Mathematical
Inequalities
& Applications
Volume 7, Number 4 (2004), 491-496

A NOTE ON BOUNDS FOR NORMS OF THE RECIPROCAL LCM MATRIX

Ercan Altinisik, Naim Tuglu and Pentti Haukkanen

(*communicated by J. Sándor*)

Abstract. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers and $[x_i, x_j]$ denote the least common multiple of x_i and x_j. The matrix $[S^{-1}] = (s_{ij})$, where $s_{ij} = \frac{1}{[x_i, x_j]}$, is called the reciprocal least common multiple (reciprocal LCM) matrix on S. In this paper, we investigate some matrix norms of the reciprocal LCM matrix and one of its generalizations on $S = \{1, 2, \ldots, n\}$ in terms of the Riemann zeta function.

1. Introduction

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers and (x_i, x_j) denote the greatest common divisor of x_i and x_j. The matrix $(S) = (s_{ij})$, where $s_{ij} = (x_i, x_j)$, is called the *greatest common divisor* (GCD) *matrix* on S [4].

Beslin and Ligh [4] showed that the GCD matrix is positive definite and

$$\det(S) = \phi(x_1)\phi(x_2)\ldots\phi(x_n),$$

where ϕ is Euler's totient, if S is factor-closed. S is said to be *factor-closed* if all positive divisors of every element of S is in S.

Z. Li [10] calculated the value of the determinant of the GCD matrix on S in terms of Euler's totient when S arbitrary.

Beslin and Ligh [5] proved that $\det(S) = B(x_1)B(x_2)\ldots B(x_n)$, where B is defined as

$$B(x_i) = \sum_{\substack{d \mid x_i \\ d \nmid x_j \\ x_j < x_i}} \phi(d)$$

on S when S is gcd-closed. $S = \{x_1, x_2, \ldots, x_n\}$ is said to be *gcd-closed* if $(x_i, x_j) \in S$ for all $x_i, x_j \in S$ [5, 6].

The $n \times n$ matrix $[S] = (s_{ij})$, where $s_{ij} = [x_i, x_j]$, the least common multiple of x_i and x_j, is called *the least common multiple* (LCM) *matrix* on S [3]. In that paper, Beslin defined the matrix $(S^{-1}) = (s_{ij})$, where $s_{ij} = \frac{1}{(x_i, x_j)}$, and called it the reciprocal

Mathematics subject classification (2000): 11C20, 15A36, 15A60, 11A25.
Key words and phrases: GCD matrices, LCM matrices, matrix norms, Riemann zeta function.

Paper MIA-07-48

GCD matrix on S. Beslin also investigated the structure of the LCM matrix on S and showed that the determinant of the LCM matrix is

$$x_1^2\, x_2^2 \dots\, x_n^2\, g(x_1)g(x_2)\dots g(x_n),$$

where the arithmetic function g is defined as $g(n) = \frac{1}{n}\sum_{d|n} d\mu(d)$, if S is factor closed.

The inverses of GCD and LCM matrices on factor-closed sets and the inverses of GCD matrices on gcd-closed sets are calculated in [6]. The inverses of LCM matrices on gcd-closed sets are calculated in [9].

Haukkanen, Wang and Sillanpää gave a brief review of papers relating to GCD and LCM matrices (see [8]).

In [1] E. Altinisik and D. Tasci presented a generalization of the reciprocal LCM matrix, and gave also some inequalities on the determinant of such matrices (see Theorem 4 in [1]). D. Bozkurt and S. Solak [7] have shown that the Euclidean norm of the reciprocal LCM matrix $[S^{-1}]$ on $S = \{1,2,\dots,n\}$ possesses the upper bound

$$\left\|[S^{-1}]\right\|_2 \leqslant \sqrt{\frac{5\pi^2}{6} - 4}.$$

The main purpose of this paper is to improve this upper bound. We apply a technique different from that by D. Bozkurt and S. Solak [7] to obtain a sharper upper bound given as

$$\left\|[S^{-1}]\right\|_2 \leqslant \frac{\sqrt{15}}{6}\pi.$$

We, in fact, obtain a more general result which gives an upper bound for the ℓ_p norm of the matrix $[S^{-r}]$ on $S = \{1,2,\dots,n\}$ with $rp > 1$.

2. The main results

DEFINITION 1. Let $S = \{x_1, x_2, \dots, x_n\}$ be a set of distinct positive integers. The $n \times n$ matrix $[S^{-1}] = \left(\frac{1}{[x_i, x_j]}\right)$ is called the reciprocal least common multiple (reciprocal LCM) matrix on S.

Throughout this paper, $[S^{-1}]$ denotes the reciprocal LCM matrix defined on the set $S = \{1,2,\dots,n\}$. Let r be a positive real number. The matrix $[S^{-r}] = \left(\frac{1}{[i,j]^r}\right)$ defined on $S = \{1,2,\dots,n\}$ is a generalization of the reciprocal LCM matrix. Now we investigate the ℓ_p norm of $[S^{-r}]$.

