

ON A NEW HARDY–HILBERT’S TYPE INEQUALITY

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Abstract. This paper deals with a new Hardy-Hilbert’s type inequality with a best constant factor. As applications, we give its equivalent form and some particular results.

1. Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n, b_n \geq 0$, such that $0 < \sum_{n=0}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=0}^{\infty} b_n^q < \infty$, then the famous Hardy-Hilbert’s inequality is given by

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m b_n}{m+n+1} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{n=0}^{\infty} a_n^p \right)^{\frac{1}{p}} \left(\sum_{n=0}^{\infty} b_n^q \right)^{\frac{1}{q}}, \quad (1)$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible (see [1]). And its equivalent form is

$$\sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} \frac{a_m}{m+n+1} \right)^p < \left[\frac{\pi}{\sin(\pi/p)} \right]^p \sum_{n=0}^{\infty} a_n^p, \quad (2)$$

where the constant factor $\left[\frac{\pi}{\sin(\pi/p)} \right]^p$ is still the best possible (see [2]).

Inequality (1) is important in analysis and its applications (see [3]). Recently, Yang and Debnath [2, 4] and Yang [5, 6] gave some new extensions and improvements of (1). Kuang and Debnath [7] considered its strengthened versions and some generalizations, and Pachpatte [8] built some new inequalities similar to (1). In 2003, Yang et al. [9] provided an extensive account of the above results.

The major objective of this paper is to obtain a new inequality similar to (1) but other than Pachpatte’s work, which relates to the double series form as:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\ln m + \ln n + \ln \alpha} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\ln \alpha m n} (\alpha \geq e^{7/6}).$$

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For this, we estimate the following weight coefficient

$$\omega(r, n, \alpha) := \sum_{m=1}^{\infty} \frac{1}{m \ln \alpha mn} \left(\frac{\ln \sqrt{\alpha n}}{\ln \sqrt{\alpha m}} \right)^{\frac{1}{r}} \quad (r > 1, n \in N, \alpha \geq e^{7/6}), \tag{3}$$

and do some preparatory works.

2. Some lemmas

If f has its first four derivatives on $[1, \infty)$, $(-1)^n f^{(n)}(x) > 0 (n = 0, 1, 2, 3, 4)$, $\int_1^{\infty} f(x) dx < \infty$ and $f(x), f'(x) \rightarrow 0 (x \rightarrow \infty)$, then (see [7], (2.1))

$$\sum_{k=1}^{\infty} f(k) < \int_1^{\infty} f(x) dx + \frac{1}{2} f(1) - \frac{1}{12} f'(1). \tag{4}$$

LEMMA 2.1. For $r > 1, n \in N$ and $\alpha \geq e^{7/6}$, define the function $R(r, n, \alpha)$ as

$$\begin{aligned} R(r, n, \alpha) : &= \left(\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}} \right)^{\frac{1}{r}} \int_0^{\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}}} \frac{1}{(1+u)u^{1/r}} du \\ &\quad - \left(\frac{7}{12} + \frac{1}{12r \ln \sqrt{\alpha}} \right) \frac{1}{\ln \alpha n} - \frac{1}{12(\ln \alpha n)^2}. \end{aligned} \tag{5}$$

Then we have $R(r, n, \alpha) > 0$.

