. *Then for* $a \leqslant c < b \leqslant d$ *we have inequality*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - \frac{1}{d-c} \int_c^d f(t)\,dt \right| \leqslant \frac{1}{2} \left(\frac{c-a}{b-a} + \frac{d-b}{d-c} + \left| \frac{c-a}{b-a} - \frac{d-b}{d-c} \right| \right) \cdot \left\| f' \right\|_1. \tag{7}$$

Proof. If we put $p = 1$, $q = \infty$, and $w(t) = \frac{1}{b-a}$, $t \in [a,b]$ and $u(t) = \frac{1}{d-c}$, $t \in [c,d]$ in the inequality from the Corollary 1 we have $\|K(t)\|_\infty = \sup_{t \in [a,d]} |K(t)|$ and

$$\sup_{t \in [a,c]} |K(t)| = \sup_{t \in [a,c]} |-W(t)| = \sup_{t \in [a,c]} W(t) = \sup_{t \in [a,c]} \frac{t-a}{b-a} = \frac{c-a}{b-a},$$

$$\sup_{t \in [b,d]} |K(t)| = \sup_{t \in [b,d]} |U(t) - 1| = \sup_{t \in [b,d]} (1 - U(t))$$

$$= \sup_{t \in [b,d]} \left(1 - \frac{t-c}{d-c}\right) = \sup_{t \in [b,d]} \frac{d-t}{d-c} = \frac{d-b}{d-c}$$

and

$$\sup_{t \in [c,b]} |K(t)| = \sup_{t \in [c,b]} |U(t) - W(t)|$$

$$= \sup_{t \in [c,b]} \left|\frac{t-c}{d-c} - \frac{t-a}{b-a}\right| = \max\left\{\frac{c-a}{b-a}, \frac{d-b}{d-c}\right\}.$$

So, we have

$$\sup_{t \in [a,d]} |K(t)| = \max\left\{\frac{c-a}{b-a}, \frac{d-b}{d-c}\right\}.$$

Since $0 \leqslant \frac{c-a}{b-a} \leqslant \frac{d-b}{d-c}$ or $0 \leqslant \frac{d-b}{d-c} \leqslant \frac{c-a}{b-a}$ we get

$$\max\left\{\frac{c-a}{b-a}, \frac{d-b}{d-c}\right\} = \frac{1}{2}\left(\frac{c-a}{b-a} + \frac{d-b}{d-c} + \left|\frac{c-a}{b-a} - \frac{d-b}{d-c}\right|\right). \quad \square$$

COROLLARY 4.   *Assume (p,q) is a pair of conjugate exponents, that is $1 < p,q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $|f'|^p : [a,d] \to \mathbb{R}$ be an R-integrable function. Then for $a \leqslant c < b \leqslant d$, we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{d-c}\int_c^d f(t)\,dt\right|$$

$$\leqslant \left[\left[\frac{(d-b)^{q+1}}{(d-c)^{q-1}} - \frac{(c-a)^{q+1}}{(b-a)^{q-1}}\right]\frac{1}{(q+1)(a-b+d-c)}\right]^{\frac{1}{q}} \cdot \|f'\|_p. \quad (8)$$

Proof. For $p,q \neq 1,\infty$ and $w(t) = \frac{1}{b-a}$, $t \in [a,b]$ and $u(t) = \frac{1}{d-c}$, $t \in [c,d]$ in the inequality from the Corollary 1 gives

$$\int_a^c |-W(t)|^q\,dt = \int_a^c \left(\frac{t-a}{b-a}\right)^q dt = \frac{(c-a)^{q+1}}{(q+1)(b-a)^q},$$

$$\int_b^d |U(t) - 1|^q\,dt = \int_b^d \left(\frac{d-t}{d-c}\right)^q dt = \frac{(d-b)^{q+1}}{(q+1)(d-c)^q}$$

and simillary as in the proof of the Corollary 2

$$
\int_c^b |-W(t) + U(t)|^q \, dt = \int_c^b \left| \frac{a-t}{b-a} + \frac{t-c}{d-c} \right|^q dt
$$

$$
= \frac{[(b-a)(d-b)]^{q+1} - [(c-a)(d-c)]^{q+1}}{(b-a)^q (d-c)^q (q+1)(a-b+d-c)}
$$

$$
= \left[\frac{(b-a)(d-b)^{q+1}}{(d-c)^q} - \frac{(d-c)(c-a)^{q+1}}{(b-a)^q} \right] \frac{1}{(q+1)(a-b+d-c)}.
$$

Cosenquently,

$$
\int_a^d |K(t)|^q \, dt = \left[\frac{(d-b)^{q+1}}{(d-c)^{q-1}} - \frac{(c-a)^{q+1}}{(b-a)^{q-1}} \right] \frac{1}{(q+1)(a-b+d-c)}.
$$

Applying Corollary 1 we have the proof. □

COROLLARY 5. *Assume (p,q) is a pair of conjugate exponents, that is $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $|f'|^p : [a,d] \to \mathbb{R}$ be an R-integrable function. Then for $x \in [a,d]$ we have*

$$
\left| \frac{1}{x-a} \int_a^x f(t) \, dt - \frac{1}{d-x} \int_x^d f(t) \, dt \right|
$$

$$
\leqslant \begin{cases} \frac{d-a}{2} \cdot \|f'\|_\infty, & f' \in L_\infty[a,b], \\[2ex] \left[\frac{d-a}{q+1} \right]^{\frac{1}{q}} \cdot \|f'\|_p, & f' \in L_p[a,b], \\[2ex] \|f'\|_1, & f' \in L_1[a,b]. \end{cases}
$$

Proof. By setting $b = c = x$ in the inequalities (6), (8), (7). □

REMARK 2. The incorrect version of the inequality from Corollary 5 was first proved by P. Cerone in $[5]$, with $\left[\frac{(x-a)^q + (d-x)^q}{q+1} \right]^{\frac{1}{q}}$ instead of $\left[\frac{d-a}{q+1} \right]^{\frac{1}{q}}$.

