Mathematical
Inequalities
& Applications
Volume 7, Number 3 (2004), 365–378

ESTIMATIONS OF THE DIFFERENCE OF TWO INTEGRAL MEANS VIA EULER–TYPE IDENTITIES

J. PEČARIĆ, I. PERIĆ AND A. VUKELIĆ

(*communicated by Z. Páles*)

Abstract. Generalizations of estimations of difference of two integral means are given, by using Euler-type identities.

1. Introduction

The following Ostrowski inequality is well known [13]:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)M,\quad x\in[a,b], \tag{1.1}$$

where $f:[a,b]\to\mathbb{R}$ is a differentiable function such that $|f'(x)|\leqslant M$, for every $x\in[a,b]$.

Note that (1.1) can be given in the equivalent form

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \frac{(x-a)^2+(b-x)^2}{2(b-a)}M.$$

The Ostrowski inequality has been generalized over the last years in a number of ways.

In this paper we will generalize the results from [2], where N. S. Barnet, P. Cerone, S. S. Dragomir and A. M. Fink estimated the difference of the two integral means for absolutely continuous mappings whose first derivative is in $L_\infty[a,b]$. We will give the results for functions whose derivative of order n, $n\geqslant 1$, is from $L_p[a,b]$ spaces.

Recently, M. Matić and J. Pečarić [12] proved the following result which is more in spirit of our results, so we use it as a initial result:

THEOREM 1. *Let $a,b,c,d\in\mathbb{R}$, be such that*

$$a\leqslant c<d\leqslant b,\ c-a+b-d>0.$$

(i) If $f:[a,b]\to\mathbb{R}$ is M-Lipschitzian on $[a,b]$, with some constant $M>0$, then

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{d-c}\int_c^d f(s)ds\right| \leqslant \frac{(c-a)^2+(b-d)^2}{2(c-a+b-d)}M. \tag{1.2}$$

Mathematics subject classification (2000): 26D15, 26D20, 26D99.
Key words and phrases: Ostrowski inequality, Bernoulli polynomials, bounded variation.

(ii) If $f_0 : [a,b] \to \mathbb{R}$ *is defined as*

$$f_0(t) = |t - s_0|, \ t \in [a,b],$$

where

$$s_0 = \frac{bc - ad}{c - a + b - d},$$

then f_0 *is* 1*-Lipschitzian on* $[a,b]$ *and we have*

$$\left| \frac{1}{b-a} \int_a^b f_0(t)dt - \frac{1}{d-c} \int_c^d f_0(s)ds \right| = \frac{(c-a)^2 + (b-d)^2}{2(c-a+b-d)}.$$

Note that for $c = d = x$ we can assume $\frac{1}{d-c} \int_c^d f(s)ds = f(x)$, as a limit case, so that (1.2) reduces to the Ostrowski inequality (1.1). So, inequality (1.2) can be regarded as a natural generalization of Ostrowski inequality (1.1).

In the recent paper [6], for every function $f : [a,b] \to \mathbb{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geqslant 1$ and for every $x \in [a,b]$, the following two formulae have been proved:

$$f(x) = \frac{1}{b-a} \int_a^b f(t)dt + T_n^{[a,b]}(x) + P_n^{[a,b]}(x), \tag{1.3}$$

and

$$f(x) = \frac{1}{b-a} \int_a^b f(t)dt + T_{n-1}^{[a,b]}(x) + R_n^{[a,b]}(x), \tag{1.4}$$

where for $1 \leqslant m \leqslant n$,

$$T_m^{[a,b]}(x) = \sum_{k=1}^{m} \frac{(b-a)^{k-1}}{k!} B_k\left(\frac{x-a}{b-a}\right) \left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] \tag{1.5}$$

with convention $T_0^{[a,b]}(x) = 0$, and

$$P_n^{[a,b]}(x) = -\frac{(b-a)^{n-1}}{n!} \int_a^b \left[B_n^*\left(\frac{x-t}{b-a}\right)\right] df^{(n-1)}(t),$$

$$R_n^{[a,b]}(x) = -\frac{(b-a)^{n-1}}{n!} \int_a^b \left[B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right] df^{(n-1)}(t).$$

Here $B_k(x)$ are Bernoulli polynomials, $B_k = B_k(0)$ are the Bernoulli numbers, and $B_k^*(x)$, $k \geqslant 0$, are periodic functions of period one, related to the Bernoulli polynomials as

$$B_k^*(x) = B_k(x), \ \text{for} \ 0 \leqslant x < 1,$$

and

$$B_k^*(x+1) = B_k^*(x), \ x \in \mathbb{R},$$

so that $B_0^* = 1$, B_1^* is a discontinuous function with a jump of -1 at each integer, and B_k^*, $k \geqslant 2$, is a continuous function. For some details on the Bernoulli polynomials and the Bernoulli numbers see for example [1] or [3].

The formulae (1.3) and (1.4) are extensions of a Euler formula [11, p. 17].