Proof. Integration by parts, we find

$$\begin{aligned} I &= \int_0^{\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}}} \frac{du}{(1+u)u^{1/r}} = \frac{r}{r-1} \int_0^{\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}}} \frac{1}{1+u} du^{1-\frac{1}{r}} \\ &= \frac{r}{r-1} \left[\frac{u^{1-\frac{1}{r}}}{1+u} \right]_0^{\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}}} - \frac{r}{r-1} \int_0^{\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}}} u^{1-\frac{1}{r}} d \frac{1}{1+u} \\ &= \frac{r \ln \sqrt{\alpha n}}{(r-1) \ln \alpha n} \left(\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}} \right)^{1-\frac{1}{r}} + \frac{r^2}{(r-1)(2r-1)} \int_0^{\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}}} \frac{du^{2-\frac{1}{r}}}{(1+u)^2} \\ &= \frac{r \ln \sqrt{\alpha n}}{(r-1) \ln \alpha n} \left(\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}} \right)^{1-\frac{1}{r}} + \frac{r^2}{(r-1)(2r-1)} \left[\frac{u^{2-\frac{1}{r}}}{(1+u)^2} \right]_0^{\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}}} \\ &\quad + \frac{2r^2}{(r-1)(2r-1)} \int_0^{\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}}} \frac{u^{2-\frac{1}{r}}}{(1+u)^3} du \\ &> \frac{r \ln \sqrt{\alpha n}}{(r-1) \ln \alpha n} \left(\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}} \right)^{1-\frac{1}{r}} + \frac{r^2}{(r-1)(2r-1)} \left[\frac{u^{2-\frac{1}{r}}}{(1+u)^2} \right]_0^{\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}}} \\ &= \left[\frac{r \ln \sqrt{\alpha n}}{(r-1) \ln \alpha n} + \frac{r^2}{(r-1)(2r-1)} \left(\frac{\ln \sqrt{\alpha}}{\ln \alpha n} \right)^2 \right] \left(\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}} \right)^{-\frac{1}{r}}. \end{aligned}$$

Hence by (5), we have

$$\begin{aligned}
 R(r, n, \alpha) &> \frac{r \ln \sqrt{\alpha n}}{(r-1) \ln \alpha n} + \frac{r^2}{(r-1)(2r-1)} \left(\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}} \right)^2 \\
 &\quad - \left(\frac{7}{12} + \frac{1}{12r \ln \sqrt{\alpha}} \right) \frac{1}{\ln \alpha n} - \frac{1}{12(\ln \alpha n)^2} \\
 &= \left(\frac{r \ln \sqrt{\alpha}}{r-1} - \frac{7}{12} - \frac{1}{12r \ln \sqrt{\alpha}} \right) \frac{1}{\ln \alpha n} \\
 &\quad + \left[\frac{r^2 \ln^2 \sqrt{\alpha}}{(r-1)(2r-1)} - \frac{1}{12} \right] \frac{1}{(\ln \alpha n)^2}. \tag{6}
 \end{aligned}$$

For $r > 1$ and $\alpha > e^{7/6}$, we find

$$\begin{aligned}
 &\frac{r \ln \sqrt{\alpha}}{r-1} - \frac{7}{12} - \frac{1}{12r \ln \sqrt{\alpha}} \\
 &= \frac{(12 \ln \sqrt{\alpha} - 7)r^2 \ln \sqrt{\alpha} + (7 \ln \sqrt{\alpha} - 1)r + 1}{12r(r-1) \ln \sqrt{\alpha}} > 0; \tag{7} \\
 &\frac{r^2 \ln^2 \sqrt{\alpha}}{(r-1)(2r-1)} - \frac{1}{12} \geq \frac{25r^2 + 36r - 12}{144(r-1)(2r-1)} > 0.
 \end{aligned}$$

Hence by (6), we have $R(r, n, \alpha) > 0$. The lemma is proved. \square

REMARK 1. If $0 < \alpha < e^{7/6}$, we obtain $12 \ln \sqrt{\alpha} - 7 < 0$. Hence (7) doesn't hold, and we can't find the result $R(r, n, \alpha) > 0$, for any $r > 1$ and $n \in N$.

