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LIONS–PEETRE TYPE COMPACTNESS RESULTS FOR SEVERAL BANACH SPACES

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Abstract. Working with interpolation methods associated to polygons, a result of Cobos and Peetre guarantees that the interpolated operator is compact provided all but two restrictions of the operator (located in adjacent vertices) are compact. We characterize here those intermediate spaces that satisfy the conclusion of Cobos-Peetre result for all operators. We also establish some results on rank-one interpolation spaces.

1. Introduction

Inequalities play a central role in interpolation theory. For instance, the interpolation property of compact operators by the real method can be stated as a convexity inequality between measures of non-compactness of the restrictions of the operator to the spaces in the couple (see the papers by Edmunds and Teixeira [25] and by Cobos, Fernández-Martínez and Martínez [6]). Central results of the article by Cobos, Cwikel and Matos [4] are also stated in terms of inequalities. They investigated to what extent the classical compactness results of Lions and Peetre [20] can hold for arbitrary intermediate spaces A. This time inequalities involve measures of non-compactness and also the functions ψ_A, ρ_A which describe the "position" of the intermediate space within the Banach couple. This research was continued by Cobos, Manzano, Martínez and Matos [10] and Cobos, Fernández-Cabrera, Martínez and Pustylnik [5]. Our aim here is to study these problems in the context of N-tuples of Banach spaces, by using the K- and J-functionals associated to a convex polygon in the plane.

In 1991, when Cobos and Peetre [11] introduced the interpolation methods $\overline{A}_{(\alpha,\beta),q;K}$, $\overline{A}_{(\alpha,\beta),q;J}$ defined by means of a polygon (we shall recall the definitions in Section 2.), they also established the following compactness result of Lions-Peetre type.

THEOREM 1.1. *Let $\Pi = \overline{P_1 \cdots P_N}$ be a convex polygon in $\mathbb{R}^2$, let P_k, P_{k+1} be two fixed adjacent vertices of Π, let (α,β) be an interior point of Π and $1 \leqslant q \leqslant \infty$. Assume that $\overline{A} = \{A_1,\dots,A_N\}$ is a Banach N-tuple, that B is a Banach space and that T is a linear operator.*

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(*i*) *If* $T : A_j \mapsto B$ *is bounded for* $1 \leqslant j \leqslant N$ *and* $T : A_j \mapsto B$ *is compact for all* $1 \leqslant j \leqslant N$ *with* $j \neq k, k+1$*, then* $T : \overline{A}_{(\alpha,\beta),q;K} \mapsto B$ *is also compact.*

(*ii*) *If* $T : B \mapsto A_j$ *is bounded for all* $1 \leqslant j \leqslant N$ *and* $T : B \mapsto A_j$ *is compact for all* $1 \leqslant j \leqslant N$ *with* $j \neq k, k+1$*, then* $T : B \mapsto \overline{A}_{(\alpha,\beta),q;J}$ *is compact as well.*

Related results can be found in [9], [3] and [7]. In this paper we investigate the validity of Theorem 1.1 for arbitrary intermediate spaces. Among other things we characterize those intermediate spaces A such that $T : A \mapsto B$ is compact for *all* operators $T \in \mathscr{L}(\overline{A}, B)$ with $T : A_j \mapsto B$ compactly for $j \neq k, k+1$. We also derive a corresponding result when the N-tuple $\overline{A}$ is the target of operators. The results extend [4], Thms. 3.15 and 3.17 to the setting of N-tuples of Banach spaces $(N \geqslant 3)$. Theorems of [4] are closely related to [10], Thms. 3.7 and 3.8. However, it is not possible to extend the results of [10] to our context, as we show here by means of examples.

Our techniques are based on ideas introduced in [4] and [5], combined with the geometrical elements which are peculiar to the functionals associated to polygons. We shall also use versions of the functions ψ_A, ρ_A with two arguments.