DEFINITION 2. Let $A = (a_{ij})_{n\times n}$ be in $M_n(\mathbb{C})$. The ℓ_p norm of A is

$$\|A\|_p = \left(\sum_{i,j=1}^{n} |a_{ij}|^p\right)^{1/p}, \quad (1 \leqslant p < \infty).$$

THEOREM 3. *Let* $[S^{-r}]$ *be a generalization of the reciprocal LCM matrix and* $rp > 1$. *Then*

$$\lim_{n\to\infty} \left\|[S^{-r}]\right\|_p = \frac{\zeta(rp)^{3/p}}{\zeta(2rp)^{1/p}},$$

where $\zeta(s)$ *is the Riemann zeta function.*

Proof. The ℓ_p norm of $[S^{-r}]$ is

$$\left\|[S^{-r}]\right\|_p = \left(\sum_{i,j=1}^{n} \frac{1}{[i,j]^{rp}}\right)^{1/p}.$$

Then

$$\lim_{n\to\infty} \left\|[S^{-r}]\right\|_p^p = \sum_{i,j=1}^{\infty} \frac{1}{[i,j]^{rp}} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{(i,j)^{rp}}{i^{rp}j^{rp}}.$$

For any positive integer m, it is well known that $m^r = \sum_{d|m} J_r(d)$, where J_r is Jordan's totient (see [2],[11]). Then

$$\begin{aligned}
\lim_{n\to\infty} \left\|[S^{-r}]\right\|_p^p &= \sum_{i=1}^{\infty} \frac{1}{i^{rp}} \sum_{j=1}^{\infty} \frac{1}{j^{rp}} \sum_{d|(i,j)} J_{rp}(d) \\
&= \sum_{i=1}^{\infty} \frac{1}{i^{rp}} \sum_{d|i} \frac{J_{rp}(d)}{d^{rp}} \sum_{k=1}^{\infty} \frac{1}{k^{rp}} \\
&= \zeta(rp) \sum_{i=1}^{\infty} \frac{1}{i^{rp}} \sum_{d|i} \frac{J_{rp}(d)}{d^{rp}} \\
&= \zeta(rp) \sum_{i=1}^{\infty} \frac{J_{rp}(i)}{i^{rp}} \sum_{k=1}^{\infty} \frac{1}{(ik)^{rp}} \\
&= \zeta(rp)^2 \sum_{i=1}^{\infty} \frac{J_{rp}(i)}{i^{2rp}}.
\end{aligned}$$

By using the relation $J_{rp}(i) = i^{rp} \sum_{d|i} \frac{\mu(d)}{d^{rp}}$, we have

$$\begin{aligned}
\lim_{n\to\infty} \left\|[S^{-r}]\right\|_p^p &= \zeta(rp)^2 \sum_{i=1}^{\infty} \frac{1}{i^{rp}} \sum_{d|i} \frac{\mu(d)}{d^{rp}} \\
&= \zeta(rp)^2 \sum_{i=1}^{\infty} \frac{\mu(i)}{i^{rp}} \sum_{k=1}^{\infty} \frac{1}{(ik)^{rp}} \\
&= \zeta(rp)^3 \sum_{i=1}^{\infty} \frac{\mu(i)}{i^{2rp}} \\
&= \frac{\zeta(rp)^3}{\zeta(2rp)}.
\end{aligned}$$

Thus

$$\lim_{n\to\infty} \left\| [S^{-r}] \right\|_p = \frac{\zeta(rp)^{3/p}}{\zeta(2rp)^{1/p}}.$$

Since $rp > 1$ it is clear that $\zeta(rp)$ and $\zeta(2rp)$ are covergent by the definition of the Riemann zeta function. □

REMARK 4. By Theorem 3 we obtain the least upper bound for the ℓ_p norm of the matrix $[S^{-r}]$ that is

$$\left\| [S^{-r}] \right\|_p \leqslant \frac{\zeta(rp)^{3/p}}{\zeta(2rp)^{1/p}},$$

where $rp > 1$.

We compare our result given in Theorem 3 with computer calculations by using Maple 6 for the ℓ_p norm of the matrix $[S^{-r}]$ for $p = 5$ and $r = 10$.

n	$\frac{\zeta(rp)^{3/p}}{\zeta(2rp)^{1/p}} - \left\| [S^{-r}] \right\|_p$
2	0.83577×10^{-24}
3	0.78886×10^{-30}
5	0.17324×10^{-38}
10	0.53541×10^{-52}
20	0.11980×10^{-65}
50	0.10720×10^{-84}
100	0.25595×10^{-99}

Table 1

COROLLARY 5. *Let* $[S^{-1}]$ *be the reciprocal LCM matrix. Then*

$$\lim_{n\to\infty} \left\| [S^{-1}] \right\|_2 = \frac{\zeta(2)^{3/2}}{\zeta(4)^{1/2}} = \frac{\sqrt{15}}{6}\pi,$$

where $\zeta(s)$ *is the Riemann zeta function.*

Proof. By letting $p = 2$ and $r = 1$ in Theorem 3, the proof is immediate. □

We compare our result in Corollary 5 with computer calculations by using Maple 6 in Table 2.

