Mathematical Inequalities & Applications
Volume 7, Number 3 (2004), 419–428

FUNCTIONAL INCLUSIONS ON SQUARE–SYMMETRIC GRUPOIDS AND HYERS–ULAM STABILITY

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(*communicated by Z. Páles*)

Abstract. In this paper we prove that a set-valued map $F : X \to P_0(Y)$ that satisfies the inclusion $F(x*y) \subset F(x) \Diamond F(y)$ under suitable conditions admits exactly one selection $f : X \to Y$ that satisfies the equation $f(x*y) = f(x) \diamond f(y)$, where $(X,*)$ and $(Y,\diamond)$ are square-symmetric grupoids and $\Diamond$ is the extension of $\diamond$ to $P_0(Y)$. This result is in connection with Hyers-Ulam stability of functional equation and generalizes a result of Z. Gajda and R. Ger.

1. Introduction

In 1941 D.H. Hyers [4] proved the following theorem:

THEOREM. *Let X be a linear normed space, Y a Banach space, ε a positive number and $f : X \to Y$ a function that satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon \tag{1}$$

for every $x, y \in X$. Then there exists a unique additive function $g : X \to Y$ such that

$$\|f(x) - g(x)\| \leqslant \varepsilon \tag{2}$$

for every $x \in X$.

This was a first answer given to a problem proposed by S.M. Ulam at the Wisconsin University in 1940 and it represents the starting point of the Hyers-Ulam stability theory of functional equations. The subject was later strongly developed by many authors, especially during the last 35 years. We recall that very important contributions at this subject were brought by G.L. Forty [2], Z. Páles [7], [8], [9], T.M. Rassias [11], J. Rätz [12], L. Székelyhidi [14]. An interesting connection between the stability of the Cauchy functional equation and subadditive set-valued functions was established by Z. Gajda and R. Ger [3]. They observed that if f satisfies the inequality (1), then the set-valued map $F : X \to P_0(Y)$,

$$F(x) = f(x) + \overline{B}(0,\varepsilon), \quad x \in X, \tag{3}$$

Mathematics subject classification (2000): 39B72, 54C60.
Key words and phrases: Hyers-Ulam stability, set-valued map, selection, square-symmetric operation.

is subadditive and the function g from the relation (2) is an additive selection of F. ($P_0(Y)$ denotes the collection of all nonempty subsets of Y and $\overline{B}(0,\varepsilon)$ is the closed ball with centre 0 and radius ε in Y).

Now one may ask under what conditions a subadditive set-valued map admits an additive selection. An answer to this question is given in [3]. Furthermore this result was generalized by the author [10], who considered a class of generalized subadditive set-valued maps. Some interesting results concerning subadditive and subquadratic set-valued maps were obtained by W. Smajdor [13].

The purpose of this paper is to give a stability result for the functional inclusion

$$F(x*y) \subset F(x) \lozenge F(y), \quad x,y \in X, \tag{4}$$

where $(X,*)$ and $(Y,\diamond)$ are square-symmetric grupoids and $\lozenge$ is a square-symmetric operation on $P_0(Y)$ determined by $\diamond$. J. Rätz [12] pointed out the role of square-symmetry for the stability of functional equations. Z. Páles [9] and Z. Páles, P. Volkmann and R.D. Luce [8] obtained nice results on stability on square-symmetric grupoids. In our paper we shall use some ideas and terminology from [9]. Let us recall some of them.

A binary operation $*$ on X is called *square-symmetric* if

$$(x*y)*(x*y) = (x*x)*(y*y)$$

for all $x,y \in X$. A grupoid endowed with a square-symmetric operation is called square-symmetric. It is obvious that a commutative semigroup is a square-symmetric grupoid.

