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JENSEN'S INEQUALITIES WITH MULTIPLE VARIABLES ON TIME SCALES

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Abstract. The Jensen inequality is of great interest in differential and difference equations, and other areas of mathematics. The purpose of this paper is to generalize the Jensen inequality to more general cases on time scales as follows:

$$F\left(\frac{\int_a^b |h(t)|g_1(t)\Delta t}{\int_a^b |h(t)|\Delta t},\dots,\frac{\int_a^b |h(t)|g_n(t)\Delta t}{\int_a^b |h(t)|\Delta t}\right) \leqslant \frac{\int_a^b |h(t)|F(g_1(t),\cdots,g_n(t))\Delta t}{\int_a^b |h(t)|\Delta t}.$$

1. Introduction

The Jensen inequality $[1,2,3,5,6,7]$ is of great interest in differential and difference equations, and other areas of mathematics. The original Jensen inequality is as follows:

THEOREM 1.A. *Let* $g \in C([a,b],(c,d))$ *and* $F \in C((c,d),\mathbb{R})$ *be convex, then*

$$F\left(\frac{\int_a^b g(t)dt}{b-a}\right) \leqslant \frac{\int_a^b F(g(t))dt}{b-a}.$$

There are many authors dealing with this renowned inequality, see, for example, Agarwal etc $[1]$, Beckenbach and Walter $[2]$, Fleming $[5]$ and Horn and Johnson $[7]$.

In 2001 Bohner and Peterson $[3]$ extended Theorem 1.A on a time scale and obtained the following:

THEOREM 1.B. *Let* $g \in Crd([a,b],(c,d))$ *and* $F \in C((c,d),\mathbb{R})$ *be convex, then*

$$F\left(\frac{\int_a^b g(t)\Delta t}{b-a}\right) \leqslant \frac{\int_a^b F(g(t))\Delta t}{b-a}.$$

The purpose of this paper is to generalize Jensen's inequality to more general cases on time scales.

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Now, we briefly introduce the time scales theory and refer to Agarwal etc [1], Bohner and Peterson [3] and Kaymakcalan [8] for further details.

DEFINITION 1.C. Let $\mathbb{T}$ be a closed subset of the real numbers $\mathbb{R}$ with the property that

$$\sigma(t) := \inf\{\tau > t : \tau \in \mathbb{T}\} \in \mathbb{T},$$

and

$$\rho(t) := \sup\{\tau < t : \tau \in \mathbb{T}\} \in \mathbb{T},$$

for all $t \in \mathbb{T}$. We assume thoughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. If $\sigma(t) > t$, we say t is right scattered, while if $\rho(t) < t$ we say t is left scattered. If $\sigma(t) = t$ we say t is right dense, while if $\rho(t) = t$ we say t is left dense.

Thoughout this paper we make the blanket assumption that $a \leqslant b$ are points in $\mathbb{T}$.

DEFINITION 1.D. Define the interval in $\mathbb{T}$ by

$$[a,b] := \{t \in \mathbb{T} \quad \text{such that} \quad a \leqslant t \leqslant b\}.$$

Other types of intervals are defined similarly.

DEFINITION 1.E. Assume $x : \mathbb{T} \to \mathbb{R}$ and fix $t \in \mathbb{T}$, then we define $x^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$|[x(\sigma(t)) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]| < \epsilon|\sigma(t) - s|,$$

for all $s \in U$. We call $x^{\Delta}(t)$ the **delta derivative** of $x(t)$ at $t \in \mathbb{T}$ and $x(t)$ delta differentiable at t. If $x(t)$ is delta diffeerentiable at every point of $\mathbb{T}$, then we say $x(t)$ is delta differentiable at $\mathbb{T}$.