In this paper we make use of the formulae (1.3) and (1.4) to prove generalizations of (1.2) using two different methods. Also, we establish our main results for functions whose derivatives are either functions of bounded variation or Lipschitzian functions or functions from the L_p -spaces.

2. Integral identities of Euler type

THEOREM 2. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is continuous function of bounded variation on $[a,b]$ for some $n\geqslant 1$. Then if $[c,d]\subset[a,b]$ for every $x\in[c,d]$*

$$\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{d-c}\int_c^d f(t)dt+T_n^{[a,b]}(x)-T_n^{[c,d]}(x)=\int_a^b K_n^1(x,t)df^{(n-1)}(t) \tag{2.1}$$

and

$$\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{d-c}\int_c^d f(t)dt+T_{n-1}^{[a,b]}(x)-T_{n-1}^{[c,d]}(x)=\int_a^b K_n^2(x,t)df^{(n-1)}(t), \tag{2.2}$$

where $T_n^{[a,b]}(x)$ and $T_n^{[c,d]}(x)$ are defined by (1.5),

$$K_n^1(x,t)=\begin{cases}\frac{(b-a)^{n-1}}{n!}B_n\left(\frac{x-t}{b-a}\right) & \text{if } t\in[a,c];\\ \frac{(b-a)^{n-1}}{n!}B_n^*\left(\frac{x-t}{b-a}\right)-\frac{(d-c)^{n-1}}{n!}B_n^*\left(\frac{x-t}{d-c}\right) & \text{if } t\in(c,d);\\ \frac{(b-a)^{n-1}}{n!}B_n\left(\frac{x-t}{b-a}+1\right) & \text{if } t\in[d,b];\end{cases}$$

and

$$K_n^2(x,t)=\begin{cases}\frac{(b-a)^{n-1}}{n!}\left[B_n\left(\frac{x-t}{b-a}\right)-B_n\left(\frac{x-a}{b-a}\right)\right] & \text{if } t\in[a,c];\\ \frac{(b-a)^{n-1}}{n!}\left[B_n^*\left(\frac{x-t}{b-a}\right)-B_n\left(\frac{x-a}{b-a}\right)\right] & \\ -\frac{(d-c)^{n-1}}{n!}\left[B_n^*\left(\frac{x-t}{d-c}\right)-B_n\left(\frac{x-c}{d-c}\right)\right] & \text{if } t\in(c,d);\\ \frac{(b-a)^{n-1}}{n!}\left[B_n\left(\frac{x-t}{b-a}+1\right)-B_n\left(\frac{x-a}{b-a}\right)\right] & \text{if } t\in[d,b].\end{cases} \tag{2.3}$$

Proof. First we write identities (1.3) and (1.4) for interval $[a,b]$ and for interval $[c,d]$. Then we subtract them and using the properties of B_n^* we get above statements. □

THEOREM 3. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is continuous function of bounded variation on $[a,b]$ for some $n\geqslant 1$. Then if $[c,d]\subset[a,b]$*

$$\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{d-c}\int_c^d f(x)dx+T_n=\int_a^b H_n^1(t)df^{(n-1)}(t) \tag{2.4}$$

and

$$\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{d-c}\int_c^d f(x)dx+T_{n-1}=\int_a^b H_n^2(t)df^{(n-1)}(t), \tag{2.5}$$

where $T_0 = 0$ *and for* $1 \leqslant m \leqslant n$,

$$T_m = \frac{1}{d-c}\sum_{k=1}^{m}\frac{(b-a)^k}{(k+1)!}\left[B_{k+1}\left(\frac{d-a}{b-a}\right) - B_{k+1}\left(\frac{c-a}{b-a}\right)\right]\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right],$$

$$H_n^1(t) = \frac{(b-a)^n}{(n+1)!(d-c)}\left[B_{n+1}^*\left(\frac{d-t}{b-a}\right) - B_{n+1}^*\left(\frac{c-t}{b-a}\right)\right]$$

and

$$\begin{aligned} H_n^2(t) = \quad & \frac{(b-a)^n}{(n+1)!(d-c)}\left[B_{n+1}^*\left(\frac{d-t}{b-a}\right) - B_{n+1}^*\left(\frac{c-t}{b-a}\right)\right. \\ & \left. - \; B_{n+1}\left(\frac{d-a}{b-a}\right) + B_{n+1}\left(\frac{c-a}{b-a}\right)\right]. \end{aligned} \tag{2.6}$$

Proof. From (1.3) we have

$$\frac{1}{d-c}\int_c^d f(x)dx = \frac{1}{b-a}\int_a^b f(t)dt + \frac{1}{d-c}\int_c^d T_n^{[a,b]}(x)dx + \frac{1}{d-c}\int_c^d P_n^{[a,b]}(x)dx.$$