COROLLARY 6. *Assume (p,q) is a pair of conjugate exponents, that is $1 \leqslant p, q \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $|f'|^p : [a,b] \to \mathbb{R}$ be a R-integrable function. Then for*

$a \leqslant c < d \leqslant b$, we have

$$\left| \int_a^b w(t) f(t)\, dt - \int_c^d u(t) f(t)\, dt \right|$$

$$\leqslant \begin{cases} \|f'\|_\infty \left[\int_a^c W(t)\, dt + \int_d^b [1 - W(t)]\, dt + \int_c^d |W(t) - U(t)|\, dt \right], \\ \qquad\qquad f' \in L_\infty[a,b], \\ \|f'\|_p \left[\int_a^c W(t)^q\, dt + \int_d^b [1 - W(t)]^q\, dt + \int_c^d |W(t) - U(t)|^q\, dt \right]^{\frac{1}{q}}, \\ \qquad\qquad f' \in L_p[a,b], \\ \|f'\|_1 \max\left\{ W(c), 1 - W(d), \sup_{t \in [c,d]} |W(t) - U(t)| \right\}. \\ \qquad\qquad f' \in L_1[a,b], \end{cases}$$

and for $a \leqslant c < b \leqslant d$

$$\left| \int_a^b w(t) f(t)\, dt - \int_c^d u(t) f(t)\, dt \right|$$

$$\leqslant \begin{cases} \|f'\|_\infty \left[\int_a^c W(t)\, dt + \int_b^d [1 - U(t)]\, dt + \int_c^b |W(t) - U(t)|\, dt \right], \\ \qquad\qquad f' \in L_\infty[a,d], \\ \|f'\|_p \left[\int_a^c W(t)^q\, dt + \int_b^d [1 - U(t)]^q\, dt + \int_c^b |W(t) - U(t)|^q\, dt \right]^{\frac{1}{q}}, \\ \qquad\qquad f' \in L_p[a,d], \\ \|f'\|_1 \max\left\{ W(c), 1 - U(b), \sup_{t \in [c,b]} |W(t) - U(t)| \right\}. \\ \qquad\qquad f' \in L_1[a,d]. \end{cases}$$

Proof. Directly from the Corollary 1. $\square$

The first inequality from the next theorem may be regarded as the weighted Ostrowsky inequality.

THEOREM 4. *Assume (p,q) is a pair of conjugate exponents, that is $1 \leqslant p, q \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $|f'|^p : [a,b] \to \mathbb{R}$ be a R-integrable function. Then*

$$\left| \int_a^b w(t) f(t)\, dt - f(x) \right|$$

$$\leqslant \begin{cases} \|f'\|_\infty \left[\int_a^x W(t)\, dt + \int_x^b [1 - W(t)]\, dt \right], & f' \in L_\infty[a,b], \\ \|f'\|_p \left[\int_a^x W(t)^q\, dt + \int_x^b [1 - W(t)]^q\, dt \right]^{\frac{1}{q}}, & f' \in L_p[a,b], \\ \|f'\|_1 \max\{ W(x), 1 - W(x) \}. & f' \in L_1[a,b]. \end{cases}$$

Proof. By setting $c = d = x$ and $u(t) = \frac{1}{d-c}$, $t \in [c,d]$ in the first inequality from the Corollary 6, and assuming $\frac{1}{d-c} \int_c^d f(t)\, dt = f(x)$ as a limit case. $\square$

THEOREM 5. *Assume (p, q) is a pair of conjugate exponents, that is $1 \leqslant p, q \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $|f'|^p : [a, d] \to \mathbb{R}$ be a R-integrable function. Then for $x \in [a, d]$ we have*

$$\left| \int_a^x w(t) f(t) \, dt - \int_x^d u(t) f(t) \, dt \right|$$

$$\leqslant \begin{cases} \|f'\|_\infty \left[\int_a^x W(t) \, dt + \int_x^d [1 - U(t)] \, dt \right], & f' \in L_\infty[a, d], \\[3mm] \|f'\|_p \left[\int_a^x W(t)^q \, dt + \int_x^d [1 - U(t)]^q \, dt \right]^{\frac{1}{q}}, & f' \in L_p[a, d], \\[3mm] \|f'\|_1 \max \{ W(x), 1 - U(x), |W(x) - U(x)| \}. & f' \in L_1[a, d]. \end{cases}$$

Proof. By setting $b = c = x$ in the second inequality from the Corollary 6. $\square$

The second method of giving the estimates of the difference between two weighted integral means for functions whose first derivatives are in L_1 or L_p or L_∞ spaces, is by integrating weighted Montgomery identity.