n	$\left\|[S^{-1}]\right\|_2$	$\frac{\zeta(2)^{3/2}}{\zeta(4)^{1/2}} = \frac{\sqrt{15}\pi}{6}$
2	1.322875656	2.027889338
3	1.462494065	"
4	1.570120307	"
5	1.618383968	"
10	1.785512467	"
30	1.923525230	"
50	1.958073310	"
100	1.988504622	"
200	2.005863321	"
300	2.012307249	"
400	2.015714494	"
500	2.017840069	"
600	2.017853121	"

Table 2

The following corollary gives us an upper bound for the spectral norm of $[S^{-r}]$.

DEFINITION 6. Let $A = (a_{ij})_{n\times n}$ be in $M_n(\mathbb{C})$. The spectral norm of A is

$$\|A\|_S = \max\left\{\sqrt{\lambda} : \lambda \text{ is an eigenvalue of } A^*A\right\}.$$

COROLLARY 7. *Let $[S^{-r}]$ be a generaliztion of the reciprocal LCM matrix and $2r > 1$. Then*

$$\left\|[S^{-r}]\right\|_S \leqslant \frac{\zeta(2r)^{3/2}}{\zeta(4r)^{1/2}}.$$

Proof. By Theorem 3 we have

$$\left\|[S^{-r}]\right\|_p \leqslant \frac{\zeta(rp)^{3/p}}{\zeta(2rp)^{1/p}}.$$

Letting $p = 2$ we have

$$\left\|[S^{-r}]\right\|_2 \leqslant \frac{\zeta(2r)^{3/2}}{\zeta(4r)^{1/2}}.$$

Since $\|A\|_S \leqslant \|A\|_2$ for any matrix $A = (a_{ij})_{n\times n}$ in $M_n(\mathbb{C})$ we obtain

$$\left\|\,[S^{-r}]\,\right\|_S \leqslant \frac{\zeta(2r)^{3/2}}{\zeta(4r)^{1/2}}. \qquad \square$$

3. Discussion

The matrix norms of GCD, LCM and their generalizations have not hitherto been studied in the literature by using some tools of number theory. We initiated the study of matrix norms of GCD and LCM matrices in a different sense. We also present an open problem on the norms of GCD and LCM matrices.

PROBLEM 8. Can values of matrix norms of GCD and LCM matrices be obtained in terms of arithmetical functions?

REFERENCES

[1] E. ALTINISIK AND D. TASCI, *On a generalization of the reciprocal lcm matrix*, Commn. Fac. Sci. Ank. Series A1 Mat. Stat. **51** (2) 37–46 (2002).
[2] T. M. APOSTOL, *Introduction to Analytic Number Theory. 1st Ed.*, New York, Springer-Verlag, 1976.
[3] S. BESLIN, *Reciprocal GCD Matrices and LCM Matrices*, Fibonacci Quart., **29** 271–274 (1991).
[4] S. BESLIN AND S. LIGH, *Greatest Common Divisor Matrices*, Linear Algebra Appl., **118** 69–76 (1989).
[5] S. BESLIN AND S. LIGH, *Another generalization of Smith's determinant*, Bull. Aust. Math. Soc. **40** (3) 413–415 (1989).
[6] K. BOURGUE AND S. LIGH, *On GCD and LCM Matrices*, Linear Algebra Appl., **174** 65–74 (1992).
[7] D. BOZKURT AND S. SOLAK, *On the norms of GCD matrices*, Math. Comput. Appl. **7** (3) 205–210 (2002).
[8] P. HAUKKANEN, J. WANG, AND J. SILLANPÄÄ, *On Smith's Determinant*, Linear Algebra Appl., **258** 251–269 (1997).
[9] I. KORKEE AND P. HAUKKANEN, *On Meet and Join Matrices Associated with Incidence Functions*, Linear Algebra Appl., **372** 127–153 (2003).
[10] Z. LI, *The Determinants of GCD Matrices*, Linear Algebra Appl., **134** 137–143 (1990).
[11] P. J. MCCARTHY, *Introduction to Arithmetical Functions*, New York, Springer-Verlag, 1986.

(Received April 14, 2003)

Ercan Altinisik
Gazi University
Faculty of Arts and Sciences
Department of Mathematics
06500 Teknikokullar – Ankara
Turkey
e-mail: altinisikercan@hotmail.com

Naim Tuglu
Gazi University
Faculty of Arts and Sciences
Department of Mathematics
06500 Teknikokullar – Ankara
Turkey
e-mail: naimtuglu@yahoo.com

Pentti Haukkanen
Department of Mathematics
University of Tampere
FIN-33014 Tampere
Finland
e-mail: mapehau@uta.fi