An operation $* : X \times X \to X$ is square-symmetric if and only if the function $\sigma_* : X \to X$ given by

$$\sigma_*(x) = x*x, \quad x \in X \tag{5}$$

is an endomorphism of $(X,*)$. The grupoid $(X,*)$ is called *divisible* if σ_* is an automorphism of $(X,*)$. The triple $(Y,*,d)$ is called a *metric grupoid* if $(Y,*)$ is a grupoid, (Y,d) is a metric space and the operation $*$ is continuous with respect to the topology of (Y,d). For a nonempty set Y we denote by $P_0(Y)$ the collection of all nonempty subsets of Y. If (Y,d) is a metric space then clY denotes the collection of all nonempty and closed subsets of Y. If $(Y,\|\cdot\|)$ is a linear normed space then

$$c(Y) := \{A \mid A \in P_0(Y),\ A \text{ is convex set}\},$$

$$cclY := \{A \mid A \in P_0(Y),\ A \text{ is closed and convex set}\},$$

$$cc(Y) := \{A \mid A \in P_0(Y),\ A \text{ is convex and compact set}\}.$$

For a nonempty subset A of a metric space (Y,d) the *diameter* of A is the extended real number $\delta(A)$ defined by

$$\delta(A) = \sup\{d(x,y) \mid x,y \in A\}.$$

Let (Y,ρ) be a metric space. The *Lipschitz modulus* of a function $f : Y \to Y$ is the smallest real extended number L with the property

$$d(f(x),f(y)) \leqslant Ld(x,y), \quad x,y \in Y. \tag{6}$$

We denote the Lipschitz modulus of f by $Lipf$.

Recall that a *selection* of a set-valued map $F : X \to P_0(Y)$ is a single valued function $f : X \to Y$ with the property $f(x) \in F(x)$ for every $x \in X$.

2. Main results

Let $(Y, \diamond, d)$ be a metric grupoid. We define an operation $\Diamond$ on $P_0(Y)$ putting

$$A \Diamond B = \{x | x = a \diamond b, a \in A, b \in B\} \tag{7}$$

Suppose in what follows that the defined operation $\Diamond$ satisfies the condition: for all $\varepsilon > 0$ there exists $\eta > 0$ such that if $\delta(A), \delta(B) < \eta$, $A, B \in P_0(Y)$, then

$$\delta(A \Diamond B) < \varepsilon \tag{8}$$

Let us remark that if $\diamond$ is square-symmetric on Y, then $\Diamond$ is not necessary square-symmetric on $P_0(Y)$. For the square-simmetry of $\Diamond$ it suffices that the operation $\diamond$ satisfies the condition of *bisimmetry* introduced by J.Aczél (see[1]). Hence the following lemma holds.

LEMMA 1. *Let $(Y, \diamond)$ be a grupoid that satisfies the condition*

$$(x_1 \diamond y_1) \diamond (x_2 \diamond y_2) = (x_1 \diamond x_2) \diamond (y_1 \diamond y_2) \tag{9}$$

for every $x_1, x_2, y_1, y_2 \in Y$.Then $\sigma_\Diamond$ is an increasing endomorphism of $(P_0(Y), \Diamond, \subset)$.

Proof. Let $A, B \in P_0(Y)$. We have to prove that

$$\sigma_\Diamond(A \Diamond B) = \sigma_\Diamond(A) \Diamond \sigma_\Diamond(B).$$

Let $x \in \sigma_\Diamond(A \Diamond B)$. Then there exist $a_1, a_2 \in A$, $b_1, b_2 \in A$ such that $x = (a_1 \diamond b_1) \diamond (a_2 \diamond b_2)$. Taking account of the condition (9) it follows $x = (a_1 \diamond a_2) \diamond (b_1 \diamond b_2) \in \sigma_\Diamond(A) \Diamond \sigma_\Diamond(B)$. Hence $\sigma_\Diamond(A \Diamond B) \subset \sigma_\Diamond(A) \Diamond \sigma_\Diamond(B)$.

The reverse inclusion can be proved analogously. Let $A, B \in P_0(Y)$, $A \subset B$. We prove that

$$\sigma_\Diamond(A) \subset \sigma_\Diamond(B).$$

Let $x \in \sigma_\Diamond(A)$. Then there exist $a_1, a_2 \in A$ such that $x = a_1 \diamond a_2 \in B \Diamond B$, hence $\sigma_\Diamond(A) \subset \sigma_\Diamond(B)$. $\square$

Now we can give the first stability result of this paper.