However,

$$\begin{aligned} \int_c^d T_n^{[a,b]}(x)dx &= \sum_{k=1}^{n}\frac{(b-a)^{k-1}}{k!}\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right]\int_c^d B_k\left(\frac{x-a}{b-a}\right)dx \\ &= \sum_{k=1}^{n}\frac{(b-a)^k}{(k+1)!}\left[B_{k+1}\left(\frac{d-a}{b-a}\right) - B_{k+1}\left(\frac{c-a}{b-a}\right)\right]\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] \end{aligned}$$

and interchanging the order of integration

$$\begin{aligned} \int_c^d P_n^{[a,b]}(x)dx &= -\frac{(b-a)^{n-1}}{n!}\int_a^b df^{(n-1)}(t)\int_c^d B_n^*\left(\frac{x-t}{b-a}\right)dx \\ &= -\frac{(b-a)^n}{(n+1)!}\int_a^b\left[B_{n+1}^*\left(\frac{d-t}{b-a}\right) - B_{n+1}^*\left(\frac{c-t}{b-a}\right)\right]df^{(n-1)}(t). \end{aligned}$$

So we get identity (2.4). Similar using identity (1.4) we get (2.5). □

3. Estimations of the difference of two integral means

THEOREM 4. *Let* $f : [a,b] \to \mathbb{R}$ *be such that* $f^{(n-1)}$ *is an* L*-Lipschitzian function on* $[a,b]$ *for some* $n \geqslant 1$*. Then for* $a \leqslant c < d \leqslant b$*, we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{d-c}\int_c^d f(t)dt + T_n^{[a,b]}(x) - T_n^{[c,d]}(x)\right| \leqslant L\int_a^b\left|K_n^1(x,t)\right|dt$$

and

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{d-c}\int_c^d f(t)dt + T_{n-1}^{[a,b]}(x) - T_{n-1}^{[c,d]}(x)\right| \leqslant L\int_a^b\left|K_n^2(x,t)\right|dt,$$

for every $x \in [c,d]$.

Proof. For integrable function $F:[a,b]\to\mathbb{R}$ we have

$$\left|\int_a^b F(t)df^{(n-1)}(t)\right| \leqslant L\int_a^b |F(t)|\,dt,$$

since $f^{(n-1)}$ is L-Lipschitzian function. This proves our assertions using identities (2.1) and (2.2). □

THEOREM 5. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[a,b]$ for some $n\geqslant 1$. Then for $a\leqslant c<d\leqslant b$, we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{d-c}\int_c^d f(x)dx+T_n\right|\leqslant L\int_a^b \left|H_n^1(t)\right|dt$$

and

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{d-c}\int_c^d f(t)dt+T_{n-1}\right|\leqslant L\int_a^b \left|H_n^2(t)\right|dt.$$

Proof. Similar as in Theorem 4 using identities (2.4) and (2.5). □

COROLLARY 1. *Let $f:[a,b]\to\mathbb{R}$ be such that f is an L-Lipschitzian function on $[a,b]$. Then for $a\leqslant c<d\leqslant b$, we have inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{d-c}\int_c^d f(t)dt\right|\leqslant\frac{(c-a)^2+(b-d)^2}{2(b-a-d+c)}L. \tag{3.1}$$

In a case $c=a$ and $d=b$ we assume that the right hand side of above inequality is equal to zero.

Proof. We give two different proofs of the inequality (3.1).

(i) For $n=1$ in the second inequality in Theorem 4 we have $T_0^{[a,b]}(x)=T_0^{[c,d]}(x)=0$. By (2.3) we get

$$\begin{aligned}\int_a^b \left|K_1^2(x,t)\right|dt &= \int_a^c \left|B_1\left(\frac{x-t}{b-a}\right)-B_1\left(\frac{x-a}{b-a}\right)\right|dt\\ &+\int_c^d\left|B_1^*\left(\frac{x-t}{b-a}\right)-B_1\left(\frac{x-a}{b-a}\right)-B_1^*\left(\frac{x-t}{d-c}\right)+B_1\left(\frac{x-c}{d-c}\right)\right|dt\\ &+\int_d^b\left|B_1\left(\frac{x-t}{b-a}+1\right)-B_1\left(\frac{x-a}{b-a}\right)\right|dt.\end{aligned}$$

However,

$$\int_a^c \left|B_1\left(\frac{x-t}{b-a}\right)-B_1\left(\frac{x-a}{b-a}\right)\right|dt=\int_a^c\left|\frac{x-t}{b-a}-\frac{x-a}{b-a}\right|dt=\int_a^c\frac{t-a}{b-a}dt=\frac{(c-a)^2}{2(b-a)},$$

$$\int_d^b \left|B_1\left(\frac{x-t}{b-a}+1\right)-B_1\left(\frac{x-a}{b-a}\right)\right|dt=\int_d^b\left|\frac{x-t}{b-a}+1-\frac{x-a}{b-a}\right|dt$$
$$=\int_d^b \frac{b-t}{b-a}dt=\frac{(b-d)^2}{2(b-a)}$$