THEOREM 6. *Let $f : [a, b] \to \mathbb{R}$ be such that $|f'|^p : [a, b] \to \mathbb{R}$ is a R-integrable function $1 \leqslant p \leqslant \infty$, and $w : [a, b] \to [0, \infty)$, $u : [c, d] \to [0, \infty)$ are normalized weighted functions, $W(t)$ and $U(t)$ as before (3). Let also (p, q) be a pair of conjugate exponents. Then, for the $[c, d] \subseteq [a, b]$ the following inequalities hold:*

$$\left| \int_a^b w(t) f(t) \, dt - \int_c^d u(t) f(t) \, dt \right|$$

$$\leqslant \begin{cases} \|f'\|_\infty \int_c^d \left[\int_a^x W(t) \, dt + \int_x^b (1 - W(t)) \, dt \right] |u(x)| \, dx, & f' \in L_\infty[a, b], \\[3mm] \|f'\|_p \int_c^d \left[\int_a^x W(t)^q \, dt + \int_x^b (1 - W(t))^q \, dt \right]^{\frac{1}{q}} |u(x)| \, dx, & f' \in L_p[a, b], \\[3mm] \|f'\|_1 \int_c^d \left[\frac{1}{2} + \left| W(x) - \frac{1}{2} \right| \right] |u(x)| \, dx, & f' \in L_1[a, b]. \end{cases}$$

Proof. Using identity (1) we obtain

$$\int_c^d u(x) f(x) \, dx - \int_a^b f(t) w(t) \, dt = \int_c^d \left(\int_a^b P(W(x), W(t)) f'(t) \, dt \right) u(x) \, dx \tag{9}$$

so

$$\left| \int_a^b f(t) w(t) \, dt - \int_c^d u(t) f(t) \, dt \right| \leqslant \int_c^d \left| \int_a^b P(W(x), W(t)) f'(t) \, dt \right| |u(x)| \, dx,$$

and therefore using the properties of modulus and Hölder inequality

$$\left| \int_a^b P\left(W\left(x\right), W\left(t\right)\right) f'\left(t\right) dt \right|$$

$$\leq \begin{cases} \|f\|_\infty \left[\int_a^x W\left(t\right) dt + \int_x^b \left(1 - W\left(t\right)\right) dt \right], & f' \in L_\infty\left[a, b\right], \\[2mm] \|f\|_p \left[\int_a^x W\left(t\right)^q dt + \int_x^b \left(1 - W\left(t\right)\right)^q dt \right]^{\frac{1}{q}}, & f' \in L_p\left[a, b\right], \\[2mm] \|f\|_1 \max\left\{W\left(x\right), 1 - W\left(x\right)\right\}, & f' \in L_1\left[a, b\right]. \end{cases}$$

The proof folows since

$$\max\left\{A, B\right\} = \frac{A + B}{2} + \left| \frac{A - B}{2} \right|. \quad \square$$

REMARK 3.   The result from Theorem 3

$$\int_a^b f\left(t\right) w\left(t\right) dt - \int_c^d u\left(t\right) f\left(t\right) dt$$

$$= \int_a^c \left[-W\left(t\right)\right] f'\left(t\right) dt + \int_c^d \left[U\left(t\right) - W\left(t\right)\right] f'\left(t\right) dt + \int_d^b \left[1 - W\left(t\right)\right] f'\left(t\right) dt$$

we could also get from formula (9) by an interchange of the order of integration in the following way

$$\int_c^d u\left(x\right) \left(\int_a^b P\left(W\left(x\right), W\left(t\right)\right) f'\left(t\right) dt \right) dx$$

$$= \int_c^d u\left(x\right) \left(\int_a^x W\left(t\right) f'\left(t\right) dt \right) dx + \int_c^d u\left(x\right) \left(\int_x^b \left[W\left(t\right) - 1\right] f'\left(t\right) dt \right) dx$$

$$= \int_a^c W\left(t\right) f'\left(t\right) \left(\int_c^d u\left(x\right) dx \right) dt + \int_c^d W\left(t\right) f'\left(t\right) \left(\int_t^d u\left(x\right) dx \right) dt$$

$$+ \int_d^b \left[W\left(t\right) - 1\right] f'\left(t\right) \left(\int_c^d u\left(x\right) dx \right) dt$$

$$+ \int_c^d \left[W\left(t\right) - 1\right] f'\left(t\right) \left(\int_c^t u\left(x\right) dx \right) dt$$

$$= \int_a^c W\left(t\right) f'\left(t\right) dt + \int_d^b \left[W\left(t\right) - 1\right] f'\left(t\right) dt$$

$$+ \int_c^d \left[W\left(t\right) \left[1 - U\left(t\right)\right] + \left[W\left(t\right) - 1\right] U\left(t\right)\right] f'\left(t\right) dt$$

$$= \int_a^c W\left(t\right) f'\left(t\right) dt + \int_d^b \left[W\left(t\right) - 1\right] f'\left(t\right) dt + \int_c^d \left[W\left(t\right) - U\left(t\right)\right] f'\left(t\right) dt$$

By substitution into (9) we get the result.

THEOREM 7.　*Let f , w and u satisfy the conditions of Theorem 6 and $[c,d] \subseteq [a,b]$. Then the following inequities are valid*

$$\left| \int_c^d u(t)f(t)\,dt - \left[\int_c^d w(t)f(t)\,dt + W(c)f(c) + [1 - W(d)]f(d) \right] \right|$$

$$\leqslant \begin{cases} \|f'\|_\infty \int_c^d |W(t) - U(t)|\,dt, & f' \in L_\infty[a,b], \\[2ex] \|f'\|_p \left[\int_c^d |W(t) - U(t)|^q\,dt \right]^{\frac{1}{q}}, & f' \in L_\infty[a,b], \\[2ex] \|f'\|_1 \sup_{t \in [c,d]} |W(t) - U(t)|, & f' \in L_1[a,b]. \end{cases}$$

