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SOME NEW PROOFS FOR THE AGM INEQUALITY

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Abstract. In this paper, using some techniques of mathematical analysis, we give some new proofs for the AGM inequality.

1. Introduction

Undoubtedly, the arithmetic mean-geometric mean inequality, or briefly the AGM inequality is the most important inequality in the classical analysis. It simply states that if $x_1, x_2, \cdots, x_n$ are nonnegative real numbers and $\lambda_1, \lambda_2, \cdots, \lambda_n > 0$ with $\sum_{i=1}^n \lambda_i = 1$, then

$$\prod_{i=1}^{n} x_i^{\lambda_i} \leqslant \sum_{i=1}^{n} \lambda_i x_i, \tag{1}$$

and equality holds if and only if $x_1 = \cdots = x_n$.

The important unweighted case occurs if we put $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \frac{1}{n}$:

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leqslant \frac{x_1 + x_2 + \cdots + x_n}{n}. \tag{2}$$

There are several interesting proofs, refinements and applications of (1) and (2), each of which has its own fascination and importance, see e.g. the references, and more than fifty proofs have been mentioned in [4] in order of their appearances. In this paper, using different methods, we give some new proofs for the AGM inequality in the special case (2).

Throughout this paper, we use the following standard notations

$$A_n = \frac{x_1 + x_2 + \cdots + x_n}{n} \quad \text{and} \quad G_n = \sqrt[n]{x_1 x_2 \cdots x_n} \tag{3}$$

for the unweighted arithmetic and geometric means of n given nonnegative numbers $x_1, x_2, \cdots, x_n$ respectively.

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2. New Proofs

In this section we give three new proofs for the AGM inequality (2). The first proof is based on a well-known inequality in analysis described in the following lemma. The second one uses the positivity of the integral operator, and the last one deals with some kinds of Maclaurin's method; see [5, p. 19].

LEMMA. *If n is a positive integer and $1+\frac{h}{n} \geqslant 0$, then*

$$\left(1+\frac{h}{n+1}\right)^{n+1} \geqslant \left(1+\frac{h}{n}\right)^{n}, \tag{4}$$

and equality holds if and only if $h=0$.

Proof. Trivially equality holds in (4) if $h=0$. Let $h \neq 0$. If $1+\frac{h}{n}=0$, strict inequality holds in (4). Suppose then $1+\frac{h}{n}>0$. Now by means of Bernouli's inequality, we have

$$\begin{aligned} \frac{(1+\frac{h}{n+1})^{n+1}}{(1+\frac{h}{n})^{n}} &= \left(1+\frac{h}{n+1}\right)\left(1+\frac{\frac{h}{n+1}-\frac{h}{n}}{1+\frac{h}{n}}\right)^{n} \\ &\geqslant \left(1+\frac{h}{n+1}\right)\left(1+n\frac{\frac{h}{n+1}-\frac{h}{n}}{1+\frac{h}{n}}\right) \\ &= 1+\frac{h^2}{(n+1)^2(n+h)}>1, \end{aligned}$$

and the proof is completed.

Now, we show that (4) implies (2).

First proof of the AGM Inequality (2). It is sufficient to show that

$$A_{n+1}^{n+1} \geqslant A_n^n x_{n+1} \tag{5}$$

with equality holding if and only if $x_{n+1}=A_n$. For, if (5) is on hand, then from the induction hypothesis $A_n \geqslant G_n$ with equality if and only if $x_1=\cdots=x_n$, we get $A_{n+1}^{n+1} \geqslant G_{n+1}^{n+1}$ with equality if and only if $x_1=\cdots=x_{n+1}$.

Since $A_{n+1}=(nA_n+x_{n+1})/(n+1)$, equality holds in (5) if $x_{n+1}=A_n$. Now let $x_{n+1} \neq A_n$. If $x_{n+1}=0$, strict inequality holds in (5). Suppose then $x_{n+1} \neq 0$. Now by means of the above lemma with $h=n\left(\frac{A_n-x_{n+1}}{x_{n+1}}\right)$, we have

$$\begin{aligned} A_{n+1}^{n+1}=\left(\frac{nA_n+x_{n+1}}{n+1}\right)^{n+1} &= x_{n+1}^{n+1}\left(1+\frac{n\frac{A_n-x_{n+1}}{x_{n+1}}}{n+1}\right)^{n+1} \\ &> x_{n+1}^{n+1}\left(1+\frac{A_n-x_{n+1}}{x_{n+1}}\right)^{n} \\ &= x_{n+1}A_n^n, \end{aligned}$$

and the proof is completed.

Now, using the positivity of integration, we give the second proof to the AGM Inequality (2). For a quite different proof of the AGM inequality using positivity of integration, see [1–2].

Second proof of the AGM Inequality (2). Again, it is sufficient to show (5) with equality holding if and only if $x_{n+1} = A_n$.