THEOREM 1. *Let $(X, *)$ be a square-symmetric divisible grupoid, $(Y, \diamond, d)$ a complete metric bisymmetric divisible grupoid and $(A, \Diamond)$ a divisible subgrupoid of $(P_0(Y), \Diamond)$.*

Suppose that $F : X \to A$ is a set-valued map that satisfies:

$$F(x * y) \subset F(x) \Diamond F(y), \quad x, y \in X. \tag{10}$$

If

$$\sigma_\Diamond^{-n} \circ F \circ \sigma_*^n(x) \in cl(Y) \tag{11}$$

for every $x \in X$ *and every* $n \in \mathbb{N}$ *and*

$$\lim_{n\to\infty} \delta(F \circ \sigma_*^n(x))Lip(\sigma_\diamond^{-n}) = 0 \tag{12}$$

for every $x \in X$*, then there exists a unique selection* $f : X \to Y$ *of* F *that satisfies the relation*

$$f(x * y) = f(x) \diamond f(y), \quad x, y \in X. \tag{13}$$

Proof. Existence. For $y = x$ the relation (10) becomes

$$F(\sigma_*(x)) \subset \sigma_\Diamond(F(x)), \quad x \in X,$$

and replacing x by $\sigma_*^n(x)$, $n \in \mathbb{N}$, we get

$$F \circ \sigma_*^{n+1}(x) \subset \sigma_\Diamond \circ F \circ \sigma_*^n(x). \tag{14}$$

Now taking into account that $\sigma_\Diamond$ is increasing, it follows that $\sigma_\Diamond^{-1}$ is increasing too, and by (13) we get

$$\sigma_\Diamond^{-n-1} \circ F \circ \sigma_*^{n+1}(x) \subset \sigma_\Diamond^{-n} \circ F \circ \sigma_*^n(x), \quad x \in X. \tag{15}$$

Let $x \in X$ be fixed. Define the sequence of sets $(F_n(x))_{n\geqslant 0}$ by

$$F_n(x) = \sigma_\Diamond^{-n} \circ F \circ \sigma_*^n(x), \quad n \geqslant 0. \tag{16}$$

The sequence $(F_n(x))_{n\geqslant 0}$ is decreasing, in view of relation (15).

We prove that

$$\lim_{n\to\infty} \delta(F_n(x)) = 0. \tag{17}$$

We have

$$\delta(F_n(x)) = \sup\{d(u,v) : \ u, v \in F_n(x)\}.$$

Let $u, v \in \sigma_\diamond^{-n} \circ F \circ \sigma_*^n(x)$. Then

$$\sigma_\diamond^n(u) \in F \circ \sigma_*^n(x) \quad \text{and} \quad \sigma_\diamond^n(v) \in F \circ \sigma_*^n(x).$$

Denote $\sigma_\diamond^n(u) = s$, $\sigma_\diamond^n(v) = t$, $s, t \in F \circ \sigma_*^n(x)$. Then

$$d(u,v) = d(\sigma_\diamond^{-n}(s), \sigma_\diamond^{-n}(t)) \leqslant Lip\sigma_\diamond^{-n} d(s,t) \leqslant Lip\sigma_\diamond^{-n}\delta(F \circ \sigma_*^n(x))$$

and

$$\delta(F_n(x)) \leqslant Lip\sigma_\diamond^{-n}\delta(F \circ \sigma_*^n(x)). \tag{18}$$

By (12) and (18) we obtain $\lim\limits_{n\to\infty} \delta(F_n(x)) = 0$. Then

$$\bigcap_{n=0}^{\infty} F_n(x) \tag{19}$$

is a singleton, in view of the Cantor theorem. Denote by $f(x)$ the single element of this intersection.

The function $f : X \to Y$ is a selection of F, since

$$f(x) \in F_0(x) = F(x) \text{ for every } x \in X.$$

Now we prove that f satisfies the equation (13).

Let us prove first that

$$F_n(x*y) \subset F_n(x) \Diamond F_n(y), \quad x,y \in X, n \geqslant 0. \tag{20}$$

Replacing x by $\sigma_*^n(x)$ and y by $\sigma_*^n(y)$, $n \geqslant 0$, in

$$F(x*y) \subset F(x) \Diamond F(y)$$

we get

$$F \circ \sigma_n^*(x*y) \subseteq (F \circ \sigma_n^*(x)) \Diamond (F \circ \sigma_n^*(y)), \quad n \geqslant 0, \tag{21}$$

in view of the square-symmetry of $*$. Since $\sigma_\Diamond^{-n}$ is an increasing automorphism by (21) it follows

$$\sigma_\Diamond^{-n} \circ F \circ \sigma_n^*(x*y) \subset (\sigma_\Diamond^{-n} \circ F \circ \sigma_n^*(x)) \Diamond (\sigma_\Diamond^{-n} \circ F * \sigma_n^n(y)),$$

hence the relation (20) is proved.

