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# A NOTE ON INTEGRAL INEQUALITIES INVOLVING TWO LOG–CONVEX FUNCTIONS

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*Abstract.* The main aim of the present note is to establish new Hadmard like integral inequalities involving two log-convex functions.

## 1. Introduction

The following inequality is well known in the literature as the Hadmard inequality:

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2} \tag{1}$$

where $f : I \to R$ is a convex function on the interval I of real numbers $a, b \in I$ with $a < b$.

For various other results related to the above inequality and the historical consideration, see [3, 4, 5] and the references cited therin. A function $f : I \to (0, \infty)$ is siad to be log-convex function, if for all $x, y \in I$ and $t \in [0, 1]$ one has the inequality (see [5, p. 7]):

$$f(tx + (1-t)y) \leqslant [f(x)]^t [f(y)]^{1-t}.$$

In [1] Dragomir and Mond proved that the following inequalities hold for log-convex

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functions:

$$f\left(\frac{a+b}{2}\right) \leqslant \exp\left[\frac{1}{b-a}\int_a^b \ln[f(x)]dx\right] \leqslant \frac{1}{b-a}\int_a^b G(f(x),f(a+b-x)) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant L(f(a),f(b)) \leqslant \frac{f(a)+f(b)}{2} \tag{2}$$

where

$$G(p,q)=\sqrt{pq}$$

is the Geometric mean and

$$L(p,q)=\frac{p-q}{\ln p - \ln q} \quad (p \neq q)$$

is the Logarithmic mean of the positive real numbers $p,q$ (for $p=q$, we put $L(p,p)=p$). For the further refinments of (1.1) for log-convex functions, see[2]. The main purpose of this note is to estblish new inequalities like(1.1) involving two log-convex functions using elementry analysis.

## 2. Main Results

We start with the following Theorem.

THEOREM 2.1. *Let $f,g : I \to (0,\infty)$ be log-convex functions on $I$ and $a,b \in I$ with $a<b$. Then the following inequality holds:*

$$\frac{4}{b-a}\int_a^b f(x)g(x)dx \leqslant [f(a)+f(b)]L(f(a),f(b)) + [g(a)+g(b)]\,L(g(a),g(b)), \tag{1}$$

*where L is a logarithmic mean of positive real numbers.*

*Proof.* Since $f,g$ are log-convex functions, we have

$$f(ta+(1-t)b) \leqslant [f(a)]^t\,[f(b)]^{1-t}, \tag{2}$$

$$g(ta+(1-t)b) \leqslant [g(a)]^t\,[g(b)]^{1-t}, \tag{3}$$

for all $t \in [0,1]$ .It is easy to observe that

$$\int_a^b f(x)g(x)dx = (b-a)\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)dt. \tag{4}$$

Using the elementry inequality $cd \leqslant \frac{1}{2}\left[c^2+d^2\right]$ ($c,d \geqslant 0$ reals), (2.2), (2.3) on the right side of (2.4) and making the change of variable we have