Proof.　From Remark 3 we have

$$\int_c^d u(t)f(t)\,dt - \int_a^b w(t)f(t)\,dt - \int_a^c W(t)f'(t)\,dt - \int_d^b [W(t) - 1]f'(t)\,dt$$

$$= \int_c^d (W(t) - U(t))f'(t)\,dt.$$

Integration by parts gives

$$\int_a^c W(t)f'(t)\,dt = W(c)f(c) - \int_a^c w(t)f(t)\,dt$$

and

$$\int_d^b [W(t) - 1]f'(t)\,dt = -[W(d) - 1]f(d) - \int_d^b w(t)f(t)\,dt.$$

So we have

$$\int_c^d u(t)f(t)\,dt - \int_a^b f(t)w(t)\,dt - W(c)f(c)$$

$$+ \int_a^c w(t)f(t)\,dt + [W(d) - 1]f(d) + \int_d^b w(t)f(t)\,dt$$

$$= \int_c^d u(t)f(t)\,dt - \left[\int_c^d f(t)w(t)\,dt + W(c)f(c) - [W(d) - 1]f(d) \right]$$

$$= \int_c^d (W(t) - U(t))f'(t)\,dt$$

Taking the modulus of the last equality and using the properties of modulus and Hölder inequality we get the proof.　□

THEOREM 8. *Let f , w and u satisfy the conditions of Theorem 7 and . Then for $[a,b] \cap [c,d] = [c,b]$ the following inequities are valid*

$$\left| \int_c^b u(t)f(t)\,dt - \left[\int_c^b w(t)f(t)\,dt + W(c)f(c) + [1 - U(b)]f(b) \right] \right|$$

$$\leqslant \begin{cases} \|f'\|_\infty \int_c^b |W(t) - U(t)|\,dt, & f' \in L_\infty [a,d], \\[2mm] \|f'\|_p \left[\int_c^b |W(t) - U(t)|^q\,dt \right]^{\frac{1}{q}}, & f' \in L_\infty [a,d], \\[2mm] \|f'\|_1 \sup_{t \in [c,b]} |W(t) - U(t)|, & f' \in L_1 [a,d]. \end{cases}$$

Proof. Similar as the proof of the previous Theorem. □

REMARK 4. In the special case when we take $b = d$, all the results from all the theorems and the corollaries from this section, i.e for the case $a \leqslant c < b \leqslant d$, coincide with all results for the case $a \leqslant c < d \leqslant b$, with $b = d$ (the only exeption are Theorem 4 and Theorem 5).

3. Generalization for functions whose derivatives of order n are in L_p space, $1 \leqslant p \leqslant \infty$

THEOREM 9. *Assume (p,q) is a pair of conjugate exponents, that is $1 \leqslant p,q \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $f : [a,b] \cup [c,d] \to \mathbb{R}$ be n-times differentiable on $[a,b] \cup [c,d]$, $n \in \mathbb{N}$ and $|f^{(n)}|^p : [a,b] \cup [c,d] \to \mathbb{R}$ is integrable on $[a,b] \cup [c,d]$. $w : [a,b] \to [0,\infty\rangle$ and $u : [c,d] \to [0,\infty\rangle$ are some normalized weighted functions, $W(t)$, $U(t)$ as before (3). Then for every $x \in [a,b] \cap [c,d]$*

$$\left| \int_a^b w(t)f(t)\,dt - \int_c^d u(t)f(t)\,dt + \sum_{k=0}^{n-2} \left(\int_a^b f^{(k+1)}(s_1) w(s_1)\,ds_1 \right) \right.$$

$$\cdot \left(\underbrace{\int_a^b \cdots \int_a^b}_{(k+1)\text{-th integral}} P(W(x), W(s_1)) \prod_{i=1}^k P(W(s_i), W(s_{i+1}))\,ds_1 \cdots ds_{k+1} \right)$$

$$\left. - \sum_{k=0}^{n-2} \left(\int_c^d f^{(k+1)}(s_1) u(s_1)\,ds_1 \right) \right.$$

$$\cdot \left(\underbrace{\int_c^d \cdot \cdot \int_c^d}_{(k+1)th\ integral} P\left(U\left(x\right), U\left(s_1\right)\right) \prod_{i=1}^{k} P\left(U\left(s_i\right), U\left(s_{i+1}\right)\right) ds_1 \cdot \cdot ds_{k+1} \right) \Bigg|$$

$$\leqslant \left\| f^{(n)} \right\|_p \left(\int_{\min\{a,c\}}^{\max\{b,d\}} |\mathbf{K}\left(x, s_n\right)|^q ds_n \right)^{\frac{1}{q}},$$

where

$$\mathbf{K}\left(x, s_n\right) = \underbrace{\int_a^b \cdot \cdot \int_a^b}_{(n-1)th\ integral} P\left(W\left(x\right), W\left(s_1\right)\right) \prod_{i=1}^{n-1} P\left(W\left(s_i\right), W\left(s_{i+1}\right)\right) ds_1 \cdot \cdot ds_{n-1}$$

$$- \underbrace{\int_c^d \cdot \cdot \int_c^d}_{(n-1)th\ integral} P\left(U\left(x\right), U\left(s_1\right)\right) \prod_{i=1}^{n-1} P\left(U\left(s_i\right), U\left(s_{i+1}\right)\right) ds_1 \cdot \cdot ds_{n-1}$$

and we suppose that

$$\underbrace{\int_a^b \cdot \cdot \int_a^b}_{(n-1)th\ integral} P\left(W\left(x\right), W\left(s_1\right)\right) \prod_{i=1}^{n-1} P\left(W\left(s_i\right), W\left(s_{i+1}\right)\right) ds_1 \cdot \cdot ds_{n-1}$$

equals zero for $s_n \notin [a, b]$ *and*

$$\underbrace{\int_c^d \cdot \cdot \int_c^d}_{(n-1)th\ integral} P\left(U\left(x\right), U\left(s_1\right)\right) \prod_{i=1}^{n-1} P\left(U\left(s_i\right), U\left(s_{i+1}\right)\right) ds_1 \cdot \cdot ds_{n-1}$$

equals zero for $s_n \notin [c, d]$.