and

$$\int_c^d \left|B_1^*\left(\frac{x-t}{b-a}\right)-B_1\left(\frac{x-a}{b-a}\right)-B_1^*\left(\frac{x-t}{d-c}\right)+B_1\left(\frac{x-c}{d-c}\right)\right|dt$$
$$=\int_c^x\left|\frac{x-t}{b-a}-\frac{x-a}{b-a}-\frac{x-t}{d-c}+\frac{x-c}{d-c}\right|dt$$
$$+\int_x^d\left|\frac{x-t}{b-a}+1-\frac{x-a}{b-a}-\frac{x-t}{d-c}-1+\frac{x-c}{d-c}\right|dt$$
$$=\int_c^d\left|\frac{a-t}{b-a}+\frac{t-c}{d-c}\right|dt=\frac{b-a-d+c}{(b-a)(d-c)}\int_c^d|t-s_0|\,dt,$$

where $s_0=\frac{bc-ad}{b-a-d+c}$. Further we have

$$s_0-c=\frac{d-c}{b-a-d+c}(c-a)\geqslant 0$$

and

$$d-s_0=\frac{d-c}{b-a-d+c}(b-d)\geqslant 0,$$

which implies that $s_0\in[c,d]$ and

$$\begin{aligned}\int_c^d|t-s_0|\,dt &= \int_c^{s_0}(s_0-t)dt+\int_{s_0}^d(t-s_0)dt\\ &= \frac{1}{2}\left[(s_0-c)^2+(d-s_0)^2\right]\\ &= \frac{(d-c)^2}{2(b-a-d+c)^2}\left[(c-a)^2+(b-d)^2\right].\end{aligned}$$

Consequently,

$$\int_a^b\left|K_1^2(x,t)\right|dt$$
$$=\frac{(c-a)^2}{2(b-a)}+\frac{d-c}{2(b-a)(b-a-d+c)}[(c-a)^2+(b-d)^2]+\frac{(b-d)^2}{2(b-a)}$$
$$=\frac{(c-a)^2+(b-d)^2}{2(b-a-d+c)}.$$

So with Theorem 4 we get (3.1).

(ii) For $n=1$ in the second inequality in Theorem 5 we have $T_0=0$. By (2.6) we get

$$\begin{aligned}
&\int_a^b \left|H_1^2(t)\right| dt \\
&= \frac{b-a}{2(d-c)}\left[\int_a^c \left|B_2\left(\frac{d-t}{b-a}\right)-B_2\left(\frac{c-t}{b-a}\right)-B_2\left(\frac{d-a}{b-a}\right)+B_2\left(\frac{c-a}{b-a}\right)\right| dt\right. \\
&+\int_c^d \left|B_2\left(\frac{d-t}{b-a}\right)-B_2\left(\frac{c-t}{b-a}+1\right)-B_2\left(\frac{d-a}{b-a}\right)+B_2\left(\frac{c-a}{b-a}\right)\right| dt \\
&\left.+\int_d^b \left|B_2\left(\frac{d-t}{b-a}+1\right)-B_2\left(\frac{c-t}{b-a}+1\right)-B_2\left(\frac{d-a}{b-a}\right)+B_2\left(\frac{c-a}{b-a}\right)\right| dt\right].
\end{aligned}$$

However,

$$\begin{aligned}
&\int_a^c \left|B_2\left(\frac{d-t}{b-a}\right)-B_2\left(\frac{c-t}{b-a}\right)-B_2\left(\frac{d-a}{b-a}\right)+B_2\left(\frac{c-a}{b-a}\right)\right| dt \\
&= \int_a^c \left|\frac{(d-t)^2}{(b-a)^2}-\frac{(c-t)^2}{(b-a)^2}-\frac{(d-a)^2}{(b-a)^2}+\frac{(c-a)^2}{(b-a)^2}\right| dt \\
&= \frac{2(d-c)}{(b-a)^2}\int_a^c (t-a)dt = \frac{(d-c)(c-a)^2}{(b-a)^2},
\end{aligned}$$

$$\begin{aligned}
&\int_d^b \left|B_2\left(\frac{d-t}{b-a}+1\right)-B_2\left(\frac{c-t}{b-a}+1\right)-B_2\left(\frac{d-a}{b-a}\right)+B_2\left(\frac{c-a}{b-a}\right)\right| dt \\
&= \int_d^b \left|\frac{(d-t)^2}{(b-a)^2}-\frac{(c-t)^2}{(b-a)^2}-\frac{(d-a)^2}{(b-a)^2}+\frac{(c-a)^2}{(b-a)^2}+2\frac{d-c}{b-a}\right| dt \\
&= \frac{2(d-c)}{(b-a)^2}\int_d^b (b-t)dt = \frac{(d-c)(b-d)^2}{(b-a)^2}
\end{aligned}$$