By (20) we get

$$d(f(x*y), f(x) \diamond f(y)) \leqslant \delta(F_n(x) \Diamond F_n(y)), \quad n \geqslant 0 \tag{22}$$

and taking account of the condition (8) we obtain

$$d(f(x*y), f(x) \diamond f(y)) = 0,$$

if n tends to infinity in (22). Hence f satisfies (13).

Uniqueness. Suppose that there exist two selections f,g of F that satisfy the relations

$$\begin{matrix} f(x*y) = f(x) \diamond f(y) \\ g(x*y) = g(x) \diamond g(y) \end{matrix}, \quad x,y \in X \tag{23}$$

By (23), we get

$$\begin{matrix} f \circ \sigma_*^n(x) = \sigma_\diamond^n \circ f(x) \\ g \circ \sigma_*^n(x) = \sigma_\diamond^n \circ g(x) \end{matrix}, \quad x \in X,\ n \in \mathbb{N}, \tag{24}$$

and taking into account that f,g are selections of F

$$\begin{matrix} f \circ \sigma_*^n(x) \in F \circ \sigma_*^n(x) \\ g \circ \sigma_*^n(x) \in F \circ \sigma_*^n(x) \end{matrix}, \quad x \in X,\ n \in \mathbb{N}. \tag{25}$$

Let $x \in X$ be fixed. Then

$$\begin{aligned} d(\sigma_\diamond^n \circ f(x), \sigma_\diamond^n \circ g(x)) &= d(f \circ \sigma_*^n(x), g \circ \sigma_*^n(x)) \\ &\leqslant \delta(F \circ \sigma_*^n(x)), \quad x \in X. \end{aligned}$$

Let $\sigma_\diamond^n \circ f(x) = s$, $\sigma_\diamond^n \circ g(x) = t$, $s,t \in F \circ F_*^n(x)$. We have $f(x) = \sigma_\diamond^{-n}(s)$, $g(x) = \sigma_\diamond^{-n}(t)$ and

$$\begin{aligned} d(f(x), g(x)) &= d(\sigma_\diamond^{-n}(s), \sigma_\diamond^{-n}(t)) \\ &\leqslant Lip\sigma_\diamond^{-n} d(s,t) \leqslant Lip\sigma_\diamond^{-n} \delta(F \circ \sigma_n^*(x)), \quad n \geqslant 0. \end{aligned}$$

Taking account of (12) it follows

$$\lim_{n\to\infty} Lip\sigma_\diamond^{-n} \delta(F \circ \sigma_n^*(x)) = 0$$

hence $f(x) = g(x)$. $\square$

THEOREM 2. *Let $(X,*)$ be a square-symmetric divisible grupoid and $(Y,\diamond,d)$ a metric bisymmetric grupoid. Suppose that $F : X \to P_0(Y)$ is a set-valued map that satisfies the relation*

$$F(x*y) \subset F(x) \Diamond F(y), \quad x,y \in X. \tag{26}$$

If

$$\lim_{n\to\infty} \delta(F\circ \sigma_*^{-n}(x))Lip(\sigma_\diamond^n) = 0 \tag{27}$$

for every $x \in X$, then F is single valued and

$$F(x*y) = F(x)\diamond F(y), \quad x,y \in X. \tag{28}$$

Proof. By the relation (26) we get

$$F(\sigma_*(x)) \subset \sigma_\Diamond(F(x)), \quad x \in X,$$

and replacing x by $\sigma_*^{-n-1}(x)$, $n \in \mathbb{N}$, we obtain

$$F\circ\sigma_*^{-n}(x) \subset \sigma_\Diamond \circ F \circ \sigma_*^{-n-1}(x), \quad x \in X,$$

and taking into account that $\sigma_\Diamond$ is increasing

$$\sigma_\Diamond^n \circ F\circ\sigma_*^{-n}(x) \subset \sigma_\Diamond^{n+1} \circ F \circ \sigma_*^{-n-1}(x), \quad x \in X. \tag{29}$$