LEMMA 2.2. *If the weight coefficient $\omega(r, n, \alpha)$ is defined by (3), then we have*

$$\omega(r, n, \alpha) < \frac{\pi}{\sin \pi(1 - 1/r)}, \text{ for } r > 1, n \in N \text{ and } \alpha > e^{7/6}. \tag{8}$$

Proof. For fixed $r > 1, n \in N$ and $\alpha > e^{7/6}$, setting $f(x)$ as:

$$f(x) := \frac{1}{x \ln \alpha n x} \left(\frac{\ln \sqrt{\alpha n}}{\ln \sqrt{\alpha x}} \right)^{1/r}, x \in [1, \infty),$$

then we find that $f(x)$ possesses all the conditions of (4), $f(1) = \frac{1}{\ln \alpha n} \left(\frac{\ln \sqrt{\alpha n}}{\ln \sqrt{\alpha}} \right)^{1/r}$ and

$$f'(1) = - \left[\frac{1 + 1/(r \ln \sqrt{\alpha})}{\ln \alpha n} + \frac{1}{\ln^2 \alpha n} \right] \left(\frac{\ln \sqrt{\alpha n}}{\ln \sqrt{\alpha}} \right)^{1/r}.$$

Putting $u = (\ln \sqrt{\alpha x})/(\ln \sqrt{\alpha n})$ in the following, we obtain

$$\begin{aligned}
 \int_1^\infty f(x) dx &= \int_1^\infty \frac{1}{x \ln \sqrt{\alpha n} [1 + (\ln \sqrt{\alpha x})/(\ln \sqrt{\alpha n})]} \left(\frac{\ln \sqrt{\alpha n}}{\ln \sqrt{\alpha x}} \right)^{\frac{1}{r}} dx \\
 &= \int_{\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}}}^\infty \frac{1}{1+u} \left(\frac{1}{u} \right)^{1/r} du = \frac{\pi}{\sin \pi(1 - \frac{1}{r})} - \int_0^{\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}}} \frac{1}{1+u} \left(\frac{1}{u} \right)^{1/r} du.
 \end{aligned}$$

Hence by (4), (5) and the above results, we have

$$\begin{aligned}
 \omega(r, n, \alpha) &= \sum_{m=1}^{\infty} f(m) < \int_1^{\infty} f(x)dx + \frac{1}{2}f(1) - \frac{1}{12}f'(1) \\
 &= \frac{\pi}{\sin \pi(1 - \frac{1}{r})} - \int_0^{\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha n}}} \frac{u^{-\frac{1}{r}}}{1+u} du + \frac{1}{2 \ln \alpha n} \left(\frac{\ln \sqrt{\alpha n}}{\ln \sqrt{\alpha}} \right)^{\frac{1}{r}} \\
 &\quad + \frac{1}{12} \left[\frac{1 + 1/(r \ln \sqrt{\alpha})}{\ln \alpha n} + \frac{1}{\ln^2 \alpha n} \right] \left(\frac{\ln \sqrt{\alpha n}}{\ln \sqrt{\alpha}} \right)^{1/r} \\
 &= \frac{\pi}{\sin \pi(1 - 1/r)} - \left(\frac{\ln \sqrt{\alpha n}}{\ln \sqrt{\alpha}} \right)^{1/r} R(r, n, \alpha).
 \end{aligned}$$

Then by Lemma 2.1, we have (8). The lemma is proved. \square

LEMMA 2.3. For $\alpha > e^{7/6}$ and $0 < \varepsilon < 1$, we have

$$\begin{aligned}
 J &:= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{mn \ln \alpha mn} \left(\frac{1}{\ln \sqrt{\alpha m}} \right)^{\frac{\varepsilon}{q} + \frac{1}{p}} \left(\frac{1}{\ln \sqrt{\alpha n}} \right)^{\frac{\varepsilon}{p} + \frac{1}{q}} \\
 &\geq \frac{1}{\varepsilon (\ln \sqrt{\alpha})^{\varepsilon}} \left[\frac{\pi}{\sin(\pi/p)} + o(1) \right] (\varepsilon \rightarrow 0^+).
 \end{aligned} \tag{9}$$