and

$$\begin{aligned}
&\int_c^d \left|B_2\left(\frac{d-t}{b-a}\right)-B_2\left(\frac{c-t}{b-a}+1\right)-B_2\left(\frac{d-a}{b-a}\right)+B_2\left(\frac{c-a}{b-a}\right)\right| dt \\
&= \int_c^d \left|\frac{(d-t)^2}{(b-a)^2}-\frac{(c-t)^2}{(b-a)^2}-2\frac{(c-t)}{(b-a)}-\frac{(d-a)^2}{(b-a)^2}+\frac{(c-a)^2}{(b-a)^2}\right| dt \\
&= \int_c^d \left|\frac{2(d-c)(a-t)}{(b-a)^2}+2\frac{t-c}{b-a}\right| dt = \frac{2(b-a-d+c)}{(b-a)^2}\int_c^d |t-s_0|\, dt,
\end{aligned}$$

where $s_0 = \frac{bc-ad}{b-a-d+c}$.

We get the same results as in (i), so with Theorem 5 we also get (3.1). □

REMARK 1. Inequality (3.1) is equal to inequality (1.2), so Theorems 4 and 5 generalize Theorem 1.

REMARK 2. Suppose that $f:[a,b]\to\mathbb{R}$ is such that $f^{(n)}$ exists and is bounded on $[a,b]$, for some $n\geqslant 1$. Therefore, the inequalities established in Theorems 4 and 5 hold with $L=\|f^{(n)}\|_\infty$.

THEOREM 6. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is continuous function of bounded variation on $[a,b]$ for some $n\geqslant 1$. Then for $a\leqslant c<d\leqslant b$, we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{d-c}\int_c^d f(t)dt+T_n^{[a,b]}(x)-T_n^{[c,d]}(x)\right| \leqslant \sup_{t\in[a,b]}\left|K_n^1(x,t)\right|\cdot V_a^b(f^{(n-1)})$$

and

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{d-c}\int_c^d f(t)dt+T_{n-1}^{[a,b]}(x)-T_{n-1}^{[c,d]}(x)\right| \leqslant \sup_{t\in[a,b]}\left|K_n^2(x,t)\right|\cdot V_a^b(f^{(n-1)}),$$

for every $x\in[c,d]$, where $V_a^b(f^{(n-1)})$ is the total variation of $f^{(n-1)}$ on $[a,b]$.

Proof. If $F:[a,b]\to\mathbb{R}$ is bounded on $[a,b]$ and the Riemann-Stieltjes integral

$$\int_a^b F(t)df^{(n-1)}(t)$$

exists, then

$$\left|\int_a^b F(t)df^{(n-1)}(t)\right|\leqslant \sup_{t\in[a,b]}|F(t)|\cdot V_a^b(f^{(n-1)}).$$

This proves our assertions. □

THEOREM 7. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is continuous function of bounded variation on $[a,b]$ for some $n\geqslant 1$. Then for $a\leqslant c<d\leqslant b$, we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{d-c}\int_c^d f(x)dx+T_n\right|\leqslant \sup_{t\in[a,b]}\left|H_n^1(t)\right|\cdot V_a^b(f^{(n-1)})$$

and

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{d-c}\int_c^d f(x)dx+T_{n-1}\right|\leqslant \sup_{t\in[a,b]}\left|H_n^2(t)\right|\cdot V_a^b(f^{(n-1)}),$$

where $V_a^b(f^{(n-1)})$ is the total variation of $f^{(n-1)}$ on $[a,b]$.

Proof. Similar as in Theorem 6 using identities (2.4) and (2.5). □

COROLLARY 2. *Let $f : [a,b] \to \mathbb{R}$ be such that f is continuous function of bounded variation on $[a,b]$. Then for $a \leqslant c < d \leqslant b$, we have inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{d-c}\int_c^d f(t)dt\right| \leqslant \frac{1}{2}\left(\frac{c-a+b-d}{b-a} + \left|\frac{c-a-b+d}{b-a}\right|\right)\cdot V_a^b(f). \tag{3.2}$$

Proof. We give two different proofs of the inequality (3.2).

(i) For $n=1$ in the second inequality in Theorem 6 we have $T_0^{[a,b]}(x) = T_0^{[c,d]}(x) = 0$. From (2.3) we get

$$\begin{aligned}\sup_{t\in[a,c]}\left|K_1^2(x,t)\right| &= \sup_{t\in[a,c]}\left|B_1\left(\frac{x-t}{b-a}\right) - B_1\left(\frac{x-a}{b-a}\right)\right| \\ &= \sup_{t\in[a,c]}\left|\frac{x-t}{b-a} - \frac{x-a}{b-a}\right| dt = \sup_{t\in[a,c]}\frac{t-a}{b-a} = \frac{c-a}{b-a},\end{aligned}$$

