Mathematical
Inequalities
& Applications
Volume 7, Number 4 (2004), 471–490

LORENTZ SPACES FOR DECREASING REARRANGEMENTS OF FUNCTIONS ON TREES

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(*communicated by L.-E. Persson*)

Abstract. We characterize the existence of a norm in the Lorentz spaces defined on trees, for non-linear decreasing rearrangements, as well as other functional properties (quasinormability, relationship with rearrangement invariant spaces, etc.). The main tool is the characterization of the saturation in Hardy-Littlewood's inequality for linearly decreasing functions.

1. Introduction

Let (X,μ) be a measure space. For every $0<p<\infty$ and every weight in the positive real line, the Lorentz space $\Lambda_X^p(u)$ is defined as the set of μ-measurable functions f such that the functional

$$\|f\|_{\Lambda_X^p(u)} = \left(\int_0^\infty f^\star(t)^p u(t)\,dt\right)^{1/p} \tag{1}$$

is finite, where

$$f^\star(t) = \inf\{\lambda : \mu(\{x\in X : |f(x)|>\lambda\}) \leqslant t\}, \qquad t>0,$$

is the classical decreasing rearrangement. The Lorentz spaces were introduced in 1951 by G.G. Lorentz ([Lo]) in the case $X=(0,l)$ and μ the Lebesgue measure, and they are generalizations of the L^p and $L^{p,q}$ spaces. In his paper, G.G. Lorentz proved that in the case $p\geqslant 1$, the functional defined in (1) is a norm if and only if the weight u is decreasing.

In [GS], we introduced a new decreasing rearrangement of functions defined in a homogeneous tree, which takes strongly into account the geometric structure of the tree. In this work, we consider the weighted Lorentz spaces related to the new decreasing rearrangement and we characterize some normability properties of these spaces in terms of the weight. It is important to remark that the classical techniques do not work in the context of a tree due to the lack of algebraic structure and the non-existence of a total order, and trivial facts for the classical rearrangement of functions become difficult

Mathematics subject classification (2000): 05C05, 46E30, 46B25.
Key words and phrases: homogeneous trees, decreasing rearrangements, Lorentz spaces.
This work has been partially supported by Grants BFM2001-3395 and 2001SGR00069.

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Paper MIA-07-47

in the tree (see for example the monotonic condition proved in [GS, Proposition 19]). Instead, we use combinatorial techniques.

In Section 2 we recall the main notations and most important results given in [GS]. In Section 3 we study the relationship between the classical and the new rearrangement, pointing out the advantages of the second (see Corollary 3.4). We also address the validity of the Hardy-Littlewood inequality and introduce the class of linearly decreasing functions, which plays a fundamental role in the theory. Lorentz spaces are defined in Section 4, where we prove our main result (Theorem 4.9): what are the conditions on the weight to obtain a norm.

We also want to mention the works of A.R. Pruss [Pr1] and [Pr2], where a decreasing rearrangement on homogeneous trees is given by means of a "spiral-like" ordering. We point out that this rearrangement is not useful in our context because it does not satisfy condition (*iii*) of [GS, Definition 1]. See also [L] for applications of this "spiral-like" order to obtain geometric information of the tree. Other recent works dealing with extensions to trees of classical results in Analysis are, for example, [CFPR], [EHL], and [NS].

2. Definitions and previous results

We adopt the definitions and some of the notations of [C] and [FTN]: a tree T is a connected graph without circuits or cycles. We identify a tree with the set of its vertices. A tree is called homogeneous of degree $q+1$ if every vertex has $q+1$ neighbor vertices.

In a tree, there exists a unique chain joining two vertices x and y. We call this chain a geodesic and we denote it by $[x,y]$ (or $[y,x]$). The distance between x and y is the number of edges in the geodesic $[x,y]$, that is, the length of $[x,y]$. As usual, we denote it by $d(x,y)$. Now, the vertices x and y are neighbors if $d(x,y)=1$.

A rooted tree T_o is a tree with a fixed reference vertex o called origin of the tree. An infinite chain is an infinite sequence $x_0, x_1, x_2, \dots$ of vertices such that x_i and x_{i+1} are neighbors and $x_i \neq x_{i+2}$ for all $i \geqslant 0$. The boundary of a rooted tree ∂T_o is the set of all infinite chains starting at o. The boundary can be viewed as the set of points at infinity.

For every x in T, we write the geodesic joining o to x by

$$\{x(0)=o, x(1), \dots, x(n)=x\} := [o,x],$$

where $k=d(o,x(k))$ and $n=d(o,x)$. The confluent vertex of the vertices x and y is the unique vertex $c(x,y)$ such that the geodesics $[o,c(x,y)]$, $[c(x,y),x]$ and $[c(x,y),y]$ meet only at $c(x,y)$. If $x \in [o,y]$, we set $c(x,y)=x$. The tent of x, $T(x)$, and its shadow in ∂T_o, $I(x)$, are defined by

$$\begin{aligned} T(x) &= \{y \in T_o : x \in [o,y]\}, \\ I(x) &= \{\omega \in \partial T_o : x \in [o,\omega)\}. \end{aligned}$$

We can define a partial order structure on T_o: the vertex x is greater than or equal to the vertex y, if y belongs to $[o,x]$. We denote it by $y \leqslant_o x$. In other words:

$$y \leqslant_o x \Leftrightarrow y \in [o,x] \Leftrightarrow x \in T(y).$$

A function defined on a tree is a discrete function evaluated on each vertex. We are interested in monotone functions. A function is decreasing if $f(x) \leqslant f(y)$ whenever $y \leqslant_o x$. A set of vertices E in T is a decreasing set if whenever $x \in E$, we have that $y \in E$, for all y such that $y \leqslant_o x$, that is, χ_E is a decreasing function. If E is a finite set of vertices in T_o, we denote by $|E|$ its cardinal.

We now give a summary of the results introduced in [GS]. In view of the so-called "Layer-cake" formula (see [LL] and (4)), to define a decreasing rearrangement of a positive function, it is enough to introduce a decreasing rearrangement for finite sets of vertices. The definition of our rearrangement initially depends on the choice of what is called an order in the boundary of the tree. It is defined by means of a suitable bijection, called admissible map (see Definition 2 in [GS]), between the boundary of the tree and an interval in the real line minus the q-adic numbers $N(q)$:

$$\sigma : \partial T_o \longrightarrow [0,(q+1)q^{-1}] \setminus N(q). \tag{2}$$

DEFINITION 2.1. Let σ be an admissible map as in (2). Given ω and ω' in ∂T_o, we define $\omega \leqslant_\sigma \omega'$ if and only if $\sigma(\omega) \leqslant \sigma(\omega')$.

In the sequel, an admissible map σ will be called an order in ∂T_o. For two given disjoint sets A and B in ∂T_o, we will write $A <_\sigma B$, if $x <_\sigma y$ for all $x \in A$ and all $y \in B$. Then, we can use this boundary order σ to introduce the order notation of the so-called boundary vertices in every finite set E,

$$\partial E := \{e_1, e_2, \cdots, e_n\}_\sigma, \tag{3}$$

where, for every $i = 1, \ldots, n$, $T(e_i) \cap E = \{e_i\}$, $I(e_i) <_\sigma I(e_{i+1})$ if $i \neq n$ and $n = n(E) = |\partial E|$. (See Figure 1 for an example.)

