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INTEGRAL CHARACTERIZATIONS FOR STABILITY OF LINEAR SKEW–PRODUCT SEMIFLOWS

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Abstract. We give necessary and sufficient conditions for uniform exponential stability of linear skew-product semiflows. Using the theory of Banach function spaces we obtain integral characterizations for this concept. We extend a stability theorem obtained by Rolewicz for evolution families, at the general case of linear skew-product semiflows.

1. Introduction

In the last decades, the theory of linear skew-product semiflows led to an important progress in the study of the asymptotic behaviour of evolution equations and to generalizations of classical results from the theory of C_0-semigroups and evolution families, respectively. New concepts of stability, expansiveness and dichotomy, respectively, have been introduced and characterized, giving significant answers to diverse questions concerning the asymptotic properties of linear skew-product semiflows (see [1]–[5], [11], [13], [15]–[18]).

Exponential stability of linear skew-product semiflows has been characterized in [11] and in [18]. The relation between stabilizability and controllability of systems associated to linear skew-product semiflows has been presented in [13]. In [16], the uniform exponential expansiveness of linear skew-product flows has been expressed in terms of the uniform complete admissibility of the pairs $(c_0(\mathbf{N},X), c_0(\mathbf{N},X))$ and $(C_0(\mathbf{R}_+,X), C_0(\mathbf{R}_+,X))$, respectively. Necessary and sufficient conditions for dichotomy of linear skew-product flows have been obtained in [1]–[5], [15], [17].

One of the most remarkable results in the stability theory of evolution equations has been obtained by Rolewicz in [20] and it is given by

THEOREM 1.1. *Let $N : \mathbf{R}_+^* \times \mathbf{R}_+ \to \mathbf{R}_+$ be a function with the properties that for every $t > 0$, $s \to N(t,s)$ is a continuous nondecreasing function, with $N(t,0) = 0$, $N(t,s) > 0$, for all $s > 0$, and for every $s \geqslant 0$, $t \to N(t,s)$ is nondecreasing. If $U = \{U(t,s)\}_{t \geqslant s \geqslant 0}$ is a strongly continuous evolution family on a Banach space X such that for every $x \in X$, there is $\alpha(x) > 0$ with*

$$\sup_{s \geqslant 0} \int_s^{\infty} N(\alpha(x), ||U(t,s)x||)\, dt < \infty$$

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then U is uniformly exponentially stable.

Since in the original proof, the continuity of N has been essentially used, Neerven proposed in [19] a similar characterization for the particular case of C_0-semigroups, giving up the continuity of N. Neerven proved that a C_0-semigroup $T = \{T(t)\}_{t\geqslant 0}$, on a Banach space X, is uniformly exponentially stable if and only if there is a nondecreasing function $N : \mathbf{R}_+ \to \mathbf{R}_+$, with $N(t) > 0$, for all $t > 0$ and $N(0) = 0$, such that

$$\int_0^\infty N(||T(t)x||)\,dt < \infty, \quad \forall x \in X.$$

In fact, this result is a consequence of another result due to Neerven (see [19]) given by

THEOREM 1.2. *A C_0-semigroup $T = \{T(t)\}_{t\geqslant 0}$ on the Banach space X is uniformly exponentially stable if and only if there exists a Banach function space B with the property $\lim\limits_{t\to\infty} F_B(t) = \infty$, such that for every $x \in X$ the mapping $t \mapsto ||T(t)x||$ belongs to B.*

Generalizations for the case of evolution families of Neerven's result have been presented in [9]. Unitary characterizations for uniform exponential stability and uniform exponential instability of C_0-semigroups, in terms of Banach function spaces, have been presented in [10]. More general characterizations have been obtained in [11], where the uniform exponential stability of linear skew-product semiflows has been expressed in terms of Banach sequence spaces and Banach function spaces, respectively. One of the main results in [11] was

THEOREM 1.3. *A linear skew-product semiflow $\pi = (\Phi, \sigma)$ on $\mathscr{E} = X \times \Theta$ is uniformly exponentially stable if and only if there is a Banach function space $B \in \mathscr{B}(\mathbf{R}_+)$ such that the following properties hold:*

(i) for every $(x, \theta) \in \mathscr{E}$ the function $f_{x,\theta} : \mathbf{R}_+ \to \mathbf{R}_+, f_{x,\theta}(t) = ||\Phi(\theta, t)x||$ belongs to B;

(ii) there exists $K : X \to (0, \infty)$ such that $|f_{x,\theta}|_B \leqslant K(x)$, for all $(x, \theta) \in \mathscr{E}$.