$$\begin{aligned}\sup_{t\in[d,b]}\left|K_1^2(x,t)\right| &= \sup_{t\in[d,b]}\left|B_1\left(\frac{x-t}{b-a}+1\right) - B_1\left(\frac{x-a}{b-a}\right)\right| \\ &= \sup_{t\in[d,b]}\left|\frac{x-t}{b-a} + 1 - \frac{x-a}{b-a}\right| = \sup_{t\in[d,b]}\frac{b-t}{b-a} = \frac{b-d}{b-a}\end{aligned}$$

and

$$\begin{aligned}\sup_{t\in[c,d]}\left|K_1^2(x,t)\right| &= \sup_{t\in[c,d]}\left|B_1^*\left(\frac{x-t}{b-a}\right) - B_1\left(\frac{x-a}{b-a}\right) - B_1^*\left(\frac{x-t}{d-c}\right) + B_1\left(\frac{x-c}{d-c}\right)\right| \\ &= \sup_{t\in[c,d]}\left|\frac{a-t}{b-a} + \frac{t-c}{d-c}\right| = \max\left\{\frac{c-a}{b-a}, \frac{b-d}{b-a}\right\}.\end{aligned}$$

Consequently,

$$\sup_{t\in[a,b]}\left|K_1^2(x,t)\right| = \max\left\{\frac{c-a}{b-a}, \frac{b-d}{b-a}\right\} = \max\{A,B\},$$

where

$$A = \frac{c-a}{b-a},\ B = \frac{b-d}{b-a}.$$

Also, $0 \leqslant A \leqslant B$ or $0 \leqslant B \leqslant A$, so that

$$\max\{A,B\} = \frac{1}{2}(A+B+|A-B|).$$

So using the above formula with Theorem 6 we get (3.2).

(ii) For $n=1$ in the second inequality in Theorem 7 we have $T_0=0$. From (2.6) we get

$$\begin{aligned}
&\sup_{t\in[a,c]}\left|H_1^2(t)\right| \\
&\quad=\frac{b-a}{2(d-c)}\sup_{t\in[a,c]}\left|B_2\left(\frac{d-t}{b-a}\right)-B_2\left(\frac{c-t}{b-a}\right)-B_2\left(\frac{d-a}{b-a}\right)+B_2\left(\frac{c-a}{b-a}\right)\right| \\
&\quad=\frac{1}{b-a}\sup_{t\in[a,c]}(t-a)=\frac{c-a}{b-a},
\end{aligned}$$

$$\begin{aligned}
&\sup_{t\in[d,b]}\left|H_1^2(t)\right| \\
&\quad=\frac{b-a}{2(d-c)}\sup_{t\in[d,b]}\left|B_2\left(\frac{d-t}{b-a}+1\right)-B_2\left(\frac{c-t}{b-a}+1\right)-B_2\left(\frac{d-a}{b-a}\right)+B_2\left(\frac{c-a}{b-a}\right)\right| \\
&\quad=\frac{1}{b-a}\sup_{t\in[d,b]}(b-t)=\frac{b-d}{b-a}
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{t\in[c,d]}\left|H_1^2(t)\right| \\
&\quad=\frac{b-a}{2(d-c)}\sup_{t\in[c,d]}\left|B_2\left(\frac{d-t}{b-a}\right)-B_2\left(\frac{c-t}{b-a}+1\right)-B_2\left(\frac{d-a}{b-a}\right)+B_2\left(\frac{c-a}{b-a}\right)\right| \\
&\quad=\sup_{t\in[c,d]}\left|\frac{a-t}{b-a}+\frac{t-c}{d-c}\right|=\max\left\{\frac{c-a}{b-a},\frac{b-d}{b-a}\right\}.
\end{aligned}$$

We get the same results as in (i), so with Theorem 7 we also get (3.2). □

REMARK 3. For $c=d=x$ we can assume $\frac{1}{d-c}\int_c^d f(s)ds=f(x)$, as a limit case, so (3.2) reduces to

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leqslant\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]V_a^b(f),$$

which is the inequality in Remark 5 from [6].

REMARK 4. Suppose that $f^{(n)}:[a,b]\to\mathbb{R}$ is R-integrable function for some $n\geqslant 1$ In this case $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ and we have

$$V_a^b(f^{(n-1)})=\int_a^b\left|f^{(n)}(t)\right|dt=\|f^{(n)}\|_1,$$

Therefore, the inequalities established in Theorem 6 and 7 hold with $\|f^{(n)}\|_1$ in place of $V_a^b(f^{(n-1)})$.

THEOREM 8. *Assume* (p,q) *is a pair of conjugate exponents, that is* $1<p,q<\infty$, $\frac{1}{p}+\frac{1}{q}=1$. *Let* $\left|f^{(n)}\right|^p:[a,b]\to\mathbb{R}$ *is* R*-integrable function for some* $n\geqslant 1$. *Then for* $a\leqslant c<d\leqslant b$, *we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{d-c}\int_c^d f(t)dt+T_n^{[a,b]}(x)-T_n^{[c,d]}(x)\right|$$
$$\leqslant\left(\int_a^b\left|K_n^1(x,t)\right|^q dt\right)^{1/q}\cdot\|f^{(n)}\|_p$$

and

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{d-c}\int_c^d f(t)dt+T_{n-1}^{[a,b]}(x)-T_{n-1}^{[c,d]}(x)\right|$$
$$\leqslant\left(\int_a^b\left|K_n^2(x,t)\right|^q dt\right)^{1/q}\cdot\|f^{(n)}\|_p,$$

for every $x\in[c,d]$.