Proof. First we give the generalized weighted Montgomery identity (see [3]):

$$f\left(x\right) - \int_a^b f\left(s_1\right) w\left(s_1\right) ds_1 - \sum_{k=0}^{n-2} \left(\int_a^b f^{(k+1)}\left(s_1\right) w\left(s_1\right) ds_1 \right)$$

$$\cdot \left(\underbrace{\int_a^b \cdot \cdot \int_a^b}_{(k+1)th\ integral} P\left(W\left(x\right), W\left(s_1\right)\right) \prod_{i=1}^{k} P\left(W\left(s_i\right), W\left(s_{i+1}\right)\right) ds_1 \cdot \cdot ds_{k+1} \right)$$

$$= \underbrace{\int_a^b \cdot \cdot \int_a^b}_{n\text{-}th\ integral} P\left(W\left(x\right), W\left(s_1\right)\right) \prod_{i=1}^{n-1} P\left(W\left(s_i\right), W\left(s_{i+1}\right)\right) f^{(n)}\left(s_n\right) ds_1 \cdot \cdot ds_n. \tag{10}$$

Now, as in Theorem 3, we subtract two weighted Montgomery identities, one for interval $[a, b]$ and the other for $[c, d]$, and the statement follows. $\square$

THEOREM 10. *Assume* (p, q) *is a pair of conjugate exponents, that is* $1 \leqslant p, q \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. *Let* $f : [a, b] \to \mathbb{R}$ *be n-times differentiable on* $[a, b]$, $n \in \mathbb{N}$ *and* $\left| f^{(n)} \right|^p : [a, b] \to \mathbb{R}$ *is integrable on* $[a, b]$. $w : [a, b] \to [0, \infty)$ *and* $u : [c, d] \to [0, \infty)$ *are some normalized weighted functions,* $W(t)$, $U(t)$ *as before. Then for* $a \leqslant c < d \leqslant b$

$$
\left| \int_c^d u(t) f(t)\, dt - \int_a^b w(t) f(t)\, dt - \int_c^d u(x) \sum_{k=0}^{n-2} \left(\int_a^b f^{(k+1)}(s_1)\, w(s_1)\, ds_1 \right) \right.
$$
$$
\left. \cdot \left(\underbrace{\int_a^b \cdots \int_a^b}_{(k+1)th\ integral} P(W(x), W(s_1)) \prod_{i=1}^k P(W(s_i), W(s_{i+1})) \cdot ds_1 \cdots ds_{k+1} \right) dx \right|
$$

$$
\leqslant \left\| f^{(n)} \right\|_p \int_c^d \left(\int_a^b \left| \underbrace{\int_a^b \cdots \int_a^b}_{(n-1)th\ integral} P(W(x), W(s_1)) \right. \right.
$$
$$
\left. \left. \cdot \prod_{i=1}^{n-1} P(W(s_i), W(s_{i+1}))\, ds_1 \cdots ds_{n-1} \right|^q ds_n \right)^{\frac{1}{q}} |u(x)|\, dx.
$$

Proof. As in Theorem 6, first we multiply weighted Montgomery identity by $u(x)$ and then integrate it. Then the statement follows. $\square$

REMARK 5. For $n = 1$ the inequality from Theorem 10 reduces to the inequality from Theorem 6.

4. Estimaties of difference between two integral means via Euler-type identities

4.1. Equivalence of Montgomery identity and Euler identity

For every function $f : [a, b] \to \mathbb{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a, b]$ for some $n \geqslant 1$ and for every $x \in [a, b]$, the following two formulae have been poved (see $[8]$):

$$
f(x) = \frac{1}{b-a} \int_a^b f(t)\, dt + T_n^{[a,b]}(x) + P_n^{[a,b]}(x), \tag{11}
$$

and

$$
f(x) = \frac{1}{b-a} \int_a^b f(t)\, dt + T_{n-1}^{[a,b]}(x) + R_n^{[a,b]}(x), \tag{12}
$$

where

$$T_m^{[a,b]}(x) = \sum_{k=1}^{m} \frac{(b-a)^{k-1}}{k!} B_k \left(\frac{x-a}{b-a} \right) \left[f^{(k-1)}(b) - f^{(k-1)}(a) \right]$$

with convention $T_0^{[a,b]}(x) = 0$, and

$$P_n^{[a,b]}(x) = -\frac{(b-a)^{n-1}}{n!} \int_a^b \left[B_n^* \left(\frac{x-t}{b-a} \right) \right] df^{(n-1)}(t),$$

$$R_n^{[a,b]}(x) = -\frac{(b-a)^{n-1}}{n!} \int_a^b \left[B_n^* \left(\frac{x-t}{b-a} \right) - B_n \left(\frac{x-a}{b-a} \right) \right] df^{(n-1)}(t).$$

Here $B_k(x)$, $k \geqslant 0$, are the Bernoulli polynomials, $B_k = B_k(0)$, $k \geqslant 0$, the Bernoulli numbers, and $B_k^*(x)$, $k \geqslant 0$, are periodic functions of period 1, related to Bernoulli polynomials as

$$B_k^*(x) = B_k(x), \quad 0 \leqslant x < 1, \qquad B_k^*(x+1) = B_k^*(x), \quad x \in \mathbb{R}.$$

From the properties of Bernoulli polynomials it follows $B_0^*(t) = 1, B_1^*$ is a discontinuous function with the jump of -1 at each integer, and B_k^*, $k \geqslant 2$, is a continuous function (see $[1]$).