Let $x\in X$ be fixed. The sequence of sets $(F_n(x))_{n\geqslant 0}$ defined by

$$F_n(x) = \sigma_\Diamond^n \circ F\circ\sigma_*^{-n}(x), \quad n \geqslant 0,$$

is increasing. Then $(\delta(F_n(x)))_{n\geqslant 0}$ is an increasing sequence of nonnegative numbers. As in the proof of Theorem 1 we obtain

$$\delta(F_n(x)) \leqslant Lip\sigma_\diamond^n \delta(F\circ\sigma_*^{-n}(x))$$

and taking account of (27) it follows

$$\lim_{n\to\infty} \delta(F_n(x)) = 0.$$

Then $\delta(F_n(x)) = 0$ for every $n\in\mathbb{N}$, hence $F_n(x)$ is single valued for every $n\in\mathbb{N}$ and $F_0(x) = F(x)$ satisfies the relation $F(x*y) = F(x)\diamond F(y)$, $x,y \in X$. $\square$

The following results are consequences of the previous theorems.

Suppose that Y is a Banach space over $\mathbb{R}$ and $\diamond$ is defined by

$$x\diamond y = px + qy, \quad x,y \in Y, \tag{30}$$

where $p,q\in\mathbb{R}$ are given numbers.

The triple $(Y,\diamond,\|\cdot\|)$ is obviously a metric groupoid that satisfies condition (9). Then for every $U,V \in P_0(Y)$ the operation $\Diamond$ is defined by

$$U \Diamond V = pU + qV, \tag{31}$$

where $+$ from the right hand side of (31) denotes the usual sum of two sets in a linear space.

COROLLARY 1. *Let $(X,*)$ be a square-symmetric divisible groupoid, Y a Banach space over $\mathbb{R}$, $p,q \in \mathbb{R}$, $p+q \neq 1$. Suppose that $F : X \to c(Y)$ is a set-valued map such that*

$$F(x*y) \subset pF(x)+qF(y), \quad x,y \in X, \tag{32}$$

and the following conditions are satisfied:

(i) $F \circ \sigma_*^n(x) \in cl(Y)$, $x \in X$, $n \in \mathbb{N}$;

(ii) there exists $M > 0$ such that

$$\delta(F(x)) \leqslant M, \quad x \in X.$$

Then there exists a unique selection $f : X \to Y$ of F such that

$$f(x*y) = pf(x)+qf(y), \quad x,y \in X. \tag{33}$$

Proof. Let $A = c(Y)$. Then $\sigma_\diamondsuit(U) = (p+q)U$ for every $U \in c(Y)$, $\sigma_\diamondsuit$ is an automorphism of $(c(Y), \diamondsuit)$ for $p+q \neq 0$ and $\sigma_\diamond^n(x) = (p+q)^n x$, $x \in X$, $n \in \mathbb{Z}$, with

$$Lip\sigma_\diamond^n = |p+q|^n, \quad n \in \mathbb{Z}.$$

1) If $|p+q| > 1$ then

$$\sigma_\diamondsuit^{-n} \circ F \circ \sigma_*^n(x) = \frac{1}{(p+q)^n} F \circ \sigma_*^n(x) \in cl(Y)$$

for every $n \in \mathbb{N}$ and

$$\delta(F \circ \sigma_*^n(x)) Lip(\sigma_\diamond^{-n}) \leqslant \frac{M}{|p+q|^n}, \; x \in X, n \in \mathbb{N}.$$

By Theorem 1 it follows that there exists a unique selection f of F that satisfies (33).

2) If $|p+q| < 1$ then

$$Lip(\sigma_\diamond^n)\delta(F \circ \sigma_*^{-n}(x)) \leqslant M|p+q|^n, \; x \in X, n \in \mathbb{N},$$

hence, in view of Theorem 2, F is single valued and satisfies the equation

$$F(x*y) = pF(x)+qF(y), \quad x,y \in X,$$

so that is its own selection. $\square$

COROLLARY 2. *Let $(X,*)$ be a square-symmetric divisible grupoid, Y a Banach space over $\mathbb{R}$, $p,q \in \mathbb{R}$, $p+q > 1$ and $B \in cc(Y)$. Suppose that $F : X \to c(Y)$ satisfies the general linear inclusion*

$$F(x*y) \subset pF(x)+qF(y)+B, \quad x,y \in X \tag{34}$$

and the following conditions are satisfied:

(i) $F \circ \sigma_*^n(x) \in cl(Y)$, $x \in X$, $n \in \mathbb{N}$;

(ii) there exists $M > 0$ *such that*

$$\delta(F(x)) \leqslant M, \quad x \in X.$$

Then there exists a unique function $f : X \to Y$ *such that*

$$f(x) \in F(x) + \frac{1}{p+q-1}B, \quad x \in X,$$

and

$$f(x*y) = pf(x) + qf(y), \quad x,y \in X.$$

Proof. Let $G : X \to c(Y)$ be defined by

$$G(x) = F(x) + \frac{1}{p+q-1}B, \quad x \in X.$$

We prove that

$$G(x*y) \subset pG(x) + qG(y), \quad x,y \in X.$$

Indeed, using the convexity of B, we get

$$\begin{aligned} G(x*y) &= F(x*y) + \frac{1}{p+q-1}B \\ &= pF(x) + qF(y) + \frac{p+q}{p+q-1}B \\ &\subset p\left(F(x) + \frac{1}{p+q-1}B\right) + q\left(F(x) + \frac{1}{p+q-1}B\right) \\ &= pG(x) + qG(y), \quad x,y \in X. \end{aligned}$$

We have

$$G \circ \sigma_*^n(x) = F \circ \sigma_*^n(x) + \frac{1}{p+q-1}B \in cl(Y)$$

for every $x \in X$ and every $n \in \mathbb{N}$, in view of the compactness of B, and

$$\begin{aligned} \delta(G \circ \sigma_*^n(x)) &\leqslant \delta(F \circ \sigma_*^n(x)) + \frac{1}{p+q-1}\delta(B) \\ &\leqslant M + \frac{1}{p+q-1}\delta(B), \quad x \in X,\ n \in \mathbb{N}. \end{aligned}$$

By the Corollary 1 it follows that there exists a selection f of G such that

$$f(x*y) = pf(x) + qf(y), \quad x,y \in X,$$

hence $f(x) \in F(x) + \frac{1}{p+q-1}B$ for every $x \in X$. □

REMARK 1. The particular case $p = q = 1$ in Corollary 1 leads to the stability result of R.Ger and Z. Gajda [3].

The previous results leads to the following stability result of the general linear equation.

COROLLARY 3. *Let* $(X,*)$ *be a square-symmetric divisible grupoid,* Y *a Banach space over* $\mathbb{R}$ *and* $B \in ccl(Y)$*. Assume that* $p,q \in \mathbb{R}$*,* $p+q>1$*, and* $b \in Y$*.*

Let $g: X \to Y$ *be a function such that*

$$g(x*y) - pg(x) - qg(y) - b \in B, \quad x,y \in X. \tag{35}$$

Then there exists a uniquely determined function $h: X \to Y$ *such that*

$$h(x*y) = ph(x) + qh(y) + b, \quad x,y \in X, \tag{36}$$

and

$$g(x) - h(x) \in \frac{1}{p+q-1}B. \tag{37}$$

Proof. Define the set-valued map $F: X \to ccl(Y)$

$$F(x) = g(x) + \frac{1}{p+q-1}A, \quad x \in X,$$

where $A := b + B \in ccl(Y)$. We have

$$\begin{aligned} F(x*y) &= g(x*y) + \frac{1}{p+q-1}A \\ &\subset pg(x) + qg(y) + A + \frac{1}{p+q-1}A \\ &= pg(x) + qg(y) + \frac{p+q}{p+q-1}A \\ &= pF(x) + qF(y), \quad x,y \in X. \end{aligned}$$

By Corollary 1 it follows that there exists a uniquely determined selection $f: X \to Y$ of F that satisfies (33). The function $h: X \to Y$ given by

$$h(x) = f(x) - \frac{1}{p+q-1}b, \quad x \in X,$$

satisfies (36) and (37). □

REMARK 2. For the particular case when B is the closed ball of center 0 and radius ε, in Corollary 3 one obtains a stability result proved by Z. Páles [9].

Acknowledgement. The author thanks to the referee for his observations and suggestions.

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(Received April 1, 2003)

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