DEFINITION 2.2. Let σ be an order in ∂T_o, and let E be a finite set of vertices in T_o with boundary $\partial E = \{e_1, e_2, \cdots, e_n\}_\sigma$. Set $\mathcal{R}_{(o,\sigma,0)}(E) := E,$ and then recursively define, for every $0 \leqslant k \leqslant n-1$, the sets

$$\mathcal{R}_{(o,\sigma,k+1)}(E) := \big(\mathcal{R}_{(o,\sigma,k)}(E) \setminus [o, e_{k+1}]\big) \cup [o, e_{k+1}(s)],$$

where $s+1 = s(k)+1 = \big|\mathcal{R}_{(o,\sigma,k)}(E) \cap [o, e_{k+1}]\big|$. Then, the final set

$$\mathcal{R}_{(o,\sigma)}(E) := \mathcal{R}_{(o,\sigma,n)}(E),$$

is the decreasing rearrangement of E.

See Figure 1 for a graphic example on a homogeneous tree of degree 3, where the chosen order σ on ∂T_o is such that the boundary of every finite set is ordered from left to right.

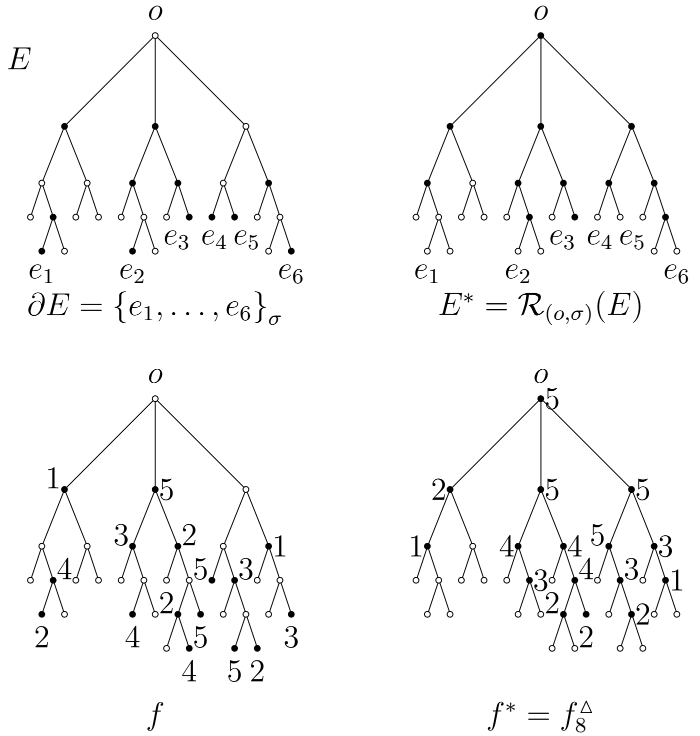

Figure 1: A set E and its decreasing rearrangement E^*, and a function f and its decreasing rearrangement f^*. We use the order σ from left to right.

REMARK 2.3. It is proved in [GS] that this rearrangement is canonical, up to automorphisms of the tree. In the sequel, we will not pronounce the dependence on either the origin of the tree or its order in the boundary, and we will simply denote $\mathcal{R}_k(E) = \mathcal{R}_{(o,\sigma,k)}(E)$, $E^* = \mathcal{R}_{(o,\sigma)}(E)$, and $y \leqslant x$ instead of $y \leqslant_o x$. Observe that $*$ refers to the new rearrangement and $\star$ to the classical one.

Let $\mathcal{M}_0$ be the set of functions f defined on the tree such that the level sets $\{x \in T : |f(x)| > \lambda\}$ are finite for all $\lambda > 0$. For a function $f \in \mathcal{M}_0$, its decreasing rearrangement is the function defined by means of the "Layer-cake" formula:

$$f^*(x) = \int_0^\infty \chi_{\{y\in T:|f(y)|>\lambda\}^*}(x)\, d\lambda, \tag{4}$$

for all $x \in T$.

By using the properties of the rearrangement proved in [GS, Proposition 23], it makes sense to extend the definition of the decreasing rearrangement to functions with non finite level sets, just by a density argument: if $(E_n)_n$ is a sequence of finite sets in T such that $E_n \subset E_{n+1}$ and $T = \cup_{n\geqslant 0} E_n$, for a general function f in the tree, consider the sequence of functions $(f_n)_n$ defined by $f_n = f\chi_{E_n}$ and then set $f^* = \lim_{n\to\infty}(f\chi_{E_n})^*$ pointwise. This shows that we can reduce our study to functions with finite support.

One of the main results in [GS] is the existence of an equivalent expression for f^* in (4), which is more intuitive and easier to handle. For a positive function f with finite support, set $E = \mathrm{supp}(f)$ and $\partial E = \{e_1, e_2, \dots, e_n\}_\sigma$, and define for each $1 \leqslant k \leqslant n$:

$$f_k^\triangle(y) = \begin{cases} f_{k-1}^\triangle(y) & \text{if } y \in \mathcal{R}_k(E) \cap [o, e_k]^c, \\ (f_{k-1}^\triangle \cdot \chi_{[o,e_k]})^*(y) & \text{if } y \in \mathcal{R}_k(E) \cap [o, e_k], \end{cases} \tag{5}$$

where $f_0^\triangle = f$. Observe that $\mathrm{supp}(f_k^\triangle) = \mathcal{R}_k(E)$. Then (see [GS, Theorem 27]),

$$f^* = f_n^\triangle. \tag{6}$$

In fact, what we are doing is to recursively rearrange the restriction of the function to each geodesic $[o, e_k]$. (See Figure 1.)

3. The Hardy-Littlewood inequality

The classical Hardy-Littlewood inequality for measurable functions f and g defined in T is

$$\sum_{x \in T} |f(x)g(x)| \leqslant \int_0^\infty f^\star(t) g^\star(t)\, dt. \tag{7}$$

A consequence of this inequality is that in the measure space $(T, |.|)$, which is *resonant* (with respect to the classical rearrangement, see [BS]), for each measurable f and g in $(T, |.|)$, the identity

$$\sup \sum_{x \in T} |f(x)h(x)| = \int_0^\infty f^\star(t) g^\star(t)\, dt, \tag{8}$$

holds, where the supremum is taken over all measurable functions h on T such that $h^\star = g^\star$. Our purpose is to give the same type of results but using our decreasing rearrangement on the tree, and our motivation is the proof of G.G. Lorentz in [Lo] of the characterization as a norm of the natural functional that defines the Lorentz spaces. We observe that we can state his result in the following way: if $p \geqslant 1$, the following are equivalent

(i) The weight u is a decreasing function in $[0, \infty)$.

(ii) For all Lebesgue measurable functions f in $(0, l)$, the equality

$$\sup_{\{h:\, h^\star = u\}} \int_0^l |f(t)h(t)|\, dt = \int_0^\infty f^\star(t) u(t)\, dt,$$

holds.

(iii) The functional $\|.\|_{\Lambda^p_{(0,l)}(u)}$ in (1), for $X = (0, l)$ and μ the Lebesgue measure, is a norm.

It is natural to ask if there is any relationship between both rearrangements. The following result gives some information about this. The proof uses standard arguments and is omitted.