Proof. For fixed $y \geq 1$, setting $u = (\ln \sqrt{\alpha x})/(\ln \sqrt{\alpha y})$ in the following, we find

$$\begin{aligned}
 \int_1^{\infty} \frac{1}{x \ln \alpha xy} \left(\frac{\ln \sqrt{\alpha y}}{\ln \sqrt{\alpha x}} \right)^{\frac{\varepsilon}{p} + \frac{1}{q}} dx &= \int_{\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha y}}}^{\infty} \frac{1}{1+u} \left(\frac{1}{u} \right)^{\frac{\varepsilon}{p} + \frac{1}{q}} du \\
 &= \int_0^{\infty} \frac{1}{1+u} \left(\frac{1}{u} \right)^{\frac{\varepsilon}{p} + \frac{1}{q}} du - \int_0^{\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha y}}} \frac{1}{1+u} \left(\frac{1}{u} \right)^{\frac{\varepsilon}{p} + \frac{1}{q}} du \\
 &\geq \int_0^{\infty} \frac{1}{1+u} \left(\frac{1}{u} \right)^{\frac{\varepsilon}{p} + \frac{1}{q}} du - \int_0^{\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha y}}} \left(\frac{1}{u} \right)^{\frac{\varepsilon}{p} + \frac{1}{q}} du \\
 &= \left[\frac{\pi}{\sin(\pi/p)} + o(1) \right] - \frac{p}{1-\varepsilon} \left(\frac{\ln \sqrt{\alpha}}{\ln \sqrt{\alpha y}} \right)^{\frac{1-\varepsilon}{p}}.
 \end{aligned}$$

Hence we have

$$\begin{aligned}
 J &> \int_1^\infty \int_1^\infty \frac{1}{xy \ln \alpha xy} \left(\frac{1}{\ln \sqrt{\alpha x}} \right)^{\frac{\varepsilon}{p} + \frac{1}{q}} \left(\frac{1}{\ln \sqrt{\alpha y}} \right)^{\frac{\varepsilon}{q} + \frac{1}{p}} dx dy \\
 &= \int_1^\infty \frac{1}{y} \left(\frac{1}{\ln \sqrt{\alpha y}} \right)^{1+\varepsilon} \left[\int_1^\infty \frac{1}{x \ln \alpha xy} \left(\frac{\ln \sqrt{\alpha y}}{\ln \sqrt{\alpha x}} \right)^{\frac{\varepsilon}{p} + \frac{1}{q}} dx \right] dy \\
 &\geq \int_1^\infty \frac{1}{y} \left(\frac{1}{\ln \sqrt{\alpha y}} \right)^{1+\varepsilon} dy \left[\frac{\pi}{\sin(\pi/p)} + o(1) \right] \\
 &\quad - \frac{p}{1-\varepsilon} (\ln \sqrt{\alpha})^{\frac{1-\varepsilon}{p}} \int_1^\infty \frac{1}{y} \left(\frac{1}{\ln \sqrt{\alpha y}} \right)^{\frac{1-\varepsilon}{p} + 1 + \varepsilon} dy \\
 &= \frac{1}{\varepsilon (\ln \sqrt{\alpha})^\varepsilon} \left[\frac{\pi}{\sin(\pi/p)} + o(1) - \frac{\varepsilon p^2}{(1-\varepsilon)(p\varepsilon + 1 - \varepsilon)} \right],
 \end{aligned}$$

and (9) holds. The lemma is proved. \square

3. Main results and some applications

THEOREM 3.1. *If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha \geq e^{7/6}, r, s \in \mathbb{R}$ and $a_n, b_n \geq 0$, such that $0 < \sum_{n=1}^\infty \left[n^{\frac{1}{q}-r} (\ln^{\frac{1}{q}-\frac{1}{p}} \sqrt{\alpha n}) a_n \right]^p < \infty$ and $0 < \sum_{n=1}^\infty \left[n^{\frac{1}{p}-s} (\ln^{\frac{1}{p}-\frac{1}{q}} \sqrt{\alpha n}) b_n \right]^q < \infty$, then*