$$\begin{aligned}
\int_a^b f(x)g(x)dx &\leqslant \frac{1}{2}(b-a)\int_0^1 \left[\{f(ta+(1-t)b)\}^2+\{g(ta+(1-t)b)\}^2\right]dt \\
&\leqslant \frac{1}{2}(b-a)\int_0^1 \left[\left\{[f(a)]^t[f(b)]^{1-t}\right\}^2+\left\{[g(a)]^t[g(b)]^{1-t}\right\}^2\right]dt \\
&= \frac{1}{2}(b-a)\left\{f^2(b)\int_0^1\left[\frac{f(a)}{f(b)}\right]^{2t}dt+g^2(b)\int_0^1\left[\frac{g(a)}{g(b)}\right]^{2t}dt\right\} \\
&= \frac{1}{4}(b-a)\left\{f^2(b)\int_0^2\left[\frac{f(a)}{f(b)}\right]^{\sigma}d\sigma+g^2(b)\int_0^2\left[\frac{g(a)}{g(b)}\right]^{\sigma}d\sigma\right\} \\
&= \frac{1}{4}(b-a)\left\{f^2(b)\left[\frac{\left[\frac{f(a)}{f(b)}\right]^{\sigma}}{\log\frac{f(a)}{f(b)}}\right]_0^2+g^2(b)\left[\frac{\left[\frac{g(a)}{g(b)}\right]^{\sigma}}{\log\frac{g(a)}{g(b)}}\right]_0^2\right\} \\
&= \frac{1}{4}(b-a)\left\{\frac{[f(a)+f(b)][f(a)-f(b)]}{\log f(a)-\log f(b)}+\frac{[g(a)+g(b)][g(a)-g(b)]}{\log g(a)-\log g(b)}\right\} \\
&= \frac{1}{4}(b-a)\left\{[f(a)+f(b)]L(f(a),f(b))+[g(a)+g(b)]L(g(a),g(b))\right\}
\end{aligned} \tag{5}$$

Rewriting (2.5) we get the required inequality in (2.1). The proof is complete. □

The following theorem also holds.

THEOREM 2.2. *Let $f,g: I \to (0,\infty)$ be differentiable log-convex functions on the interval of real numbers $\overset{0}{I}$ (the interior of I) and $a,b \in \overset{0}{I}$ with $a<b$. Then the following inequality holds:*

$$\begin{aligned}
\frac{2}{b-a}\int_a^b f(x)g(x)dx \geqslant& \frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^b g(x)\exp\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx \\
&+\frac{1}{b-a}g\left(\frac{a+b}{2}\right)\int_a^b f(x)\exp\left[\frac{g'\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx
\end{aligned} \tag{6}$$

*Proof.* Since $f, g$ are differentiable and log-convex functions on $\overset{0}{I}$, we have that

$$\log f(x) - \log f(y) \geqslant \frac{d}{dt}(\log f(y))(x-y), \tag{7}$$

$$\log g(x) - \log g(y) \geqslant \frac{d}{dt}(\log g(y))(x-y), \tag{8}$$

for all $x, y \in \overset{0}{I}$ gives that

$$log\left(\frac{f(x)}{f(y)}\right) \geqslant \frac{f'(y)}{f(y)}(x-y), \tag{9}$$

$$\log\left(\frac{g(x)}{g(y)}\right) \geqslant \frac{g'(y)}{g(y)}(x-y), \tag{10}$$

for all $x, y \in \overset{0}{I}$.That is

$$f(x) \geqslant f(y)\exp\left[\frac{f'(y)}{f(y)}(x-y)\right], \tag{11}$$

$$g(x) \geqslant g(y)\exp\left[\frac{g'(y)}{g(y)}(x-y)\right]. \tag{12}$$

Multiplying both sides of $(2.11)$ and $(2.12)$ by $g(x)$ and $f(x)$ respectively and adding the resulting inequalities we have

$$2f(x)g(x) \geqslant g(x)f(y)\exp\left[\frac{f'(y)}{f(y)}(x-y)\right] + f(x)g(y)\exp\left[\frac{g'(y)}{g(y)}(x-y)\right] \tag{13}$$

Now, if we choose $y = \frac{a+b}{2}$ ,from $(2.13)$ we obtain

$$\begin{aligned} 2f(x)g(x) \; &\geqslant g(x)f\left(\tfrac{a+b}{2}\right)\exp\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(x-\tfrac{a+b}{2}\right)\right] \\ &+ f(x)g\left(\tfrac{a+b}{2}\right)\exp\left[\frac{g'\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)}\left(x-\tfrac{a+b}{2}\right)\right]. \end{aligned} \tag{14}$$

Integrating both sides of $(2.14)$ with respect to $x$ from $a$ to $b$ and dividing both sides of the resulting inequality by $b-a$ in, we get the desired inequality $(2.6)$. The proof is complete. $\square$

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