PROPOSITION 3.1. *For every measurable function f in $\mathcal{M}_0$, we have $(f^*)^\star(t) = f^\star(t)$, for all $t > 0$.*

This proposition shows that if $f^* = g^*$ then $f^\star = g^\star$, and it is trivial to show that the converse is not true in general.

PROPOSITION 3.2. *For all $f \in \mathcal{M}_0$ and for all finite sets of vertices $E \subset T$ the inequality*

$$\sum_{x\in E} |f(x)| \leqslant \sum_{x\in E^*} f^*(x)$$

holds.

Proof. We use the notation $f^*(x) = \sum_{n=1}^{\infty} b_n \chi_{F_n^*}(x)$, (whenever $f(x) = \sum_{n=1}^{\infty} b_n \chi_{F_n}(x)$, where $F_n \subset F_{n+1}$, $b_n > 0$ and $\sum_{n=1}^{\infty} b_n < \infty$), and the monotony of the rearrangement proved in [GS, Proposition 19]:

$$\sum_{x\in E} |f(x)| = \sum_{n=1}^{\infty} b_n |E \cap F_n| = \sum_{n=1}^{\infty} b_n |(E\cap F_n)^*| \leqslant \sum_{n=1}^{\infty} b_n |E^* \cap F_n^*| = \sum_{x\in E^*} f^*(x).$$

□

We obtain for our rearrangement the Hardy-Littlewood inequality.

THEOREM 3.3. (Hardy-Littlewood inequality) *For all f and g in $\mathcal{M}_0$, the inequality*

$$\sum_{x\in T} |f(x)g(x)| \leqslant \sum_{x\in T} f^*(x)g^*(x)$$

holds.

Proof. We use Proposition 3.2:

$$\sum_{x\in T} |f(x)g(x)| = \sum_{n=1}^{\infty} b_n \left(\sum_{x\in F_n} |g(x)|\right) \leqslant \sum_{n=1}^{\infty} b_n \left(\sum_{x\in F_n^*} g^*(x)\right) = \sum_{x\in T} f^*(x)g^*(x).$$

□

As a consequence of this theorem, Proposition 3.1 and (7), we obtain:

COROLLARY 3.4. *For all measurable functions f and g in $\mathcal{M}_0$, we have*

$$\sum_{x\in T} |f(x)g(x)| \leqslant \sum_{x\in T} f^*(x)g^*(x) \leqslant \int_0^{\infty} f^\star(t)g^\star(t)\,dt.$$

Observe that the new rearrangement gives better estimates than the classical one. Another important consequence of Theorem 3.3 is that the inequality

$$\sum_{x\in T} |f(x)h(x)| \leqslant \sum_{x\in T} f^*(x)g^*(x)$$

holds for all functions $h \in \mathcal{M}_0$ such that $h^* = g^*$. In view of (8), it is natural to consider the following question: is it possible to get the equality

$$\sup \sum_{x\in T} |f(x)h(x)| = \sum_{x\in T} f^*(x)g^*(x), \tag{9}$$

where the supremum is taken over all functions $h \in \mathcal{M}_0$ such that $h^* = g^*$? Now the answer is negative for a general decreasing function g, even for a characteristic function of a decreasing set (this fact reflects that the classical rearrangement heavily depends on the total order structure of the real line). But it is not difficult to see that there exist decreasing functions g in the tree such that (9) holds. Our purpose now is to identify the case of equality in (9). We first fix our attention to the functions with finite support, and we begin by looking at functions with support in one geodesic from the origin o to a fixed vertex e. In this case, our rearrangement is equivalent to the classical rearrangement of discrete functions defined on $\mathbb{N}$, with support in the interval $[0,N]$ for a fixed N, simply by considering the bijection

$$[o,e] = \{o = e(0), e(1), \dots, e(N) = e\} \equiv [0,N],$$

such that $e(i) \longleftrightarrow i$. The measure space $([0,N], |.|)$ is *strongly resonant* and this means that, for fixed f and g, the equality

$$\sum_{x\in[o,e]} |f(x)h(x)| = \sum_{x\in[o,e]} f^*(x)g^*(x), \tag{10}$$

holds for a suitable h. In fact, this h can be constructed by permuting the values of g^*, by using the permutation that takes f into f^*. To be precise, consider the permutation $\varphi_f : [o,e] \longrightarrow [o,e]$ such that

$$|f(x)| = f^*(\varphi_f(x)),$$

for all $x \in [o,e]$. Then, $h(x) = g^*(\varphi_f(x))$ satisfies equality (10) (by considering a change of variable $x = \varphi_f(y)$), and trivially $h^* = g^*$. We observe that the permutation φ_f, for a measurable f, plays an important role in order to obtain equality (10). In the "linear" case of $[o,e]$, we trivially have that if g is decreasing, then $g(\varphi_f(.))^* = g$ for all f. In fact, the reverse implication is also true, and we can trivially state that g is decreasing if and only if $g(\varphi_f(.))^* = g$ for all f. Thus, a monotony property of g is equivalent to an invariancy notion of g. We fix now our attention on functions with general finite support not necessarily contained in a geodesic.

DEFINITION 3.5. Let f be a positive function with finite support $E \subset T$. A rearranging transformation for f is a bijection $\varphi_f : E \longrightarrow E^*$ such that $f(x) = f^*(\varphi_f(x))$, for all $x \in E$.

In view of (5) and (6), we can decompose φ_f into the composition of rearranging transformations for every geodesic from o to each vertex in the boundary of E. To be precise, if $n = |\partial E|$,

$$\varphi_f = \varphi_{f,n} \circ \varphi_{f,n-1} \circ \cdots \circ \varphi_{f,1},$$

where each $\varphi_{f,k}$ is a mapping $\varphi_{f,k} : \mathcal{R}_{k-1}(E) \longrightarrow \mathcal{R}_k(E)$ such that

- $\varphi_{f,k}$ is the identity out of $[o,e_k]$, that is

$$\varphi_{f,k} \cdot \chi_{\mathcal{R}_{k-1}(E)\setminus[o,e_k]} = \mathrm{Id}.$$

- Each $\varphi_{f,k}$ is the rearranging transformation for $f_k^{\triangle}$ restricted to $[o,e_k]$, that is:
  (i) $\varphi_{f,k} : \mathcal{R}_{k-1}(E) \cap [o,e_k] \longleftrightarrow \mathcal{R}_k(E) \cap [o,e_k]$.

$$\text{(ii)}\ \ f_k^{\triangle}(y) = \begin{cases} f_{k-1}^{\triangle}(y) & \text{if } \ y \in \mathcal{R}_k(E) \setminus [o,e_k], \\ \\ f_{k-1}^{\triangle}(\varphi_{f,k}^{-1}(y)) & \text{if } \ y \in \mathcal{R}_k(E) \cap [o,e_k]. \end{cases}$$

In other words, each $\varphi_{f,k}$ is the rearranging transformation for $f_k^{\triangle} \cdot \chi_{[o,e_k]}$, extended to all $\mathcal{R}_{k-1}(E)$ as the identity. These bijections are not unique in general, unless we require that $\varphi_{f,k}(x) \leqslant \varphi_{f,k}(y)$ if $x \leqslant y$, whenever $f_{k-1}^{\triangle}(x) = f_{k-1}^{\triangle}(y)$. We keep this condition for granted, so that φ_f is also unique for every f. For a finite set of vertices E, we define

$$\Phi(E) = \{\varphi : E \longleftrightarrow E^* : \exists f \text{ s.t. } \varphi = \varphi_f\},$$

and

$$\Phi = \bigcup_{\{E \subset T : |E| < \infty\}} \Phi(E).$$

Thus, Φ is the set of all the rearranging transformations in the tree. In general, for a decreasing positive function g, the equality

$$g(\varphi_f(.))^* = g \tag{11}$$

does not hold. Consequently, g must be something better than decreasing in order to have (9) or (11). The unexpected solution is given by considering a new order structure in the tree. We recall that for two given disjoint sets A and B in ∂T_o, we write $A <_\sigma B$, if $x <_\sigma y$ for all $x \in A$ and all $y \in B$.