$$\begin{aligned}
 \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m^r n^s \ln \alpha mn} &< \frac{\pi}{\sin(\frac{\pi}{p})} \left\{ \sum_{n=1}^\infty \left[n^{\frac{1}{q}-r} (\ln^{\frac{1}{q}-\frac{1}{p}} \sqrt{\alpha n}) a_n \right]^p \right\}^{\frac{1}{p}} \\
 &\quad \times \left\{ \sum_{n=1}^\infty \left[n^{\frac{1}{p}-s} (\ln^{\frac{1}{p}-\frac{1}{q}} \sqrt{\alpha n}) b_n \right]^q \right\}^{\frac{1}{q}}, \tag{10}
 \end{aligned}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. In particular,

(a) for $r = 1/q$ and $s = 1/p$, we have

$$\begin{aligned}
 &\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m^{1/q} n^{1/p} \ln \alpha mn} \\
 &< \frac{\pi}{\sin(\pi/p)} \left\{ \sum_{n=1}^\infty (\ln^{p-2} \sqrt{\alpha n}) a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty (\ln^{q-2} \sqrt{\alpha n}) b_n^q \right\}^{\frac{1}{q}}; \tag{11}
 \end{aligned}$$

(b) for $r = s = 1$, we have

$$\begin{aligned}
 &\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{mn \ln \alpha mn} \\
 &< \frac{\pi}{\sin(\pi/p)} \left\{ \sum_{n=1}^\infty \frac{1}{n} (\ln^{p-2} \sqrt{\alpha n}) a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty \frac{1}{n} (\ln^{q-2} \sqrt{\alpha n}) b_n^q \right\}^{\frac{1}{q}}; \tag{12}
 \end{aligned}$$

(c) for $r = s = 0$, we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{\ln \alpha mn} < \frac{\pi}{\sin(\pi/p)} \left\{ \sum_{n=1}^{\infty} n^{p-1} (\ln^{p-2} \sqrt{\alpha n}) a_n^p \right\}^{\frac{1}{p}} \times \left\{ \sum_{n=1}^{\infty} n^{q-1} (\ln^{q-2} \sqrt{\alpha n}) b_n^q \right\}^{\frac{1}{q}}. \tag{13}$$

Proof. By Hölder’s inequality and (3), we have

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m^r n^s \ln \alpha mn} &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\frac{a_m}{(\ln \alpha mn)^{1/p}} \frac{(\ln \alpha m)^{1/q^2}}{(\ln \alpha n)^{1/p^2}} \left(\frac{m^{\frac{1}{q}-r}}{n^{1/p}} \right) \right] \\ &\quad \times \left[\frac{b_n}{(\ln \alpha mn)^{1/q}} \frac{(\ln \alpha n)^{1/p^2}}{(\ln \alpha m)^{1/q^2}} \left(\frac{n^{\frac{1}{p}-s}}{m^{1/q}} \right) \right] \\ &\leq \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m^p}{\ln \alpha mn} \frac{(\ln \alpha m)^{p/q^2}}{(\ln \alpha n)^{1/p}} \left[\frac{m^{p(\frac{1}{q}-r)}}{n} \right] \right\}^{1/p} \\ &\quad \times \left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{b_n^q}{\ln \alpha mn} \frac{(\ln \alpha n)^{q/p^2}}{(\ln \alpha m)^{1/q}} \left[\frac{n^{q(\frac{1}{p}-s)}}{m} \right] \right\}^{1/q} \\ &= \left\{ \sum_{m=1}^{\infty} \omega(p, m, \alpha) \left[m^{\frac{1}{q}-r} (\ln^{\frac{1}{q}-\frac{1}{p}} \sqrt{\alpha m}) a_m \right]^p \right\}^{1/p} \\ &\quad \times \left\{ \sum_{n=1}^{\infty} \omega(q, n, \alpha) \left[n^{\frac{1}{p}-s} (\ln^{\frac{1}{p}-\frac{1}{q}} \sqrt{\alpha n}) b_n \right]^q \right\}^{1/q}. \end{aligned}$$

By (8), since $\sin(\pi/p) = \sin(\pi/q)$, we have (10).