In the present paper we extend the study begun in [11] and we generalize the results obtained in the last mentioned paper. We will obtain very general necessary and sufficient conditions for uniform exponential stability of linear skew-product semiflows. As a consequence of our results, we will deduce the version of the theorem of Rolewicz, for the case of linear skew-product semiflows. In this manner, we will also present a new approach for the theorems of Rolewicz type.

2. Definitions and notations

2.1. Linear skew-product semiflows

Let X be a Banach space, let Θ be a compact Haussdorf space and let $\mathscr{E} = X \times \Theta$. We shall denote by $\mathscr{L}(X)$ the Banach algebra of all bounded linear operators from X into itself.

DEFINITION 2.1. A continuous mapping $\sigma : \Theta \times \mathbf{R}_+ \to \Theta$ is called a *semiflow* on Θ if $\sigma(\theta, 0) = \theta$ and $\sigma(\theta, t+s) = \sigma(\sigma(\theta, s), t)$, for all $(\theta, s, t) \in \Theta \times \mathbf{R}_+^2$.

DEFINITION 2.2. A pair $\pi = (\Phi, \sigma)$ is called *linear skew-product semiflow* on $\mathscr{E} = X \times \Theta$ if σ is a semiflow on Θ and $\Phi : \Theta \times \mathbf{R}_+ \to \mathscr{L}(X)$ satisfies the following conditions:

(i) $\Phi(\theta, 0) = I$, the identity operator on X, for all $\theta \in \Theta$;

(ii) $\Phi(\theta, t+s) = \Phi(\sigma(\theta, t), s)\Phi(\theta, t)$, for all $(\theta, t, s) \in \Theta \times \mathbf{R}_+^2$ (the cocycle identity);

(iii) $\lim\limits_{t \to 0_+} \Phi(\theta, t)x = x$, uniformly in θ.

REMARK 2.1. If $\pi = (\Phi, \sigma)$ is a linear skew-product semiflow on $\mathscr{E} = X \times \Theta$, then there are $M, \omega > 0$ such that $||\Phi(\theta, t)|| \leqslant Me^{\omega t}$, for all $(\theta, t) \in \Theta \times \mathbf{R}_+$ and for every $x \in X$ the mapping $t \mapsto \Phi(\theta, t)x$ is right-continuous.

Important examples of linear skew-product semiflows can be found in [1]–[5], [13], [15], [17], [18].

DEFINITION 2.3. A linear skew-product semiflow $\pi = (\Phi, \sigma)$ on $\mathscr{E} = X \times \Theta$ is said to be *uniformly exponentially stable* if there are $N, \nu > 0$ such that

$$||\Phi(\theta, t)|| \leqslant Ne^{-\nu t}, \quad \forall (\theta, t) \in \Theta \times \mathbf{R}_+.$$

2.2. Banach function spaces

Let $\mathscr{M}$ be the linear space of all Lebesgue measurable functions $f : \mathbf{R}_+ \to \mathbf{C}$ identifying the functions which are equal almost everywhere.

DEFINITION 2.4. A *Banach function norm* is a function $N : \mathscr{M} \to [0, \infty]$ with the following properties:

(i) $N(f) = 0$ if and only if $f = 0$ a.e.;

(ii) if $|f| \leqslant |g|$ a.e. then $N(f) \leqslant N(g)$;

(iii) $N(\alpha f) = |\alpha| N(f)$, for all $\alpha \in \mathbf{C}$ and all f with $N(f) < \infty$;

(iv) $N(f + g) \leqslant N(f) + N(g)$, for all $f, g \in \mathscr{M}$.