DEFINITION 3.6. Given two vertices x and y in T, we define $x \trianglelefteq y$ if and only if $x \leqslant y$ or $I(x) \geqslant_\sigma I(y)$.

It is very important to observe that this is a total order, compatible with the natural partial order (although it is not a locally finite order), and that it depends on the choice of σ. We give now some lemmas that will lead to the final result of this section. In what follows, we will use the notation $(x,y] = [x,y] \setminus \{x\}$, or $[x,y) = [x,y] \setminus \{y\}$, for two vertices x and y in T.

LEMMA 3.7. *Let f be a positive function in T with finite support E, and $\partial E = \{e_1, e_2, \dots, e_n\}_\sigma$ its boundary. If $x, y \in E \cap [o,e_1]$ satisfy that $f(x) \geqslant f(y)$, then $\varphi_f(x) \trianglelefteq \varphi_f(y)$.*

Proof. If $\varphi_f(x)$ and $\varphi_f(y)$ lie in the same geodesic, we trivially have that $\varphi_f(x) \leqslant \varphi_f(y)$, because of the hypothesis and that f^* is decreasing. Suppose that $\varphi_f(x)$ and $\varphi_f(y)$ do not lie in the same geodesic. Take the decomposition

$$\varphi_f = \varphi_{f,n} \circ \varphi_{f,n-1} \circ \cdots \circ \varphi_{f,1},$$

and write $y_j = \varphi_{f,j} \circ \cdots \circ \varphi_{f,1}(y)$ and $x_j = \varphi_{f,j} \circ \cdots \circ \varphi_{f,1}(x)$. By hypothesis, there exists $1 \leqslant k \leqslant n$ such that x_k and y_k are vertices of the same geodesic and $x_k \leqslant y_k$, but x_{k+1} and y_{k+1} are not in the same geodesic. This can only happen if (see Figure 2)

$$y_k \in \mathcal{R}_k(E) \cap (c(e_k, e_{k+1}), e_k],$$

and

$$x_{k+1} \in \mathcal{R}_{k+1}(E) \cap (c(e_k, e_{k+1}), e_{k+1}],$$

since then, by definition of $\varphi_{f,k+1}$ we have

$$y_{k+1} = \varphi_{f,k+1}(y_k) = y_k \in (c(e_k, e_{k+1}), e_k],$$

and then x_{k+1} and y_{k+1} do not lie in the same geodesic. Hence, by construction $x_{k+1} \trianglelefteq y_{k+1}$.

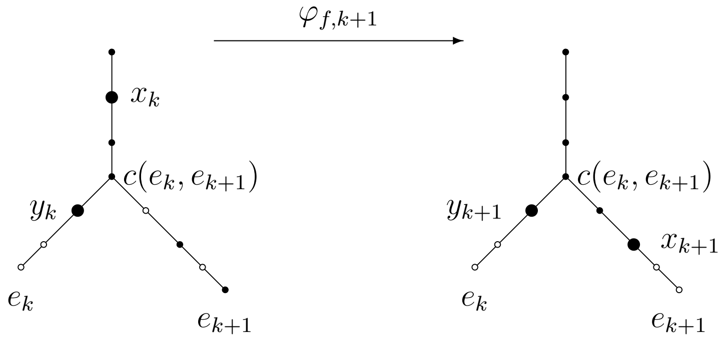

Figure 2: The situation of x_k and y_k, and the action of $\varphi_{f,k+1}$.

Now, we claim that

$$\varphi_f(x) \leqslant x_{k+1}. \tag{12}$$

Thus,

$$\varphi_f(x) \leqslant x_{k+1} \trianglelefteq y_{k+1} = \varphi_f(y),$$

where the last equality is due to the fact that $\varphi_{f,j} = \mathrm{Id}$ in $(c(e_k, e_{k+1}), e_k]$ for all $j \geqslant k+1$. We now prove the claim. Two possibilities can happen: first, if $x_{k+1} \in (c(e_{k+1}, e_{k+2}), e_{k+1}]$, then $\varphi_f(x) = x_{k+1}$, because $\varphi_{f,j} = \mathrm{Id}$ in $(c(e_{k+1}, e_{k+2}), e_{k+1}]$ for all $j \geqslant k+2$, and so we have an equality in (12). Second, if $x_{k+1} \in [o, c(e_{k+1}, e_{k+2})]$ and if it does not exist $y \in (c(e_{k+1}, e_{k+2}), e_{k+2}]$ such that $f^{\triangle}_{k+1}(x_{k+1}) \leqslant f^{\triangle}_{k+1}(y)$, then

$$x_{k+2} = \varphi_{f,k+2}(x_{k+1}) = x_{k+1},$$

and nothing changes in this rearrangement. If there exists $y \in (c(e_{k+1}, e_{k+2}), e_{k+2}]$ such that $f^{\triangle}_{k+1}(x_{k+1}) \leqslant f^{\triangle}_{k+1}(y)$, then necessarily

$$x_{k+2} = \varphi_{f,k+2}(x_{k+1}) \in T(x_{k+1}),$$

and thus $x_{k+2} \leqslant x_{k+1}$. Repeating this argument, we obtain

$$\varphi_f(x) = x_n \leqslant x_{n-1} \leqslant \ldots x_{k+1}. \qquad \square$$

LEMMA 3.8. *Let E be a finite set in T, and $n = |\partial E|$. If*

$$\varphi = \varphi_n \circ \varphi_{n-1} \circ \cdots \circ \varphi_2 \circ \varphi_1 \in \Phi(E),$$

then $\varphi' := \varphi_n \circ \varphi_{n-1} \circ \cdots \circ \varphi_2 \in \Phi(D)$, where $D = \mathcal{R}_1(E) \setminus (c(e_1, e_2), e_1]$.

Proof. Observe that $|D| = n-1$ since $\partial D = \partial E \setminus \{e_1\}$. Hence, there exists f supported in E such that $\varphi = \varphi_f$. If we set $g = f_1^{\triangle} \cdot \chi_D$, then we get

$$g^* = f^* \cdot \chi_{E^* \setminus (c(e_1,e_2),e_1]},$$

and $g(x) = g^*(\varphi'(x))$, for all $x \in D$. $\square$

Before proving the next important result, we define a new decreasing property for functions in the tree.

DEFINITION 3.9. A function g is linearly decreasing if $g(x) \geqslant g(y)$, whenever $x \trianglelefteq y$.

We observe that if g is linearly decreasing, then g is decreasing. The basic result of this section is the following:

THEOREM 3.10. *If g is a linearly decreasing positive function, then*

$$(g \circ \varphi)^*(y) = g(y), \tag{13}$$

for all $\varphi \in \Phi$, and for all y in the support of φ.