REMARK 6.   In the special case, if we take $w(t) = \frac{1}{b-a}$, $t \in [a, b]$ and $u(t) = \frac{1}{d-c}$, $t \in [c, d]$ the weighted Montgomery identity reduces to the Euler identity (12), see $[2]$. For the readers conviniens we give the shorter proof of this fact in the next Lemma.

LEMMA 1.   *For all $k \in \{0, 1, 2, 3, ..\}$ we have*

$$\underbrace{\int_a^b \cdots \int_a^b}_{(k+1)\text{-th integral}} P(x, s_1) P(s_1, s_2) \cdots P(s_k, s_{k+1}) \, ds_1 \cdots ds_{k+1}$$

$$= \frac{(b-a)^{k+1}}{(k+1)!} B_{k+1} \left(\frac{x-a}{b-a} \right). \tag{13}$$

Proof. We will prove this by induction with respect to k. For $k = 0$ and $x \in [a, b\rangle$ we have for the left side of (13) by integration by parts

$$\int_a^b P(x, s_1) \, ds_1 = P(x, b) \cdot b - P(x, a) \cdot a - \int_a^b s_1 dP(x, s_1)$$

$$= -\left(\int_a^b \frac{s_1}{b-a} ds_1 \right) - (-1)x = -\frac{b^2 - a^2}{2(b-a)} + x = x - \frac{a+b}{2},$$

since $P(x, b) = P(x, a) = 0$ and $P(x, s_1)$ is differentiable on $[a, b] \setminus \{x\}$ and its derivative is equal to $\frac{1}{b-a}$, and at x it has a jump of -1. Since $B_1(t) = t - \frac{1}{2}$, the right side of (13) is

$$\frac{(b-a)^1}{1!} B_1 \left(\frac{x-a}{b-a} \right) = (b-a) \left[\frac{x-a}{b-a} - \frac{1}{2} \right] = x - \frac{a+b}{2}.$$

For $x = b$ we have $P(b, s_1) = \frac{s_1 - a}{b - a}$, $s_1 \in [a, b]$ and equality (13) obviously holds. Thus, we have proved the base of induction. Let's denote

$$Q_{k+1}(x) = \underbrace{\int_a^b \cdots \int_a^b}_{(k+1)\text{-th integral}} P(x, s_1) P(s_1, s_2) \cdots P(s_k, s_{k+1}) \, ds_1 \cdots ds_{k+1}$$

We can suppose now for $k > 0$ that

$$Q_k(y) = \frac{(b-a)^k}{k!} B_k \left(\frac{y - a}{b - a} \right), \quad y \in [a, b].$$

Then we have for $k + 1$ and for $x \in [a, b\rangle$

$$Q_{k+1}(x) = \int_a^b P(x, s_1) Q_k(s_1) \, ds_1 = \int_a^b P(x, s_1) \frac{(b-a)^k}{k!} B_k \left(\frac{s_1 - a}{b - a} \right) ds_1$$

Since $B_n'(t) = nB_{n-1}(t)$ we have

$$\frac{d}{ds_1} \left[\frac{b - a}{k + 1} B_{k+1} \left(\frac{s_1 - a}{b - a} \right) \right] = B_k \left(\frac{s_1 - a}{b - a} \right),$$

and by integration by parts

$$
\begin{aligned}
Q_{k+1}(x) &= \frac{(b-a)^{k+1}}{(k+1)!} \int_a^b P(x, s_1) \, dB_{k+1} \left(\frac{s_1 - a}{b - a} \right) \\
&= \frac{(b-a)^{k+1}}{(k+1)!} \left[-\int_a^b B_{k+1} \left(\frac{s_1 - a}{b - a} \right) dP(x, s_1) \right] \\
&= \frac{(b-a)^{k+1}}{(k+1)!} \left[-\int_a^b \frac{1}{b - a} B_{k+1} \left(\frac{s_1 - a}{b - a} \right) ds_1 - B_{k+1} \left(\frac{x - a}{b - a} \right) (-1) \right] \\
&= \frac{(b-a)^{k+1}}{(k+1)!} \left[-\left[\frac{1}{k + 2} B_{k+2} \left(\frac{s_1 - a}{b - a} \right) \right|_a^b \right] - B_{k+1} \left(\frac{x - a}{b - a} \right) (-1) \right] \\
&= \frac{(b-a)^{k+1}}{(k+1)!} B_{k+1} \left(\frac{x - a}{b - a} \right),
\end{aligned}
$$

since $B_k(1) = B_k(0)$ for all $k \geqslant 2$. It is easy to see that

$$Q_{k+1}(b) = \int_a^b \frac{s_1 - a}{b - a} \cdot \frac{(b-a)^k}{k!} B_k \left(\frac{s_1 - a}{b - a} \right) ds_1 = \frac{(b-a)^{k+1}}{(k+1)!} B_{k+1} \left(\frac{b - a}{b - a} \right)$$

The statement is valid also for $k + 1$, so it is valid for all $k \in \{0, 1, 2, 3, ...\}$ □

From this lemma we have the equality of the left sides of Montgomery identity and the Euler identity, so the equality of the right sides follows. So we cite the next lemma from [2] without the proof.