If $B := \{f \in \mathscr{M} : |f|_B := N(f) < \infty\}$, then $(B, |\cdot|_B)$ is a normed linear space. If B is complete then it is called *Banach function space*.

For a Banach function space B we define

$$F_B : \mathbf{R}_+^* \to [0, \infty], \quad F_B(t) = \begin{cases} |\chi_{[0,t)}|_B, & \text{if} \quad \chi_{[0,t)} \in B \\ \infty, & \text{if} \quad \chi_{[0,t)} \notin B \end{cases}$$

where $\chi_{[0,t)}$ denotes the characteristic function of the set $[0, t)$. The function F_B is called *the fundamental function* of the Banach function space B.

In what follows we denote by $\mathscr{B}(\mathbf{R}_+)$ the set of all Banach function spaces B with the property that $\lim\limits_{t \to \infty} F_B(t) = \infty$ and there is a strictly increasing sequence $(t_n) \subset \mathbf{R}_+$

such that $t_n \to \infty$,

$$\sup_{n\in\mathbf{N}} (t_{n+1} - t_n) < \infty \quad \text{and} \quad \inf_{n\in\mathbf{N}} |\chi_{[t_n,t_{n+1})}|_B > 0.$$

EXAMPLE 2.1. Let $N : \mathbf{R}_+ \to [0,\infty]$ be a nondecreasing and left-continuous function which is not identically 0 or ∞ on $(0,\infty)$. *The Young function* associated to N is

$$Y_N(t) := \int_0^t N(s)ds.$$

For $f \in \mathscr{M}$ we consider

$$M_N(f) := \int_0^\infty Y_N(|f(s)|)ds.$$

The set $O_N := \{f \in \mathscr{M} : \exists\, k > 0 \text{ with } M_N(kf) < \infty\}$ is a Banach function space with respect to the norm $|f|_N := \inf\{k > 0 : M_N(\frac{1}{k}f) \leqslant 1\}$. $(O_N, |\cdot|_N)$ is called *the Orlicz space* associated to N. Trivial examples of Orlicz spaces are $L^p(\mathbf{R}_+, \mathbf{C})$, with $p \in [1,\infty]$ (see [9], [19]).

REMARK 2.2. If $0 < N(t) < \infty$, for all $t > 0$, then $O_N \in \mathscr{B}(\mathbf{R}_+)$ (see [9]).

3. Main results

In this section we obtain characterizations for uniform exponential stability of linear skew-product semiflows. We generalize the theorem of Rolewicz, for the case of linear skew-product semiflows.

Let X be a Banach space, let Θ be a compact Hausdorff space and let $\mathscr{E} = X \times \Theta$. For $x \in X$ and $r > 0$ we denote by $D(x,r) = \{y \in X : ||x-y|| \leqslant r\}$.

We denote by $\mathscr{F}$ the set of all nondecreasing functions $N : \mathbf{R}_+ \to \mathbf{R}_+$ with $N(0) = 0$ and $N(t) > 0$, for all $t > 0$.

THEOREM 3.1. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product semiflow on $\mathscr{E} = X \times \Theta$. Then π is uniformly exponentially stable if and only if there exist a function $N \in \mathscr{F}$, $x_0 \in X$ and two constants $K, \delta \in (0,\infty)$ such that*

$$\int_0^\infty N(||\Phi(\theta,t)x||)dt \leqslant K, \quad \forall (x,\theta) \in D(x_0,\delta) \times \Theta.$$

Proof. Necessity. It results for $x_0 = 0, \delta > 0$ and $N(t) = t$, for all $t \geqslant 0$.

Sufficiency. Let $M, \omega > 0$ be the constants given by Remark 2.1. Let $t_0 > 0$ such that $K < t_0 N(1)$ and let $M_1 = Me^{\omega t_0}$.