Proof. Take $\varphi \in \Phi(E)$ for a certain finite set E. We prove the theorem by induction on $|\partial E|$. If $|\partial E| = 1$, then E is contained into a geodesic $[o,e]$, and we are done, because if g is linearly decreasing, it is decreasing. Suppose it is true for $|\partial E| = n-1$. Fix E such that $\partial E = \{e_1, e_2, \dots, e_n\}_\sigma$. If $\varphi \in \Phi(E)$, we know that

$$\varphi = \varphi_n \circ \varphi_{n-1} \circ \cdots \circ \varphi_1,$$

where $\varphi_i = \varphi_{f,i}$ for a certain f supported in E. Set $h = g \circ \varphi$, which is supported in E. Then, there exists $\varphi_h \in \Phi(E)$ such that $h(x) = h^*(\varphi_h(x))$, for all $x \in E$. Its decomposition is $\varphi_h = \varphi_{h,n} \circ \varphi_{h,n-1} \circ \cdots \circ \varphi_{h,1}$. We claim that $\varphi_1 = \varphi_{h,1}$. Take $x, y \in E \cap [o,e_1]$ and suppose that $\varphi_1(x) < \varphi_1(y)$. Then, $f(x) \geqslant f(y)$, and by Lemma 3.7, we have that $\varphi(x) \trianglelefteq \varphi(y)$, and then

$$h(x) = g(\varphi(x)) \geqslant g(\varphi(y)) = h(y),$$

because g is linearly decreasing. Therefore,

$$\varphi_{h,1}(x) < \varphi_{h,1}(y).$$

Since this is true for all x and y in $E \cap [o,e_1]$, which is a finite set, we have proved the claim. Using the claim and that $\varphi_1 = \mathrm{Id}$ in $\mathcal{R}_1(E) \setminus [o,e_1] \subset E \setminus [o,e_1]$, we have:

$$\begin{aligned} h_1^{\triangle}(y) &= \begin{cases} h(y) & \text{if } y \in \mathcal{R}_1(E) \setminus [o,e_1], \\ h(\varphi_{h,1}^{-1}(y)) & \text{if } y \in \mathcal{R}_1(E) \cap [o,e_1] \end{cases} \\ &= g(\varphi_n \circ \cdots \circ \varphi_2(y)), \end{aligned} \tag{14}$$

for all $y \in \mathcal{R}_1(E)$. So, for $y \in E^* \setminus (c(e_1,e_2),e_1]$, we get

$$h^*(y) = (h_1^{\triangle})^*(y) = (g(\varphi_n \circ \cdots \circ \varphi_2(.))^*(y). \tag{15}$$

If $\mathcal{R}_1(E) \cap (c(e_1,e_2),e_1] \neq \emptyset$, using (14) and that $\varphi_j = \mathrm{Id}$ in $\mathcal{R}_1(E) \cap (c(e_1,e_2),e_1]$ for $j \geqslant 2$, we get for $y \in \mathcal{R}_1(E) \cap (c(e_1,e_2),e_1]$:

$$h^*(y) = h_1^{\Delta}(y) = g(\varphi_n \circ \cdots \circ \varphi_2(y)) = g(y).$$

This equality, (15) and the fact that $E^* \cap (c(e_1,e_2),e_1] = \mathcal{R}_1(E) \cap (c(e_1,e_2),e_1]$ finally lead to:

$$\begin{aligned} h^*(y) &= h^*(y) \cdot \chi_{E^*\cap(c(e_1,e_2),e_1]}(y) + h^*(y) \cdot \chi_{E^*\setminus(c(e_1,e_2),e_1]}(y) \\ &= g(y) \cdot \chi_{E^*\cap(c(e_1,e_2),e_1]}(y) + (g(\varphi_n \circ \cdots \circ \varphi_2(.)))^*(y) \cdot \chi_{E^*\setminus(c(e_1,e_2),e_1]}(y) \\ &= g(y) \cdot \chi_{E^*\cap(c(e_1,e_2),e_1]}(y) + g(y) \cdot \chi_{E^*\setminus(c(e_1,e_2),e_1]}(y) \\ &= g(y), \end{aligned}$$

where we have used in the third equality Lemma 3.8 and the hypothesis of induction. □

The last result of this section is connected to the question equipped with equality (9).

THEOREM 3.11. *If g is a positive function in T such that $(g \circ \varphi)^* = g$, in the support of φ, for all $\varphi \in \Phi$, then*

$$\sum_{x\in T} f^*(x)g(x) = \sup_{\{h:\, h^*=g\}} \sum_{x\in T} |f(x)h(x)|\,,$$

for all measurable functions f.

Proof. Set $C = \sum_{x\in T} f^*(x)g(x)$. It is enough to see that for all $\varepsilon > 0$, there exists a function h such that $h^* = g$ and

$$C - \varepsilon < \sum_{x\in T} |f(x)h(x)|\,.$$

Set $E_k = \{x \in T : |x| \leqslant k\}$ and $f_k = |f| \cdot \chi_{E_k}$. By [GS, Proposition 23 $(viii)$], we know that

$$f_k^* \nearrow f^*,$$

since $f_k \nearrow |f|$. By the Monotone Convergence Theorem, and observing that $E_k^* = E_k$ for all $k \geqslant 0$, there exists n such that

$$C - \varepsilon < \sum_{x\in E_n} f_n^*(x)g(x). \tag{16}$$

By hypothesis, we know that $h_n = g \circ \varphi_{f_n}$ satisfies $h_n^* = g \cdot \chi_{E_n}$, and

$$\sum_{x\in E_n} f_n(x)h_n(x) = \sum_{x\in E_n} f_n^*(x)g(x),$$

because trivially

$$\sum_{x\in E_n} f_n^*(x)g(x) = \sum_{y\in E_n} f_n^*(\varphi_{f_n}(y))g(\varphi_{f_n}(y)) = \sum_{y\in E_n} f_n(y)h_n(y).$$

This equality and (16) lead to

$$C - \varepsilon < \sum_{x \in E_n} f_n(x) h_n(x).$$

Define $h = h_n \cdot \chi_{E_n} + g \cdot \chi_{T \setminus E_n}$. We claim that $h^* = g$, and using the last inequality, we get

$$C - \varepsilon < \sum_{x \in E_n} f_n(x) h_n(x) \leqslant \sum_{x \in T} f(x) h(x),$$

as we wanted to prove. We now prove the claim. Since we know that

$$h^* = \lim_k (h \cdot \chi_{E_k})^*,$$

it is enough to prove that $(h \cdot \chi_{E_m})^* = g \cdot \chi_{E_m}$, for all $m \geqslant n$. We denote $\partial E_n = T_n = \{e_1, e_2, \dots, e_r\}_\sigma$, with $r = (q+1)q^{n-1}$. If we decompose

$$\varphi_{f_n} = \varphi_r \circ \varphi_{r-1} \circ \cdots \circ \varphi_1,$$

then $(g \circ \varphi_{f_n})_k^{\triangle}(e_k) = g(e_k)$, because we know that $(g \circ \varphi_{f_n})^* = g$ and $\varphi_j = \mathrm{Id}$ in $\mathcal{R}_{j-1}(E) \setminus [o, e_j]$ for all j, and in our case $e_k \in \mathcal{R}_{j-1}(E) \setminus [o, e_j]$ for all $j \geqslant k+1$. Now, to finish the proof, we only need to observe that g is a decreasing function because of the hypothesis, and to consider the following trivial fact at each geodesic rearrangement: let $\{a_1, \dots, a_k\}$ be a sequence of positive real numbers and $\{a_1^*, \dots, a_k^*\}$ its decreasing rearrangement; let us add some new values $\{b_1, \dots, b_m\}$ to the sequence satisfying $b_i \geqslant b_{i+1}$, for all $1 \leqslant i \leqslant m$ and $b_1 \leqslant a_k^*$. Then, the rearrangement of $\{a_1, \dots, a_k, b_1, \dots, b_m\}$ is $\{a_1^*, \dots, a_k^*, b_1, \dots, b_m\}$. □

As a consequence of the last two results, we can state the following corollary:

COROLLARY 3.12. *If g is a positive linearly decreasing function, then*

$$\sum_{x \in T} f^*(x) g(x) = \sup_{\{h:\, h^* = g\}} \sum_{x \in T} |f(x) h(x)|,$$

for all measurable functions f.