It can be shown that if $x : \mathbb{T} \to \mathbb{R}$ is continuous at $t \in \mathbb{T}$ and t is right scattered, then

$$x^{\Delta}(t) = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t}.$$

DEFINITION 1.F. A function $F : \mathbb{T}^k \to \mathbb{R}$ is an antiderivative of $f : \mathbb{T} \to \mathbb{R}$, if $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^k$. We define the integral of f by

$$\int_s^t f(\tau)\ \Delta\tau = F(t) - F(s)$$

for $s, t \in \mathbb{T}^k$.

DEFINITION 1.G. If $f : \mathbb{T} \to \mathbb{R}$, then the function $f^{\sigma} : \mathbb{T} \to \mathbb{R}$ is defined by

$$f^{\sigma}(t) = f(\sigma(t))$$

for all $t \in \mathbb{T}$.

DEFINITION 1.H. A mapping $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it satisfies:

(A) f is continuous at each right-dense or maximal element $t \in \mathbb{T}$ and
(B) if the left-side limit $\lim_{s \to t^-} f(s) = f(t^-)$ exists at each left-dense point $t \in \mathbb{T}$.

In this paper, we denote

$$C_{rd}(\mathbb{T}, \mathbb{R}) = \{f \mid f : \mathbb{T} \to \mathbb{R} \text{ is a rd-continuous function}\}.$$

2. Main Results

In order to treat our main results, we need the following:

DEFINITION 2.A. $S \subseteq \mathbb{R}^n$ is a convex subset of $\mathbb{R}$ if it satisfies:

$$\lambda \vec{x} + (1-\lambda)\vec{y} \in S \quad \text{for all} \quad \vec{x}, \vec{y} \in S \quad \text{and} \quad \lambda \in [0,1].$$

DEFINITION 2.B. Suppose that $S \subseteq \mathbb{R}^n$. $F : S \to \mathbb{R}$ is a convex function on S if it satisfies:

$$F(\lambda \vec{x} + (1-\lambda)\vec{y}) \leqslant \lambda F(\vec{x}) + (1-\lambda)F(\vec{y}) \quad \text{for all} \quad \vec{x}, \vec{y} \in S \quad \text{and} \quad \lambda \in [0,1].$$

DEFINITION 2.C. Suppose that $S \subseteq \mathbb{R}^n$. $F : S \to \mathbb{R}$ is a weakly convex function on S if it satisfies: for each given $\vec{y} = (y_1, \cdots, y_n) \in S$, there exists a vector $(\beta_1, \cdots, \beta_n) \in \mathbb{R}^n$ such that

$$F(x_1, \cdots, x_n) - F(y_1, \cdots, y_n) \geqslant \sum_{i=1}^{n} \beta_i (x_i - y_i) \quad \text{for all} \quad \vec{x} \equiv (x_1, \cdots, x_n) \in S.$$

LEMMA 2.D. (Theorem 3.6 – 3.6a of W. Fleming [5]) *Suppose that $F : S \to \mathbb{R}$ is twice differentiable in convex set $S \subseteq \mathbb{R}^n$. Then, we have the following equivalent statements:*

(R_1) *F is convex on S;*
(R_2) *for each given $\vec{y} = (y_1, \cdots, y_n) \in S$, the Jacobian matrix*

$$J_F(\vec{y}) \equiv \left(\frac{\partial^2 F}{\partial x_i \partial_j}(\vec{y}) \right)_{n \times n}$$

is positive semi-definite;
(R_3) *for each given $\vec{y} = (y_1, \cdots, y_n) \in S$,*

$$F(x_1, \cdots, x_n) - F(y_1, \cdots, y_n) \geqslant \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(\vec{y})(x_i - y_i)$$

for all $\vec{x} \equiv (x_1, \cdots, x_n) \in S$.

Moreover, one of (R_1), (R_2) and (R_3) implies
(R_4) *F is weakly convex on S holds.*

We now can state and prove our main results.