For $0 < \varepsilon < 1$, setting $a'_n = n^{r-1}/(\ln^{\frac{\varepsilon}{p}+\frac{1}{q}} \sqrt{\alpha n})$ and $b'_n = n^{s-1}/(\ln^{\frac{\varepsilon}{q}+\frac{1}{p}} \sqrt{\alpha n})$, $n \in N$, then we have

$$\begin{aligned} &\left\{ \sum_{n=1}^{\infty} \left[n^{\frac{1}{q}-r} (\ln^{\frac{1}{q}-\frac{1}{p}} \sqrt{\alpha n}) a'_n \right]^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \left[n^{\frac{1}{p}-s} (\ln^{\frac{1}{p}-\frac{1}{q}} \sqrt{\alpha n}) b'_n \right]^q \right\}^{\frac{1}{q}} \\ &= \sum_{n=1}^{\infty} \frac{1}{n (\ln \sqrt{\alpha n})^{1+\varepsilon}} = \sum_{n=1}^2 \frac{1}{n (\ln \sqrt{\alpha n})^{1+\varepsilon}} + \sum_{n=3}^{\infty} \frac{1}{n (\ln \sqrt{\alpha n})^{1+\varepsilon}} \\ &< \sum_{n=1}^2 \frac{1}{n (\ln \sqrt{\alpha n})^{1+\varepsilon}} + \int_{\frac{\varepsilon}{\sqrt{\alpha}}}^{\infty} \frac{1}{x (\ln \sqrt{\alpha x})^{1+\varepsilon}} dx \\ &= \frac{1}{(\ln \sqrt{\alpha})^{1+\varepsilon}} + \frac{1}{2(\ln 2\sqrt{\alpha})^{1+\varepsilon}} + \frac{1}{\varepsilon}. \end{aligned} \tag{14}$$

If the constant factor $\pi/\sin(\pi/p)$ in (10) is not the best possible, then, there exists $0 < K < \pi/\sin(\pi/p)$, such that (10) is still valid if we replace $\pi/\sin(\pi/p)$ by K . In particular, we have

$$\begin{aligned} \varepsilon J &= \varepsilon \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a'_m b'_n}{m^r n^s \ln \alpha mn} \\ &< \varepsilon K \left\{ \sum_{n=1}^{\infty} \left[n^{\frac{1}{q}-r} (\ln^{\frac{1}{q}-\frac{1}{p}} \sqrt{\alpha n}) a'_n \right]^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \left[n^{\frac{1}{p}-s} (\ln^{\frac{1}{p}-\frac{1}{q}} \sqrt{\alpha n}) b'_n \right]^q \right\}^{\frac{1}{q}}. \end{aligned}$$

Hence by (9) and (14), we find

$$\frac{1}{(\ln \sqrt{\alpha})^\varepsilon} \left[\frac{\pi}{\sin(\pi/p)} + o(1) \right] < K \left[\frac{\varepsilon}{(\ln \sqrt{\alpha})^{1+\varepsilon}} + \frac{\varepsilon}{2(\ln 2\sqrt{\alpha})^{1+\varepsilon}} + 1 \right].$$

It follows that $\pi/\sin(\pi/p) \leq K$, for $\varepsilon \rightarrow 0^+$. This contradicts the fact that $K < \pi/\sin(\pi/p)$. Hence the constant factor $\pi/\sin(\pi/p)$ is the best possible. The theorem is proved. \square

REMARK 2. Inequality (12) is similar to the Mulholland's inequality as (see [10]):

$$\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{a_m b_n}{mn \ln mn} < \frac{\pi}{\sin(\pi/p)} \left\{ \sum_{n=2}^{\infty} \frac{1}{n} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=2}^{\infty} \frac{1}{n} b_n^q \right\}^{\frac{1}{q}}, \tag{15}$$

and (13) is similar to (1). Inequality (10) is a new Hardy-Hilbert's type inequality with three parameters α, r and s . For $p=q=2$ and $\alpha \geq e^{7/6}$, (10) reduces to