We will show in Theorem 4.9 that the converse also holds.

4. The Lorentz spaces

We consider now the study of the functional properties of the Lorentz spaces on the tree, defined in terms of the new rearrangement. Our goal is to characterize the normability of the defining functional (see Theorem 4.9).

DEFINITION 4.1. Let $0 < p < \infty$ be a real number and u a positive function defined in T, that is, a weight. The Lorentz space $\Delta_T^p(u)$ is the set of measurable functions f defined in T such that the functional

$$\|f\|_{\Delta_T^p(u)} = \left(\sum_{x \in T} f^*(x)^p \, u(x) \right)^{1/p}$$

is finite.

We observe that the simple functions with finite support are in $\Delta_T^p(u)$. If $u \in L^1(T)$, then $L^\infty(T) \subset \Delta_T^p(u)$ and all simple functions are in $\Delta_T^p(u)$. We give some basic properties, with trivial proof derived from [GS, Proposition 23].

PROPOSITION 4.2. *For measurable functions f, g and f_k, $k \geqslant 1$, defined in T, we have:*

(i) If $|f| \leqslant |g|$, then $\|f\|_{\Delta_T^p(u)} \leqslant \|g\|_{\Delta_T^p(u)}$.

(ii) $\|\lambda f\|_{\Delta_T^p(u)} = |\lambda|\, \|f\|_{\Delta_T^p(u)}$.

(iii) If $0 \leqslant f_k \nearrow f$ pointwise, then $\|f_k\|_{\Delta_T^p(u)} \nearrow \|f\|_{\Delta_T^p(u)}$.

(iv) $\|\liminf_k |f_k|\|_{\Delta_T^p(u)} \leqslant \liminf_k \|f_k\|_{\Delta_T^p(u)}$.

We will use the notation $U(E) = \sum_{x \in E} u(x)$, for every set $E \subset T$ and for every weight u in T. We describe the functional in a new way that will be useful later on.

LEMMA 4.3. *For all $f \in \Delta_T^p(u)$, we have*

$$\|f\|_{\Delta_T^p(u)} = \left(\int_0^\infty p\, \lambda^{p-1} U(\{|f| > \lambda\}^*)\, d\lambda \right)^{1/p}.$$

Proof. By [GS, Proposition 23 (vii)], we have:

$$\|f\|_{\Delta_T^p(u)} = \left(\sum_{x \in T} (|f|^p)^*(x)) u(x) \right)^{1/p}.$$

We use the definition (4) of the decreasing rearrangement and then we apply Fubini's Theorem obtaining:

$$\begin{aligned} \|f\|_{\Delta_T^p(u)} &= \left(\sum_{x \in T} \left(\int_0^\infty \chi_{\{|f|^p > \lambda\}^*}(x)\, d\lambda \right) u(x) \right)^{1/p} \\ &= \left(\sum_{x \in T} \left(\int_0^\infty p\, \xi^{p-1} \chi_{\{|f| > \xi\}^*}(x)\, d\xi \right) u(x) \right)^{1/p} \\ &= \left(\int_0^\infty p\, \xi^{p-1} \left(\sum_{x \in \{|f| > \xi\}^*} u(x) \right) d\xi \right)^{1/p}. \qquad \square \end{aligned}$$

Our Lorentz spaces have the property of completeness. The proof is standard and is omitted.

PROPOSITION 4.4. *Suppose $u(o) \neq 0$, and let $\{f_k : k \geqslant 0\}$ be a sequence of measurable functions defined in T. If*

$$\lim_{m,n} \|f_m - f_n\|_{\Delta_T^p(u)} = 0,$$

then there exists a function $f \in \Delta_T^p(u)$ defined in T such that

$$\lim_n \|f - f_n\|_{\Delta_T^p(u)} = 0.$$

The classical Lorentz spaces are generalizations of the classical Lebesgue spaces, in the sense that $\Lambda_X^p(1) = L^p(X,\mu)$. In view of this, it is logical to ask if this relation holds true in the case of our Lorentz spaces. Next proposition gives an answer to this question.

PROPOSITION 4.5. *For $0<p<\infty$, we have $\Delta_T^p(1) = L^p(T,|.|)$.*

Proof. Using [GS, Proposition 23 (vi), (vii)], that $|E| = |E^*|$ for all $E \subset T$, and Fubini's Theorem, we get

$$\begin{aligned} \|f\|_{L^p(T,|.|)} &= \sum_{x\in T} |f(x)|^p = \int_0^\infty \left|\{|f|^p > \lambda\}\right| d\lambda = \int_0^\infty |\{(f^*)^p > \lambda\}|\, d\lambda \\ &= \sum_{x\in T} (f^*(x))^p = \|f\|_{\Delta_T^p(1)}. \qquad \square \end{aligned}$$

As a consequence we have $\Lambda_T^p(1) = \Delta_T^p(1)$. However, the spaces $\Lambda_T^p(v)$ (v a weight in $[0,\infty)$) and $\Delta_T^p(u)$ (u a weight in T) are not equal in general. The classical Lorentz spaces are *rearrangement invariant spaces*, that is $\|f\|_{\Lambda_X^p(u)} = \|g\|_{\Lambda_X^p(u)}$, whenever f and g are *equimeasurable* functions, in the sense that $\mu(\{|f| > \lambda\}) = \mu(\{|g| > \lambda\})$, for all $\lambda > 0$. In fact, two functions f and g are equimeasurable if and only if $f^\star = g^\star$. The Lorentz spaces $\Delta_T^p(u)$ are not rearrangement invariant spaces in this sense in general, because it is easy to find f and g in T such that $f^\star = g^\star$ but $f^* \neq g^*$. Furthermore:

PROPOSITION 4.6. *The space $\Delta_T^p(u)$ is a rearrangement invariant space if and only if the weight u is constant in $T \setminus \{o\}$.*