THEOREM 2.1. (The generalized Jensen inequaity with multiple variables) *Suppose that*

(C_1) $F: S \to \mathbb{R}$ *is continuous and weakly convex on* $S \subseteq \mathbb{R}^n$,

(C_2) $h \in C_{rd}(\mathbb{T}, \mathbb{R})$ *satisfies* $\int_a^b |h(t)|\Delta t > 0$,

(C_3) $g_1, \cdots, g_n \in C_{rd}(\mathbb{T}, \mathbb{R})$ *satisfy* $X_{i=1}^n g_i([a,b]) \subseteq S$ *hold. Then, we have*

$$F\left(\frac{\int_a^b |h(t)|g_1(t)\Delta t}{\int_a^b |h(t)|\Delta t}, \cdots, \frac{\int_a^b |h(t)|g_n(t)\Delta t}{\int_a^b |h(t)|\Delta t}\right) \leqslant \frac{\int_a^b |h(t)|F(g_1(t), \cdots, g_n(t))\Delta t}{\int_a^b |h(t)|\Delta t}.$$

Proof. Let

$$\vec{y} = (y_1, \cdots, y_n) = \left(\frac{\int_a^b |h(t)|g_1(t)\Delta t}{\int_a^b |h(t)|\Delta t}, \cdots, \frac{\int_a^b |h(t)|g_n(t)\Delta t}{\int_a^b |h(t)|\Delta t}\right).$$

Follows from (C_1) that there exists a vector $(\beta_1, \cdots, \beta_n) \in \mathbb{R}^n$ such that

$$F(x_1, \cdots, x_n) - F(y_1, \cdots, y_n) \geqslant \sum_{i=1}^n \beta_i(x_i - y_i) \quad \text{for all} \quad \vec{x} \equiv (x_1, \cdots, x_n) \in S.$$

Therefore, we have

$$\begin{aligned}
&\int_a^b |h(t)|F(g_1(t), \cdots, g_n(t))\Delta t \\
&\quad - \left(\int_a^b |h(t)|\Delta t\right) F\left(\frac{\int_a^b |h(t)|g_1(t)\Delta t}{\int_a^b |h(t)|\Delta t}, \cdots, \frac{\int_a^b |h(t)|g_n(t)\Delta t}{\int_a^b |h(t)|\Delta t}\right) \\
&\quad = \int_a^b |h(t)|F(g_1(t), \cdots, g_n(t))\Delta t - \left(\int_a^b |h(t)|\Delta t\right) F(y_1, \cdots, y_n) \\
&\quad = \int_a^b |h(t)|\{F(g_1(t), \cdots, g_n(t)) - F(y_1, \cdots, y_n)\}\Delta t \\
&\quad \geqslant \int_a^b |h(t)| \left\{\sum_{i=1}^n \beta_i(g_i(t) - y_i)\right\} \Delta t \\
&\quad = \sum_{i=1}^n \beta_i \left\{\int_a^b |h(t)|g_i(t)\Delta t - \left(\int_a^b |h(t)|\Delta t\right) y_i\right\} \\
&\quad = \sum_{i=1}^n \beta_i \left\{\int_a^b |h(t)|g_i(t)\Delta t - \left(\int_a^b |h(t)|\Delta t\right) \frac{\int_a^b |h(t)|g_i(t)\Delta t}{\int_a^b |h(t)|\Delta t}\right\} \\
&\quad = 0,
\end{aligned}$$

which complete the proof.

THEOREM 2.2. *Suppose that* $S \subseteq \mathbb{R}^n$ *is convex,* (C_2), (C_3) *and*
(C_4) $A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ *is positive semi-definite hold. Then, we have*

$$\sum_{i=1}^{n} a_{ii} \left(\frac{\int_a^b |h(t)| g_i(t)\Delta t}{\int_a^b |h(t)|\Delta t} \right)^2 + \sum_{i\neq j} a_{ij} \left(\frac{\int_a^b |h(t)| g_i(t)\Delta t}{\int_a^b |h(t)|\Delta t} \right) \left(\frac{\int_a^b |h(t)| g_j(t)\Delta t}{\int_a^b |h(t)|\Delta t} \right)$$
$$\leqslant \frac{\int_a^b |h(t)| \left\{ \sum_{i=1}^{n} a_{ii} g_i^2(t) + \sum_{i\neq j} a_{ij} g_i(t) g_j(t) \right\} \Delta t}{\int_a^b |h(t)|\Delta t}.$$