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m^r n^s \ln \alpha mn} < \pi \left\{ \sum_{n=1}^{\infty} n^{1-2r} a_n^2 \sum_{n=1}^{\infty} n^{1-2s} b_n^2 \right\}^{\frac{1}{2}}. \tag{16}$$

The more accurate results is for $\alpha = e^{3/4}$ in (16) (see [11]).

THEOREM 3.2. If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha \geq e^{7/6}, r \in R$ and $a_n \geq 0$, such that $0 < \sum_{n=1}^{\infty} \left[n^{\frac{1}{q}-r} (\ln^{\frac{1}{q}-\frac{1}{p}} \sqrt{\alpha n}) a_n \right]^p < \infty$, then we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{\ln^{p-2} \sqrt{\alpha n}}{n} \left(\sum_{m=1}^{\infty} \frac{a_m}{m^r \ln \alpha mn} \right)^p \\ &< \left[\frac{\pi}{\sin(\pi/p)} \right]^p \sum_{n=1}^{\infty} \left[n^{\frac{1}{q}-r} (\ln^{\frac{1}{q}-\frac{1}{p}} \sqrt{\alpha n}) a_n \right]^p, \end{aligned} \tag{17}$$

where the constant factor $\left[\frac{\pi}{\sin(\pi/p)} \right]^p$ is the best possible. Inequality (17) is equivalent to (10). In particular,

(a) for $r = 1/q$, we have

$$\sum_{n=1}^{\infty} \frac{\ln^{p-2} \sqrt{\alpha n}}{n} \left(\sum_{m=1}^{\infty} \frac{a_m}{m^{\frac{1}{q}} \ln \alpha mn} \right)^p < \left[\frac{\pi}{\sin(\frac{\pi}{p})} \right]^p \sum_{n=1}^{\infty} (\ln^{p-2} \sqrt{\alpha n}) a_n^p; \tag{18}$$

(b) for $r=1$, we have

$$\sum_{n=1}^{\infty} \frac{\ln^{p-2} \sqrt{\alpha n}}{n} \left(\sum_{m=1}^{\infty} \frac{a_m}{m \ln \alpha mn} \right)^p < \left[\frac{\pi}{\sin(\frac{\pi}{p})} \right]^p \sum_{n=1}^{\infty} \frac{\ln^{p-2} \sqrt{\alpha n}}{n} a_n^p, \tag{19}$$

(c) for $r=0$, we have

$$\sum_{n=1}^{\infty} \frac{\ln^{p-2} \sqrt{\alpha n}}{n} \left(\sum_{m=1}^{\infty} \frac{a_m}{\ln \alpha mn} \right)^p < \left[\frac{\pi}{\sin(\frac{\pi}{p})} \right]^p \sum_{n=1}^{\infty} n^{p-1} (\ln^{p-2} \sqrt{\alpha n}) a_n^p. \tag{20}$$

Proof. Since $0 < \sum_{n=1}^{\infty} \left[n^{\frac{1}{q}-r} (\ln^{\frac{1}{q}-\frac{1}{p}} \sqrt{\alpha n}) a_n \right]^p < \infty$, there exists $k_0 > 1$, such for $k > k_0$, that $\sum_{n=1}^k \left[n^{\frac{1}{q}-r} (\ln^{\frac{1}{q}-\frac{1}{p}} \sqrt{\alpha n}) a_n \right]^p > 0$, and $b_n(k) := \frac{\ln^{p-2} \sqrt{\alpha n}}{n^{1/q}} \cdot \left(\sum_{m=1}^k \frac{a_m}{m^r \ln \alpha mn} \right)^{p-1} > 0$. By (10), for $s=1/p$, setting $b_n(k) = a_n = 0$, for $n > k$, we have