Proof. Suppose first that u is constant in $T\setminus\{o\}$. Take two equimeasurable functions f and g in $(T,|.|)$, that is

$$|\{|f| = \lambda\}| = |\{|g| = \lambda\}|. \tag{17}$$

We can assume that their supports are finite. We have:

$$\|f\|_{\Delta_T^p(u)} = (f^*(o))^p u(o) + C\sum_{x\neq o}(f^*(x))^p,$$

$$\|g\|_{\Delta_T^p(u)} = (g^*(o))^p u(o) + C\sum_{x\neq o}(g^*(x))^p.$$

By (17), $f^*(o) = g^*(o)$ and using also Proposition 4.5, we get:

$$\begin{aligned} \sum_{x\neq o}(f^*(x))^p &= \|f\|_{L^p(T,|.|)} - (f^*(o))^p \\ &= \|g\|_{L^p(T,|.|)} - (g^*(o))^p \\ &= \sum_{x\neq o}(g^*(x))^p, \end{aligned}$$

and thus

$$\|f\|_{\Delta_T^p(u)} = \|g\|_{\Delta_T^p(u)}.$$

Let us see the converse implication. Suppose that $\Delta_T^p(u)$ is a rearrangement invariant space. We first show that necessarily, u is radial. Take x and y two different vertices such that $d(o,x) = d(o,y)$. Then $f = \chi_{[o,x]}$ and $g = \chi_{[o,y]}$ are equimeasurable functions and thus

$$U([o,x]) = \|f\|_{\Delta_T^p(u)} = \|g\|_{\Delta_T^p(u)} = U([o,y]).$$

This equality implies that u is radial. Now take $x \in T$, and y such that $y \notin [o,x]$ and $d(o,y) = 1$. Set $E = ([o,x] \setminus \{x\}) \cup \{y\}$, $f = \chi_E$ and $g = \chi_{[o,x]}$. Then f and g are equimeasurable functions satisfying $f^* = f$ and $g^* = g$, and thus

$$U([o,x]) = \|g\|_{\Delta_T^p(u)} = \|f\|_{\Delta_T^p(u)} = U([o,x]) - u(x) + u(y),$$

that is, $u(x) = u(y)$. This equality and the fact that u is radial lead to $u = C$ in $T \setminus \{o\}$. $\square$

However, we always have an inclusion between these two spaces.

PROPOSITION 4.7. *If u is a weight in T, then $\Lambda_T^p(u^\star)$ is a subspace of $\Delta_T^p(u^*)$.*

Proof. We simply apply Corollary 3.4:

$$\begin{aligned} \|f\|^p_{\Delta_T^p(u^*)} &= \sum_{x\in T} (f^*(x))^p\, u^*(x) = \sum_{x\in T} (|f|^p)^*(x)\, u^*(x) \\ &\leqslant \int_0^\infty (|f|^p)^\star(t)\, u^\star(t)\, dt = \int_0^\infty (f^\star(t))^p\, u^\star(t)\, dt \\ &= \|f\|_{\Lambda_T^p(u^\star)}. \end{aligned}$$

We have used the well-known fact $(|f|^p)^\star(x) = (f^\star(x))^p$, also valid for the new rearrangement (see [GS, Proposition 23 (vii)]). $\square$

We focus our attention on the functional $\|.\|_{\Delta_T^p(u)}$, and we study what kind of conditions are required on the weight u such that it becomes a quasi-norm or a norm. We observe that we trivially have $\|f\|_{\Delta_T^p(u)} = 0 \Leftrightarrow f \equiv 0$. In the classical context, M.J. Carro and J. Soria ([CS]) characterized the weights u such that the functional $\|.\|_{\Lambda_X^p(u)}$ in (1) is a quasi-norm, if X is non-atomic. Later, J.A. Raposo ([R] or [CRS]) completed this result for all X. In our case, we have the following characterization.

THEOREM 4.8. *The functional $\|.\|_{\Delta_T^p(u)}$ is a quasi-norm if and only if there exists a constant $C > 0$ such that*

$$0 < U((E \cup D)^*) \leqslant C(U(E^*) + U(D^*)), \tag{18}$$

for all sets E and D such that $E \cup D \neq \emptyset$.

Proof. Suppose that condition (18) holds. By Lemma 4.3, if $\|f\|_{\Delta^p_T(u)} = 0$, then

$$U(\{|f| > \lambda\}^*) = 0,$$

for all $\lambda > 0$, and by hypothesis,

$$\{|f| > \lambda\} = \emptyset,$$

for all λ, that is, $f \equiv 0$. Also by Lemma 4.3 and applying our hypothesis, we have:

$$\begin{aligned}
\|f+g\|^p_{\Delta^p_T(u)} &= \int_0^\infty p\,\lambda^{p-1} U(\{|f+g| > \lambda\}^*)\,d\lambda \\
&\leqslant \int_0^\infty p\,\lambda^{p-1} U((\{|f| > \lambda/2\} \cup \{|g| > \lambda/2\})^*)\,d\lambda \\
&\leqslant C\Big(\int_0^\infty p\,\lambda^{p-1} U(\{|f| > \lambda/2\}^*)\,d\lambda + \int_0^\infty p\,\lambda^{p-1} U(\{|g| > \lambda/2\}^*)\,d\lambda\Big) \\
&= 2C\Big(\int_0^\infty p\,\lambda^{p-1} U(\{|f| > \lambda\}^*)\,d\lambda + \int_0^\infty p\,\lambda^{p-1} U(\{|g| > \lambda\}^*)\,d\lambda\Big) \\
&= 2C(\|f\|^p_{\Delta^p_T(u)} + \|g\|_{\Delta^p_T(u)})^p,
\end{aligned}$$

where we have used the monotonic property $E^* \subset D^*$ if $E \subset D$ (see [GS, Proposition 19]). Now, suppose that the functional is a quasi-norm. Take E and D such that $E \cup D \neq \emptyset$. Then,

$$\begin{aligned}
U((E\cup D)^*)^{1/p} &= \|\chi_{E\cup D}\|_{\Delta^p_T(u)} \leqslant C(\|\chi_E\|_{\Delta^p_T(u)} + \|\chi_D\|_{\Delta^p_T(u)}) \\
&= C(U(E^*)^{1/p} + U(D^*)^{1/p}). \qquad \square
\end{aligned}$$

We study now when the functional $\|.\|_{\Delta^p_T(u)}$ is a norm. In the classical context, this problem was solved by G.G. Lorentz ([Lo]) in the case $X = (0,l)$ and $p \geqslant 1$. The general case is completely characterized by J.A. Raposo ([R] or the bookmanuscript [CRS]). In all cases, the necessary and sufficient condition on the weight so that $\|.\|_{\Lambda^p_X(u)}$ becomes a norm is that u must be a decreasing function. It is not difficult to see that, in our context, the weight u must be also a decreasing function on the tree in order to get a norm. But a very simple example shows that this condition on u is not enough. So, it seems natural to think that u must be something better than decreasing. In view of the results of the previous section, we guess that u has to be a linearly decreasing function. The positive answer is given in the next theorem. Recall that Φ is the set of all the rearranging transformations in the tree.

THEOREM 4.9. *Let u be a weight in T.*

(i) If $0 < p < 1$, the functional $\|.\|_{\Delta^p_T(u)}$ is a norm if and only if $\operatorname{supp}(u) = \{o\}$.

(ii) If $p \geqslant 1$, the following are equivalent:

(a) u is linearly decreasing in T.

(b) For all $\varphi \in \Phi$, the equality

$$(u \circ \varphi)^*(y) = u(y)$$

holds for all y in the support of φ.

(c) For all measurable functions f in T, the equality

$$\sup_{\{h:\, h^*=u\}} \sum_{x\in T} |f(x)h(x)| = \sum_{x\in T} f^*(x)u(x)$$

holds.