Proof. Setting

$$F(x_1, \cdots, x_n) \equiv \sum_{i=1}^{n} a_{ii} x_i^2 + \sum_{i\neq j} a_{ij} x_i x_j \quad \text{on convex subset} \quad S \subseteq \mathbb{R}^n.$$

It is clear that for each given $\vec{y} = (y_1, \cdots, y_n) \in S$,

$$J_F(\vec{y}) \equiv \left(\frac{\partial^2 F}{\partial x_i \partial_j}(\vec{y}) \right)_{n\times n} = A.$$

is positive semi-definite. Therefore, it follows from Lemma 2.D and our Theorem 2.1, we obtain the desired result.

REMARK. (cf: page 401 – ex 16, 17, 18 of Horn and Johnson [7])
(a) $A \equiv (a_{ij})_{n\times n}$ via $a_{ij} = \frac{1}{i+j-1}$ for $i, j = 1, \cdots, n$ is positive semi-definite;
(b) $A \equiv (a_{ij})_{n\times n}$ via $a_{ij} = \frac{1}{i+j}$ for $i, j = 1, \cdots, n$ is positive semi-definite;
(c) $A \equiv (a_{ij})_{n\times n}$ via $a_{ij} = min\{i, j\}$ for $i, j = 1, \cdots, n$ is positive semi-definite.

THEOREM 2.3. *Suppose that* $S \equiv \mathbb{R}^n$, $p \geqslant \frac{1}{2}$, (C_2) *and* $g_1, \cdots, g_n \in C_{rd}(\mathbb{T}, \mathbb{R})$ *hold. Then, we have*

$$\frac{\left[\sum_{i=1}^{n} \left(\int_a^b |h(t)| g_i(t)\Delta t \right)^2 \right]^p}{(\int_a^b |h(t)|\Delta t)^{2p}} \leqslant \frac{\int_a^b |h(t)| \left\{ \sum_{i=1}^{n} g_i^2(t) \right\}^p \Delta t}{\int_a^b |h(t)|\Delta t}.$$

Proof. Setting

$$F(x_1, \cdots, x_n) \equiv \left(\sum_{i=1}^{n} x_i^2 \right)^p \quad \text{on convex set} \quad S = \mathbb{R}^n.$$

It is clear that for each given $\vec{y} \in \mathbb{R}^n$, $J_F(\vec{y})$ is positive semi-definite (cf: page 118 – ex 5 of W. Fleming [5]). Therefore, it follows from Lemma 2.D and our Theorem 2.1, we obtain the desired result.

The following is an extension of the Čhebyšev inequality (cf: Čhebyšev [4]):

THEOREM 2.4. (The generalized Čhebyšev inequality)
Suppose that $S \equiv [0,\infty)^n \subseteq \mathbb{R}^n$, (C_2) *and* $g_1,\cdots,g_n \in C_{rd}(\mathbb{T},\mathbb{R})$ *hold. Then, we have*

$$\frac{\prod_{i=1}^{n}(\int_a^b |h(t)||g_i(t)|\Delta t)}{(\int_a^b |h(t)|\Delta t)^n} \leqslant \frac{\int_a^b |h(t)| \left\{ \prod_{i=1}^{n} |g_i(t)| \right\} \Delta t}{\int_a^b |h(t)|\Delta t}.$$

Proof. Setting

$$F(x_1,\cdots,x_n) \equiv \prod_{i=1}^{n} x_i \quad \text{on convex set} \quad S=[0,\infty)^n.$$

It is clear that for each given $\vec{y} \in [0,\infty)^n$, $J_F(\vec{y})$ is positive semi-definite (cf: page 177 of Beckenbach and Walter [2]). Therefore, it follows from Lemma 2.D and our Theorem 2.1, we obtain the desired result.

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