$$\begin{aligned} 0 &< \left[\sum_{n=1}^k (\ln^{q-2} \sqrt{\alpha n}) b_n^q(k) \right]^p \\ &= \left[\sum_{n=1}^k \frac{\ln^{p-2} \sqrt{\alpha n}}{n} \left(\sum_{m=1}^k \frac{a_m}{m^r \ln \alpha mn} \right)^p \right]^p \\ &= \left[\sum_{n=1}^k \sum_{m=1}^k \frac{a_m b_n(k)}{m^r n^{1/p} \ln \alpha mn} \right]^p < \left[\frac{\pi}{\sin(\frac{\pi}{p})} \right]^p \\ &\quad \times \sum_{n=1}^k \left[n^{\frac{1}{q}-r} (\ln^{\frac{1}{q}-\frac{1}{p}} \sqrt{\alpha n}) a_n \right]^p \left[\sum_{n=1}^k (\ln^{q-2} \sqrt{\alpha n}) b_n^q(k) \right]^{p-1}. \end{aligned} \tag{21}$$

Thus we find

$$\begin{aligned} 0 &< \sum_{n=1}^k \frac{\ln^{p-2} \sqrt{\alpha n}}{n} \left(\sum_{m=1}^k \frac{a_m}{m^r \ln \alpha mn} \right)^p = \sum_{n=1}^k (\ln^{q-2} \sqrt{\alpha n}) b_n^q(k) \\ &< \left[\frac{\pi}{\sin(\frac{\pi}{p})} \right]^p \sum_{n=1}^k \left[n^{\frac{1}{q}-r} (\ln^{\frac{1}{q}-\frac{1}{p}} \sqrt{\alpha n}) a_n \right]^p. \end{aligned} \tag{22}$$

It follows that

$$\sum_{n=1}^{\infty} (\ln^{q-2} \sqrt{\alpha n}) b_n^q(\infty) \leq \left[\frac{\pi}{\sin(\frac{\pi}{p})} \right]^p \sum_{n=1}^{\infty} \left[n^{\frac{1}{q}-r} (\ln^{\frac{1}{q}-\frac{1}{p}} \sqrt{\alpha n}) a_n \right]^p < \infty.$$

Hence by (10), for $k \rightarrow \infty$, neither (21) and (22) takes the form of equality, and we have (17).

On the other hand, if (17) holds, by Hölder's inequality, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m^r n^s \ln \alpha mn} \\ &= \sum_{n=1}^{\infty} \left[\frac{\ln^{\frac{1}{q}-\frac{1}{p}} \sqrt{\alpha n}}{n^{1/p}} \sum_{m=1}^{\infty} \frac{a_m}{m^r \ln \alpha mn} \right] \left[n^{\frac{1}{p}-s} (\ln^{\frac{1}{p}-\frac{1}{q}} \sqrt{\alpha n}) b_n \right] \\ &\leq \left\{ \sum_{n=1}^{\infty} \frac{\ln^{p-2} \sqrt{\alpha n}}{n} \left(\sum_{m=1}^{\infty} \frac{a_m}{m^r \ln \alpha mn} \right)^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} [n^{\frac{1}{p}-s} (\ln^{\frac{1}{p}-\frac{1}{q}} \sqrt{\alpha n}) b_n]^q \right\}^{\frac{1}{q}}. \quad (23) \end{aligned}$$

By (17), we have (10). Hence (10) and (17) are equivalent.

If the constant factor $[\pi/\sin(\pi/p)]^p$ in (17) is not the best possible, we may conclude that the constant factor $\pi/\sin(\pi/p)$ in (10) is not the best possible by using (23). This is a contradiction. The theorem is proved. \square

REMARK 3. Inequality (20) is similar to (2), and equivalent to (13). Since Inequality (10) and its equivalent form (17) are all with the best constant factor, we give some new results.

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