(d) The functional $\|.\|_{\Delta_T^p(u)}$ is a norm.

Proof.

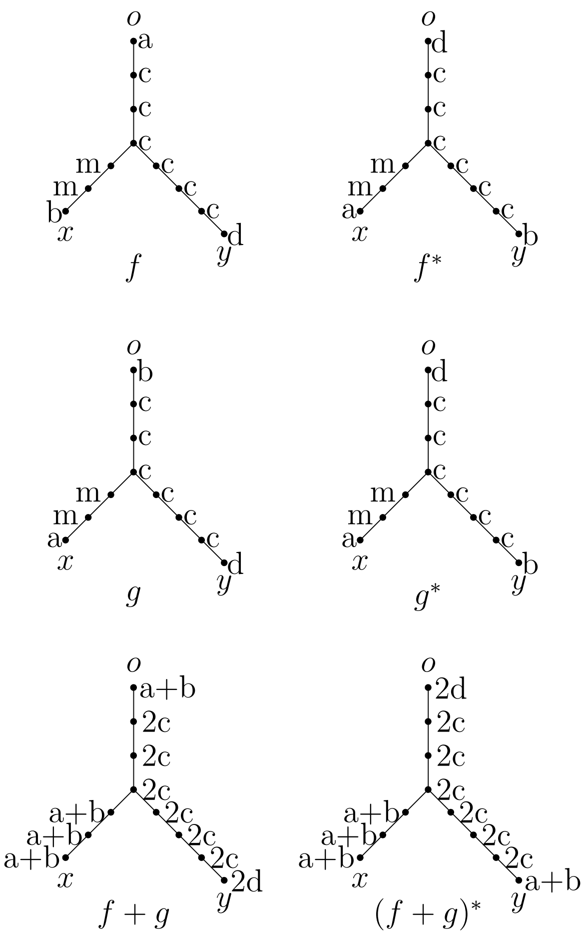

Figure 3: The functions f, g, $f+g$ and their rearrangements.

We first prove that if $\|.\|_{\Delta^p_T(u)}$ is a norm, then u is linearly decreasing, for all $0 < p < \infty$. If $x \geqslant y$, it is well-known that in this “linear" case we have that $u(x) \leqslant u(y)$ (see [Lo] or [R]). Take $x \trianglerighteq y$ and $0 < a < b < c < d$, set $2m = a+b$ and consider the functions f and g of Figure 3. Observe that $f^* = g^*$. The triangle inequality gives, after cancelations

$$((a+b)^p - (2a)^p)\, u(x) \leqslant ((2b)^p - (a+b)^p)\, u(y).$$

If we set $1 < \lambda = b/a$, the inequality becomes

$$((1+\lambda)^p - 2^p)\, u(x) \leqslant ((2\lambda)^p - (1+\lambda)^p)\, u(y).$$

Now, observe that

$$\lim_{\lambda \to 1} \frac{(2\lambda)^p - (1+\lambda)^p}{(1+\lambda)^p - 2^p} = 1,$$

and thus, $u(x) \leqslant u(y)$. This proves $(d) \Rightarrow (a)$ in part (ii).

(i) The sufficiency is obvious. Suppose that the functional is a norm. Set $f = \chi_{\{o\}}$ and $g = \lambda \chi_{\{x\}}$ with $0 < \lambda < 1$ and x a neighbor vertex of o such that $x \trianglelefteq y$ for all $y \neq 0$ in T. Then $f^* = f$, $g^* = \lambda \chi_{\{o\}}$ and $(f+g)^* = f + g = \chi_{\{o\}} + \lambda \chi_{\{x\}}$. The triangle inequality gives

$$\|f+g\|_{\Delta^p_T(u)} = \Big(u(o) + \lambda^p u(x)\Big)^{1/p} \leqslant u(o)^{1/p} + (\lambda^p u(o))^{1/p} = \|f\|_{\Delta^p_T(u)} + \|g\|_{\Delta^p_T(u)}.$$

From this, we have

$$\frac{\Big(u(o) + \lambda^p u(x)\Big)^{1/p} - u(o)^{1/p}}{\lambda} \leqslant u(o)^{1/p} < \infty.$$

If $u(x) \neq 0$, then

$$\lim_{\lambda \to 0} \frac{\Big(u(o) + \lambda^p u(x)\Big)^{1/p} - u(o)^{1/p}}{\lambda} = \infty,$$

because $p < 1$, getting a contradiction. Thus, $u(x) = 0$ and since u is linearly decreasing, $u = u(o)\chi_{\{o\}}$.

(ii) $(a) \Rightarrow (b)$ This is Theorem 3.10.

$(b) \Rightarrow (c)$ This is Theorem 3.11.
$(c) \Rightarrow (d)$ We apply [GS, Proposition 23 (vii)], and the hypothesis (twice):

$$\begin{aligned}
\|f+g\|_{\Delta_T^p(u)} &= \left(\sum_{x\in T}(f+g)^*(x)^p u(x)\right)^{1/p} \\
&= \left(\sum_{x\in T}(|f+g|^p)^*(x)u(x)\right)^{1/p} \\
&= \sup_{\{h:\, h^*=u\}}\left(\sum_{x\in T}|f(x)+g(x)|^p\, h(x)\right)^{1/p} \\
&\leqslant \sup_{\{h:\, h^*=u\}}\left(\sum_{x\in T}|f(x)|^p\, h(x)\right)^{1/p} + \sup_{\{h:\, h^*=u\}}\left(\sum_{x\in T}|g(x)|^p\, h(x)\right)^{1/p} \\
&= \left(\sum_{x\in T}(f^*(x))^p u(x)\right)^{1/p} + \left(\sum_{x\in T}(g^*(x))^p u(x)\right)^{1/p} \\
&= \|f\|_{\Delta_T^p(u)} + \|g\|_{\Delta_T^p(u)}. \qquad \square
\end{aligned}$$

Taking into account Propositions 4.4 and 4.7, we obtain:

COROLLARY 4.10. *For* $1 \leqslant p < \infty$,
(i) $\Delta_T^p(u)$ *is a Banach space if and only if* u *is linearly decreasing in* T.
(ii) If u *is a linearly decreasing weight in* T, $\Lambda_T^p(u^\star)$ *is a proper Banach subspace of the Banach space* $\Delta_T^p(u)$.

In view of Theorem 4.9, it is now clear that the property of being Banach depends on the choice of the origin and the order of the rearrangement, since the linearly decreasing property on a weight is not invariant under the change of either the origin or the order in ∂T_o.

REMARK 4.11. It is possible to extend these results to the cases of finite or regular trees. A tree is finite if the number of edges is finite, and it is regular if, for every vertex, the number of neighbor vertices is greater than or equal to 2, and is bounded above by an absolute constant. To handle these cases, the main idea is to embed the tree into a suitable homogeneous tree and to rearrange the embedded sets of vertices.

In the case of a finite rooted tree, it is easy to choose an order in ∂T_o simply by listing the boundary vertices (without considering the compatibility conditions on the boundary of the tree of [GS, Definition 2]). So, we can use this listing order to rearrange sets and, therefore, functions. A natural question arises: can we have the same previous theorem for this rearrangement? If the listing order coincides with an admissible order, the result remains true. But it is easy to see that there are listing orders that are not admissible, for which Theorem 4.9 fails to be true, as stated.

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(Received November 5, 2003)

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