LEMMA 2.   *For all $n \in \mathbb{N}$ we have*

a)

$$\underbrace{\int_a^b \cdots \int_a^b}_{n\text{-th integral}} P\left(x, s_1\right) P\left(s_1, s_2\right) \cdots P\left(s_{n-1}, s_n\right) f^{(n)}\left(s_n\right) ds_1 \cdots ds_n$$

$$= \frac{(b-a)^{n-1}}{n!} \int_a^b \left[B_n\left(\frac{x-a}{b-a}\right) - B_n^*\left(\frac{x-t}{b-a}\right) \right] f^{(n)}(t)\, dt.$$

b)

$$\underbrace{\int_a^b \cdots \int_a^b}_{(n-1)\text{-th integral}} P\left(x, s_1\right) P\left(s_1, s_2\right) \cdots P\left(s_{n-1}, s_n\right) ds_1 \cdots ds_{n-1}$$

$$= \frac{(b-a)^{n-1}}{n!} \left[B_n\left(\frac{x-a}{b-a}\right) - B_n^*\left(\frac{x-s_n}{b-a}\right) \right].$$

4.2.   Case $[c, d] \subseteq [a, b]$ (i.e. $a \leqslant c < d \leqslant b$)

In the recent paper $[10]$, using formulae (11) and (12), following two theorems were obtained:

THEOREM 11.   *Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a, b]$ for some $n \geqslant 1$. Then for $x \in [c, d]$, if $[c, d] \subset [a, b]$ we have*

$$\frac{1}{b-a} \int_a^b f(t)\, dt - \frac{1}{d-c} \int_c^d f(t)\, dt + T_n^{[a,b]}(x) - T_n^{[c,d]}(x) = \int_a^b K_n^1(x, t)\, df^{(n-1)}(t)$$

and

$$\frac{1}{b-a} \int_a^b f(t)\, dt - \frac{1}{d-c} \int_c^d f(t)\, dt + T_{n-1}^{[a,b]}(x) - T_{n-1}^{[c,d]}(x) = \int_a^b K_n^2(x, t)\, df^{(n-1)}(t),$$

where

$$K_n^1(x, t) = \begin{cases} \dfrac{(b-a)^{n-1}}{n!} B_n\left(\frac{x-t}{b-a}\right), & t \in [a, c], \\[3mm] \dfrac{(b-a)^{n-1}}{n!} B_n^*\left(\frac{x-t}{b-a}\right) - \dfrac{(d-c)^{n-1}}{n!} B_n^*\left(\frac{x-t}{d-c}\right), & t \in \langle c, d \rangle, \\[3mm] \dfrac{(b-a)^{n-1}}{n!} B_n\left(\frac{x-t}{b-a} + 1\right), & t \in [d, b], \end{cases}$$

and

$$K_n^2(x,t) = \begin{cases} \frac{(b-a)^{n-1}}{n!}\left[B_n\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right], & t \in [a,c], \\[2ex] \frac{(b-a)^{n-1}}{n!}\left[B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right] \\[1ex] \quad - \frac{(d-c)^{n-1}}{n!}\left[B_n^*\left(\frac{x-t}{d-c}\right) - B_n\left(\frac{x-c}{d-c}\right)\right], & t \in \langle c,d \rangle, \\[2ex] \frac{(b-a)^{n-1}}{n!}\left[B_n\left(\frac{x-t}{b-a}+1\right) - B_n\left(\frac{x-a}{b-a}\right)\right], & t \in [d,b]. \end{cases}$$

THEOREM 12. *Assume (p,q) is a pair of conjugate exponents, that is $1 \leqslant p,q \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $\left|f^{(n)}\right|^p : [a,b] \to \mathbb{R}$ be an R-integrable function for some $n \geqslant 1$. Then for $a \leqslant c < d \leqslant b$, we have*

$$\left| \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{d-c}\int_c^d f(t)\,dt + T_n^{[a,b]}(x) - T_n^{[c,d]}(x) \right|$$

$$\leqslant \left(\int_a^b \left|K_n^1(x,t)\right|^q dt \right)^{\frac{1}{q}} \left\|f^{(n)}\right\|_p \tag{14}$$

and

$$\left| \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{d-c}\int_c^d f(t)\,dt + T_{n-1}^{[a,b]}(x) - T_{n-1}^{[c,d]}(x) \right|$$

$$\leqslant \left(\int_a^b \left|K_n^2(x,t)\right|^q dt \right)^{\frac{1}{q}} \left\|f^{(n)}\right\|_p, \tag{15}$$

for every $x \in [c,d]$.

Now we will prove that inequalities in the Theorem 12 are sharp.

THEOREM 13. *The constants $\left(\int_a^b \left|K_n^1(x,t)\right|^q dt\right)^{\frac{1}{q}}$ and $\left(\int_a^b \left|K_n^2(x,t)\right|^q dt\right)^{\frac{1}{q}}$ in the inequalities (14) and (15) are sharp for $1 < p \leqslant \infty$ and the best possible for $p = 1$.*

Proof. Similar as the proof of the Corollary 1 (instead of $K(t)$ we have $K_n^1(x,t)$ and $K_n^2(x,t)$, instead of $f'(t)$, $f^{(n)}(t)$). $\square$

REMARK 7. If we take $n = 1$ we get the results from the Theorem 1. All these results were obtained in $[10]$.

4.3. Case $[\mathbf{a}, \mathbf{b}] \cap [\mathbf{c}, \mathbf{d}] = [\mathbf{c}, \mathbf{b}]$ (i.e. $\mathbf{a} \leqslant \mathbf{c} < \mathbf{b} \leqslant \mathbf{d}$)

Now, we establish result from the Theorem 11 (which is valid for $[c,d] \subset [a,b]$, i.e. $a \leqslant c < d \leqslant b$), for the other case $[a,b] \cap [c,d] = [c,b]$, i.e. $a \leqslant c < b \leqslant d$.