For $(x,\theta) \in D(x_0,\delta) \times \Theta$ and $t \geqslant t_0$, setting $y_x = x/M_1$ and using the cocycle identity we have that

$$||\Phi(\theta,t)y_x|| \leqslant ||\Phi(\theta,s)x||, \quad \forall s \in [t-t_0, t]$$

so

$$t_0 N(||\Phi(\theta,t)y_x||) \leqslant \int_{t-t_0}^t N(||\Phi(\theta,s)x||)ds \leqslant K.$$

It follows that

$$||\Phi(\theta,t)x|| \leqslant M_1, \quad \forall (x,\theta) \in D(x_0,\delta) \times \Theta, \forall t \geqslant t_0.$$

Let $x \in X$ with $||x|| \leqslant 1$. Since

$$||\Phi(\theta,t)\delta x|| \leqslant ||\Phi(\theta,t)(\delta x + x_0)|| + ||\Phi(\theta,t)x_0|| \leqslant 2M_1$$

we obtain that

$$||\Phi(\theta,t)x|| \leqslant \frac{2M_1}{\delta}, \quad \forall (x,\theta) \in D(0,1) \times \Theta, \forall t \geqslant t_0,$$

so

$$||\Phi(\theta,t)|| \leqslant \frac{2M_1}{\delta}, \quad \forall \theta \in \Theta, \forall t \geqslant t_0.$$

Denoting $M_2 = M_1(2/\delta + 1)$ it follows that $||\Phi(\theta,t)|| \leqslant M_2$, for all $(\theta,t) \in \Theta \times \mathbf{R}_+$.

We may assume that N is left continuous - if not we can consider the function $\tilde{N}(t) = \lim\limits_{s \nearrow t} N(s)$ and the proof is similar.

Let $(O_N, |\cdot|_N)$ be the Orlicz space associated to N and let Y_N be the Young function. For every $(x,\theta) \in \mathscr{E}$ let

$$f_{x,\theta} : \mathbf{R}_+ \to \mathbf{R}_+, \quad f_{x,\theta}(t) = ||\Phi(\theta,t)x||.$$

Let $x \in X$ with $||x|| \leqslant \delta$ and let $\gamma = 1/[M_2(K+1)(||x_0|| + \delta + 1)]$. Taking into account that $\gamma \in (0,1)$ and N is nondecreasing we have that

$$\begin{aligned} Y_N(\gamma f_{x+x_0,\theta}(t)) &\leqslant \gamma\, f_{x+x_0,\theta}(t)\, N(\gamma f_{x+x_0,\theta}(t)) \\ &\leqslant \frac{1}{K+1}\, N(||\Phi(\theta,t)(x+x_0)||),\ \forall t \geqslant 0 \end{aligned}$$

and hence $M_N(\gamma f_{x+x_0,\theta}) < 1$. It follows that $f_{x+x_0,\theta} \in O_N$ and $|f_{x+x_0,\theta}|_N \leqslant 1/\gamma$, for every $x \in D(0,\delta)$.

Let $x \in X \setminus \{0\}$ and $z_x = \delta x/||x||$. From

$$|f_{z_x,\theta}|_N \leqslant |f_{z_x+x_0,\theta}|_N + |f_{x_0,\theta}|_N \leqslant \frac{2}{\gamma}$$

it follows that

$$|f_{x,\theta}|_N \leqslant \frac{2}{\gamma\delta}\, ||x||, \quad \forall (x,\theta) \in \mathscr{E}.$$

Applying Theorem 1.3 we obtain that π is uniformly exponentially stable. □

In what follows we denote by $\mathscr{C}$ the set of all continuous functions $N \in \mathscr{F}$.

THEOREM 3.2. *Let $\pi = (\Phi,\sigma)$ be a linear skew-product semiflow on $\mathscr{E} = X \times \Theta$ and let $(N_m)_{m \in \mathbf{N}^*} \subset \mathscr{C}$. If for every $x \in X$ there is $m \in \mathbf{N}^*$ such that*

$$\sup_{\theta \in \Theta} \int_0^\infty N_m(||\Phi(\theta,t)x||)dt < \infty$$

then π is uniformly exponentially stable.