Proof. Use the identity (2.1) and apply the Hölder inequality to obtain

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{d-c}\int_c^d f(t)dt+T_n^{[a,b]}(x)-T_n^{[c,d]}(x)\right|$$
$$\leqslant\int_a^b\left|K_n^1(x,t)\right|\left|f^{(n)}(t)\right|dt\leqslant\left(\int_a^b\left|K_n^1(x,t)\right|^q dt\right)^{1/q}\cdot\|f^{(n)}\|_p,$$

which proves first inequality and similar we prove the second inequality. □

THEOREM 9. *Assume* (p,q) *is a pair of conjugate exponents, that is* $1<p,q<\infty$, $\frac{1}{p}+\frac{1}{q}=1$. *Let* $\left|f^{(n)}\right|^p:[a,b]\to\mathbb{R}$ *is* R*-integrable function for some* $n\geqslant 1$. *Then for* $a\leqslant c<d\leqslant b$, *we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{d-c}\int_c^d f(x)dx+T_n\right|\leqslant\left(\int_a^b\left|H_n^1(t)\right|^q dt\right)^{1/q}\cdot\|f^{(n)}\|_p$$

and

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{d-c}\int_c^d f(x)dx+T_{n-1}\right|\leqslant\left(\int_a^b\left|H_n^2(t)\right|^q dt\right)^{1/q}\cdot\|f^{(n)}\|_p.$$

Proof. Similar as in Theorem 8 using identities (2.4) and (2.5). □

REMARK 5. For $p=\infty$ results from Theorem 8 and Theorem 9 coincide with the results of Theorem 4 and Theorem 5 with $L=\|f^{(n)}\|_\infty$. For $p=1$ results from

Theorem 8 and Theorem 9 coincide with the results of Theorem 6 and Theorem 7 with $V_a^b(f^{(n-1)}) = \|f^{(n)}\|_1$.

COROLLARY 3. *Assume (p,q) is a pair of conjugate exponents, that is $1 < p,q < \infty$, $\frac{1}{p}+\frac{1}{q}=1$. Let $|f'|^p : [a,b] \to \mathbb{R}$ is R-integrable function. Then for $a \leqslant c < d \leqslant b$, we have*

$$\left| \frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{d-c}\int_c^d f(t)dt \right| \leqslant \left[\frac{(c-a)^{q+1}+(b-d)^{q+1}}{(q+1)(b-a)^{q-1}(b-a-d+c)} \right]^{1/q} \cdot \|f'\|_p. \tag{3.3}$$

Proof. We give two different proofs of the inequality (3.3).

(i) For $n=1$ in the second inequality in Theorem 8 we have $T_0^{[a,b]}(x) = T_0^{[c,d]}(x) = 0$. From (2.3), similar as in the proof of Corollary 1, we get

$$\int_a^c \left| B_1\left(\frac{x-t}{b-a}\right) - B_1\left(\frac{x-a}{b-a}\right)\right|^q dt = \frac{(c-a)^{q+1}}{(q+1)(b-a)^q},$$

$$\int_d^b \left| B_1\left(\frac{x-t}{b-a}+1\right) - B_1\left(\frac{x-a}{b-a}\right)\right|^q dt = \frac{(b-d)^{q+1}}{(q+1)(b-a)^q}$$

and

$$\int_c^d \left| B_1^*\left(\frac{x-t}{b-a}\right) - B_1\left(\frac{x-a}{b-a}\right) - B_1^*\left(\frac{x-t}{d-c}\right) + B_1\left(\frac{x-c}{d-c}\right)\right|^q dt$$
$$= \frac{d-c}{(q+1)(b-a)^q(b-a-d+c)}\left[(c-a)^{q+1}+(b-d)^{q+1}\right],$$

Consequently,

$$\int_a^b \left|K_1^2(x,t)\right|^q dt = \frac{(c-a)^{q+1}+(b-d)^{q+1}}{(q+1)(b-a)^{q-1}(b-a-d+c)}.$$

So with Theorem 8 we get (3.3).