THEOREM 14. *Let $f : [a, d] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a, d]$ for some $n \geqslant 1$. Then if $[a, b] \cap [c, d] = [c, b]$, i.e. $a \leqslant c < b \leqslant d$ for $x \in [c, b]$ we have*

$$\frac{1}{b-a} \int_a^b f(t)\, dt - \frac{1}{d-c} \int_c^d f(t)\, dt + T_n^{[a,b]}(x) - T_n^{[c,d]}(x) = \int_a^d \widetilde{K}_n^1(x, t)\, df^{(n-1)}(t)$$

(16)

and

$$\frac{1}{b-a} \int_a^b f(t)\, dt - \frac{1}{d-c} \int_c^d f(t)\, dt + T_{n-1}^{[a,b]}(x) - T_{n-1}^{[c,d]}(x) = \int_a^d \widetilde{K}_n^2(x, t)\, df^{(n-1)}(t),$$

(17)

where

$$\widetilde{K}_n^1(x, t) = \begin{cases} \frac{(b-a)^{n-1}}{n!} B_n\left(\frac{x-t}{b-a}\right), & t \in [a, c], \\[2ex] \frac{(b-a)^{n-1}}{n!} B_n^*\left(\frac{x-t}{b-a}\right) - \frac{(d-c)^{n-1}}{n!} B_n^*\left(\frac{x-t}{d-c}\right), & t \in \langle c, b \rangle, \\[2ex] -\frac{(d-c)^{n-1}}{n!} B_n\left(\frac{x-t}{d-c} + 1\right), & t \in [b, d], \end{cases}$$

and

$$\widetilde{K}_n^2(x, t) = \begin{cases} \frac{(b-a)^{n-1}}{n!}\left[B_n\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right], & t \in [a, c], \\[2ex] \frac{(b-a)^{n-1}}{n!}\left[B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right] \\ -\frac{(d-c)^{n-1}}{n!}\left[B_n^*\left(\frac{x-t}{d-c}\right) - B_n\left(\frac{x-c}{d-c}\right)\right], & t \in \langle c, b \rangle, \\[2ex] -\frac{(d-c)^{n-1}}{n!}\left[B_n\left(\frac{x-t}{d-c} + 1\right) - B_n\left(\frac{x-c}{d-c}\right)\right], & t \in [b, d]. \end{cases}$$

Proof. We subtract identities (11) for interval $[a, b]$ and $[c, d]$, and then using the properties of B_n^*, we get the first formula. By doing the same with identitie (12), we get the second formula. $\square$

THEOREM 15. *Assume (p, q) is a pair of conjugate exponents, that is $1 \leqslant p, q \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $\left|f^{(n)}\right|^p : [a, b] \to \mathbb{R}$ be an R-integrable function for some $n \geqslant 1$. Then for $a \leqslant c < b \leqslant d$, we have*

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - \frac{1}{d-c} \int_c^d f(t)\, dt + T_n^{[a,b]}(x) - T_n^{[c,d]}(x) \right|$$

$$\leqslant \left(\int_a^d \left|\widetilde{K}_n^1(x, t)\right|^q dt \right)^{\frac{1}{q}} \left\|f^{(n)}\right\|_p$$

and

$$\left| \frac{1}{b-a} \int_a^b f\,(t)\,dt - \frac{1}{d-c} \int_c^d f\,(t)\,dt + T_{n-1}^{[a,b]}(x) - T_{n-1}^{[c,d]}(x) \right|$$

$$\leqslant \left(\int_a^d \left| \widetilde{K}_n^2(x,t) \right|^q dt \right)^{\frac{1}{q}} \left\| f^{(n)} \right\|_p,$$

for every $x \in [c,b]$. *The constants* $\left(\int_a^d \left| \widetilde{K}_n^1(x,t) \right|^q dt \right)^{\frac{1}{q}}$ *and* $\left(\int_a^d \left| \widetilde{K}_n^2(x,t) \right|^q dt \right)^{\frac{1}{q}}$ *are sharp for* $1 < p \leqslant \infty$ *and the best possible for* $p = 1$.

Proof. Use the identity (16) and apply the Hölder inequality to obtain

$$\left| \frac{1}{b-a} \int_a^b f\,(t)\,dt - \frac{1}{d-c} \int_c^d f\,(t)\,dt + T_n^{[a,b]}(x) - T_n^{[c,d]}(x) \right|$$

$$\leqslant \int_a^d \left| \widetilde{K}_n^1(x,t) \right| \left| f^{(n)}(t) \right| dt \leqslant \left(\int_a^d \left| \widetilde{K}_n^1(x,t) \right|^q dt \right)^{\frac{1}{q}} \left\| f^{(n)} \right\|_p,$$

which proves the first inequality and similary we prove the second inequality. The proof for sharpness and the best possibility are similar as in Corollary 1. $\square$

REMARK 8. If we take $n = 1$ and for $p = \infty$, we get the same result as in the Corollay 2; for $p = 1$, the Corollary 3; and for $1 < p < \infty$, the Corollary 4.

REMARK 9. We have showed that in the special case, if we take $w\,(t) = \frac{1}{b-a}$, $t \in [a,b]$ and $u\,(t) = \frac{1}{d-c}$, $t \in [c,d]$, in (10) the weighted Montgomery identity reduces to the Euler identity. Consequently, the inequality from the Theorem 9, for $[c,d] \subset [a,b]$ reduces to the second inequality from the Theorem 12; and for $[a,b] \cap [c,d] = [c,b]$ to the second inequality from the Theorem 15.

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