Proof. Let $M, \omega > 0$ be the constants given by Remark 2.1. From hypothesis we have that

$$X = \bigcup_{m,k \in \mathbf{N}^*} \mathscr{V}_{m,k}.$$

where

$$\mathscr{V}_{m,k} = \{x \in X : \sup_{\theta \in \Theta} \int_0^\infty N_m(||\Phi(\theta,t)x||)dt \leqslant k\}, \quad \forall (m,k) \in \mathbf{N}^* \times \mathbf{N}^*.$$

Let $(m,k) \in \mathbf{N}^* \times \mathbf{N}^*$. For every $(T,\theta) \in (0,\infty) \times \Theta$ let

$$\mathscr{W}_{T,\theta}^{m,k} = \{x \in X : \int_0^T N_m(||\Phi(\theta,s)x||)ds \leqslant k\}.$$

Then

$$\mathscr{V}_{m,k} = \bigcap_{\theta \in \Theta} \bigcap_{T>0} \mathscr{W}_{T,\theta}^{m,k}$$

We show that $\mathscr{V}_{m,k}$ is closed. Therefore, we prove that $\mathscr{W}_{T,\theta}^{m,k}$ is closed. Let $(x_n) \subset \mathscr{W}_{T,\theta}^{m,k}$ with $x_n \to x$ and let $\varepsilon > 0$. Let $T_x = Me^{\omega T}(||x|| + 1)$. Using the continuity of N_m on $[0, T_x]$ it follows that there exists $\xi \in (0,1)$ such that for every $s_1, s_2 \in [0, T_x]$ with $|s_1 - s_2| \leqslant \xi$ we have $|N_m(s_1) - N_m(s_2)| < \varepsilon/T.$

Let $n_0 \in \mathbf{N}$ such that $||x_{n_0} - x|| \leqslant \xi/Me^{\omega T}$. Then we deduce that $||\Phi(\theta,s)x||$, $||\Phi(\theta,s)x_{n_0}|| \in [0,T_x]$ and

$$|\, ||\Phi(\theta,s)x|| - ||\Phi(\theta,s)x_{n_0}||\, | \leqslant ||\Phi(\theta,s)(x - x_{n_0})|| \leqslant \xi,\ \forall s \in [0,T].$$

It follows that

$$\int_0^T N_m(||\Phi(\theta,s)x||)ds \leqslant \int_0^T N_m(||\Phi(\theta,s)x_{n_0}||)ds + \varepsilon \leqslant k + \varepsilon,\ \forall \varepsilon > 0.$$

We obtain that $x \in \mathscr{W}_{T,\theta}^{m,k}$, so $\mathscr{W}_{T,\theta}^{m,k}$ is closed, for every $(T,\theta) \in (0,\infty) \times \Theta$. Then we deduce that $\mathscr{V}_{m,k}$ is closed, for every $(m,k) \in \mathbf{N}^* \times \mathbf{N}^*$.

Now, using the theorem of Baire it results that there exists $(m_0,k_0) \in \mathbf{N}^* \times \mathbf{N}^*$ such that there are $x_0 \in \mathscr{V}_{m_0,k_0}$ and $\delta > 0$ with $D(x_0,\delta) \subset \mathscr{V}_{m_0,k_0}$. Thus, from Theorem 3.1 applied for $N = N_{m_0}$ we conclude that π is uniformly exponentially stable. □

As a consequence of the above result we obtain the version of Rolewicz's theorem, for the case of uniform exponential stability of linear skew-product semiflows.

THEOREM 3.3. *Let $N : \mathbf{R}_+^* \times \mathbf{R}_+ \to \mathbf{R}_+$ be a function such that for every $t > 0, N(t,\cdot) \in \mathscr{C}$ and for every $s \geqslant 0, N(\cdot,s)$ is nondecreasing. Let $\pi = (\Phi,\sigma)$ be a linear skew-product semiflow on $\mathscr{E} = X \times \Theta$. If for every $x \in X$ there is $\alpha(x) > 0$ such that*

$$\sup_{\theta \in \Theta} \int_0^\infty N(\alpha(x), ||\Phi(\theta,t)x||)dt < \infty$$

then π is uniformly exponentially stable.

Proof. It immediately follows from Theorem 3.2, taking $N_m = N(1/m,\cdot)$, for all $m \in \mathbf{N}^*$. □

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