(ii) For $n=1$ in the second inequality in Theorem 9 we have $T_0 = 0$. From (2.6), similar as in the proof of Corollary 1, we get

$$\int_a^c \left| B_2\left(\frac{d-t}{b-a}\right) - B_2\left(\frac{c-t}{b-a}\right) - B_2\left(\frac{d-a}{b-a}\right) + B_2\left(\frac{c-a}{b-a}\right)\right|^q dt$$
$$= \frac{2^q(d-c)^q(c-a)^{q+1}}{(q+1)(b-a)^{2q}},$$

$$\int_d^b \left| B_2\left(\frac{d-t}{b-a}+1\right) - B_2\left(\frac{c-t}{b-a}+1\right) - B_2\left(\frac{d-a}{b-a}\right) + B_2\left(\frac{c-a}{b-a}\right)\right|^q dt$$
$$= \frac{2^q(d-c)^q(b-d)^{q+1}}{(q+1)(b-a)^{2q}}$$

and

$$\int_c^d \left| B_2\left(\frac{d-t}{b-a}\right) - B_2\left(\frac{c-t}{b-a}+1\right) - B_2\left(\frac{d-a}{b-a}\right) + B_2\left(\frac{c-a}{b-a}\right)\right|^q dt$$
$$= \frac{2^q(d-c)^{q+1}}{(q+1)(b-a)^{2q}(b-a-d+c)}\left[(c-a)^{q+1} + (b-d)^{q+1}\right].$$

We get the same results as in (i), so with Theorem 9 we also get (3.3). □

REMARK 6. Inequality (3.3) was also obtained by P. Cerone and S. S. Dragomir in [5] by using different method.

REMARK 7. For $c = d = x$ we can assume $\frac{1}{d-c}\int_c^d f(s)ds = f(x)$, as a limit case, so (3.3) reduces to

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leqslant \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{(q+1)(b-a)^q}\right]^{1/q} \cdot \|f'\|_p.$$

This inequality was proved by A.M. Fink [10] (see also [7], [8] and [9]).

REMARK 8. The results from Corollaries 1, 2 and 3 were also obtained in [4].

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN (EDS), *Handbook of mathematical functions with formulae, graphs and mathematical tables*, National Bureau of Standards, Applied Math. Series 55, 4th printing, Washington 1965.

[2] N. S. BARNETT, P. CERONE, S. S. DRAGOMIR AND A. M. FINK, *Comparing two integral means for absolutely continuous mappings whose derivatives are in $L_\infty[a,b]$ and applications*, RGMIA Res. Rep. Coll., Vol. **3**, No. 2, (2000).

[3] I. S. BEREZIN AND N. P. ZHIDKOV, *Computing methods*, Vol. **I**. Pergamon Press, Oxford-London-Edinburgh-New York-Paris-Frankfurt, 1965.

[4] P. CERONE AND S. S. DRAGOMIR, *Diferences between means with bounds from a Riemann-Stieltjes integral*, RGMIA Res. Rep. Coll., Vol. **4**, No. 2, (2001).

[5] P. CERONE AND S. S. DRAGOMIR, *On some inequalities arising Montogomery's identity*, RGMIA Res. Rep. Coll., Vol. **3**, No. 2, (2000).

[6] LJ. DEDIĆ, M. MATIĆ AND J. PEČARIĆ, *On generalization of Ostrowski inequality via some Euler-type identities*, Math. Inequal. Appl., Vol. **3**, No. 3, (2000), 337–353.

[7] LJ. DEDIĆ, M. MATIĆ, J. PEČARIĆ AND A. VUKELIĆ, *On generalization of Ostrowski inequality via Euler harmonic identities*, J. of Inequal. and Appl., **7**(6) (2002), 787–805.

[8] S. S. DRAGOMIR AND S. WANG, *A new inequality of Ostrowski's type in L_p norm and applications to some special means and to some numerical quadrature rules*, Indian Journal of Mathematics, **40** (3)(1998), 299–304.

[9] S. S. DRAGOMIR, R. P. AGARWAL AND N. S. BARNETT, *Inequalities for Beta and Gamma functions via some classical and new inequalities*, Journal of Inequalities and Applications **5**(2000), 103–165.

[10] A. M. FINK, *Bounds of the deviation of a function from its avereges*, Czechoslovak Math. J., **42**(117) (1992), 289–310.

[11] V. I. KRYLOV, *Approximate calculation of integrals*, Macmillan, New York-London, 1962

[12] M. MATIĆ AND J. PEČARIĆ, *Two-point Ostrowski inequality*, Math. Inequal. Appl., Vol. **4**, No. 2, (2001), 215–221.

[13] A. OSTROWSKI, *Über die Absolutabweichung einer differentiebaren Funktion von ihren Integralmittelwert*, Comment. Math. Helv. **10** (1938), pp. 226–227.

(Received December 4, 2002)

J. Pečarić
Faculty of Textile Technology
University of Zagreb, Pierottijeva 6
10000 Zagreb, Croatia
e-mail: pecaric@mahazu.hazu.hr

I. Perić
Faculty of Food Technology and Biotechnology
Mathematics Department
University of Zagreb, Pierottijeva 6
10000 Zagreb, Croatia e-mail: iperic@pbf.hr

A. Vukelić
Faculty of Food Technology and Biotechnology
Mathematics Department
University of Zagreb, Pierottijeva 6
10000 Zagreb, Croatia e-mail: avukelic@pbf.hr