Clearly equality holds in (5) if $x_{n+1} = A_n$. Suppose then $x_{n+1} \neq A_n$. If $x_{n+1} > A_n$, integrating both sides of the trivial inequality

$$A_n^n < \left(\frac{nA_n + t}{n+1}\right)^n \qquad (A_n < t) \tag{6}$$

with respect to t from A_n to x_{n+1} yields (5) with strict inequality. Similarly, if $x_{n+1} < A_n$ it is sufficient to integrate both sides of the trivial inequality

$$A_n^n > \left(\frac{nA_n + t}{n+1}\right)^n \qquad (A_n > t) \tag{7}$$

with respect to t from x_{n+1} to A_n.

Finally, we give a proof of (2) depending on Maclaurin's method [5, p. 19].

Third proof of the AGM Inequality (2). Consider the continuous real-valued function f defined by

$$f(x) = \frac{x_1 + \cdots + x_n}{n} \qquad (x = (x_1, \cdots, x_n) \in [0, \infty)^n).$$

For any two nonnegative real numbers M and a, put

$$C_{M,a} = \left\{ x = (x_1, \cdots, x_n) \in [0, M]^n : \prod_{i=1}^{n} x_i = a \right\}.$$

Let f takes its absolute minimum on the compact set $C_{M,a}$ at a point $u = (u_1, \cdots, u_n) \in C_{M,a}$. We show that $u_1 = \cdots = u_n$. Let, on the contrary, there exist two u_i's, say u_1 and u_2, such that $u_1 \neq u_2$. Put $u_1 u_2 = \alpha$. Two cases may occur.

Case 1. $\alpha \neq 0$. Take $x = (x_1, \cdots, x_n)$ such that

$$x_1 = x_2 = \alpha^{1/2},\ x_3 = u_3,\ \cdots,\ x_n = u_n.$$

Clearly, $x \in C_{M,a}$. But since $u_1 \neq \alpha^{1/2}$, we have

$$x_1 + x_2 = 2\alpha^{1/2} < u_1 + \frac{\alpha}{u_1} = u_1 + u_2,$$

and so $f(x) < f(u)$, which is a contradiction.

Case 2. $\alpha = 0$. So $a = 0$. We can assume $u_1 = 0 < u_2$. Now take $x = (x_1, \cdots, x_n)$ such that

$$x_1 = 0,\ x_2 = \frac{u_2}{2},\ x_3 = u_3,\ \cdots,\ x_n = u_n.$$

Clearly, $x \in C_{M,a}$. But since

$$x_1 + x_2 = \frac{u_2}{2} < u_1 + u_2,$$

we have $f(x) < f(u)$, which again is a contradiction.

Thus, $u_1 = \cdots = u_n = a^{1/n}$, and so

$$A_n = f(x) \geqslant f(u) = a^{1/n} = G_n \qquad (x = (x_1, \cdots, x_n) \in C_{M,a}),$$

with equality holding if and only if $x_1 = \cdots = x_n$. Now, since $[0,\infty)^n = \bigcup_{M,a \geqslant 0} C_{M,a}$, the proof is completed.

REMARK. (i) Actually (4) is equivalent to the unweighted AGM inequality (2). For, if (2) is on hand, then from

$$\sqrt[n+1]{1 \cdot \left(1 + \frac{h}{n}\right)^n} < \frac{1 + n\left(1 + \frac{h}{n}\right)}{n+1} = 1 + \frac{h}{n+1} \qquad \left(1 + \frac{h}{n} \geqslant 0,\ h \neq 0\right),$$

we obtain (4) with strict inequality.

(ii) The second proof can be easily modified in order to include the general case (1). For, if $\lambda_1, \lambda_2, \cdots, \lambda_{n+1} > 0$ with $\sum_{i=1}^{n+1} \lambda_i = 1$, then putting

$$A_{n+1} = \sum_{i=1}^{n+1} \lambda_i x_i, \qquad G_{n+1} = \prod_{i=1}^{n+1} x_i^{\lambda_i},$$

and

$$A_n = \sum_{i=1}^{n} \frac{\lambda_i}{1 - \lambda_{n+1}} x_i, \qquad G_n = \prod_{i=1}^{n} x_i^{\frac{\lambda_i}{1-\lambda_{n+1}}},$$

and integrating both sides of the trivial inequalities

$$A_n^{\frac{1-\lambda_{n+1}}{\lambda_{n+1}}} < [(1-\lambda_{n+1})A_n + \lambda_{n+1} t]^{\frac{1-\lambda_{n+1}}{\lambda_{n+1}}} \qquad (A_n < t) \tag{8}$$

and

$$A_n^{\frac{1-\lambda_{n+1}}{\lambda_{n+1}}} > [(1-\lambda_{n+1})A_n + \lambda_{n+1} t]^{\frac{1-\lambda_{n+1}}{\lambda_{n+1}}} \qquad (t < A_n) \tag{9}$$

with respect to t on $[A_n, x_{n+1}]$ and $[x_{n+1}, A_n]$ in the case of $A_n < x_{n+1}$ and $A_n > x_{n+1}$ respectively, we obtain

$$A_{n+1} > A_n^{1-\lambda_{n+1}} x_{n+1}^{\lambda_{n+1}} \qquad (x_{n+1} \neq A_n).$$

Now, (1) and its equality condition follows by induction on n.

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