In the case of Banach couples, the functions ψ_A, ρ_A are connected with the notion of *rank-one interpolation space*. In fact, a necessary and sufficient condiciton for A to be a rank-one interpolation space with respect to the couple $\overline{A}$ is that for some $C > 0$ it holds $\psi_A(t) \leqslant C\rho_A(t)$ for all $t > 0$ (see the papers by Dmitriev [13] and by Pustylnik [23]). However, in our setting of N-tuples, we show that the corresponding inequality characterizes rank-one interpolation spaces only if $\overline{A}$ is a triple, that is, $N = 3$.

The organization of the paper is as follows. In Section 2. we recall some basic facts on interpolation methods associated to polygons and on measure of non-compactness. Section 3. contains the compactness results. Finally, in Section 4., we study rank-one interpolation spaces.

2. Preliminaries

Let $\Pi = \overline{P_1 \cdots P_N}$ be a convex polygon in the affine plane $\mathbb{R}^2$, with vertices $P_j = (x_j, y_j)$, $j = 1, \ldots, N$. Let $\overline{A} = \{A_1, \ldots, A_N\}$ be a Banach N-tuple, that is, a family of N Banach spaces A_j all of them continuously embedded in a common Hausdorff topological vector space. It will be useful to imagine each space A_j as sitting on the vertex P_j. By means of the polygon Π we define the following family of equivalent norms on $\Sigma(\overline{A}) = A_1 + \cdots + A_N$

$$K(t,s;a) = \inf\Big\{ \sum_{j=1}^{N} t^{x_j} s^{y_j} \|a_j\|_{A_j} : \ a = \sum_{j=1}^{N} a_j, \ a_j \in A_j \Big\}, \ t,s > 0.$$

Similarly, in $\Delta(\overline{A}) = A_1 \cap \cdots \cap A_N$, we consider the family of equivalent norms

$$J(t,s;a) = \max\big\{ t^{x_j} s^{y_j} \|a\|_{A_j} : \ 1 \leqslant j \leqslant N \big\}, \ t,s > 0.$$

Note that $\|\cdot\|_{\Sigma(\overline{A})} = K(1,1;\cdot)$ and $\|\cdot\|_{\Delta(\overline{A})} = J(1,1;\cdot)$.

Given (α,β) in the interior of Π, $(\alpha,\beta)\in \operatorname{Int}\Pi$, and $1\leqslant q\leqslant\infty$, the K-space $\overline{A}_{(\alpha,\beta),q;K}$ is formed by all $a\in\Sigma(\overline{A})$ for which the norm

$$\|a\|_{(\alpha,\beta),q;K}=\left(\int_0^\infty\int_0^\infty\left(t^{-\alpha}s^{-\beta}K(t,s;a)\right)^q\frac{dt}{t}\frac{ds}{s}\right)^{1/q}$$

is finite (the integral should be replaced by the supremum if $q=\infty$). On the other hand, the J-space $\overline{A}_{(\alpha,\beta),q;J}$ consists of all those $a\in\Sigma(\overline{A})$ which can be represented as

$$a=\int_0^\infty\int_0^\infty u(t,s)\frac{dt}{t}\frac{ds}{s},$$

where $u(t,s)$ is a strongly measurable function with values in $\Delta(\overline{A})$ and satisfies

$$\left(\int_0^\infty\int_0^\infty\left(t^{-\alpha}s^{-\beta}J(t,s;u(t,s))\right)^q\frac{dt}{t}\frac{ds}{s}\right)^{1/q}<\infty.$$

The norm in $\overline{A}_{(\alpha,\beta),q;J}$ is

$$\|a\|_{(\alpha,\beta),q;J}=\inf\left\{\left(\int_0^\infty\int_0^\infty\left(t^{-\alpha}s^{-\beta}J(t,s;u(t,s))\right)^q\frac{dt}{t}\frac{ds}{s}\right)^{1/q}\right\},$$

where the infimum is taken over all representations of a as before.

These spaces were introduced by Cobos and Peetre in [11]. When Π is equal to the simplex $\{(0,0),(1,0),(0,1)\}$, then the spaces coincide with (the first non-trivial case of) spaces studied by Sparr [24]. If Π coincides with the unit square $\{(0,0),(1,0),(1,1),(0,1)\}$, we recover spaces investigated by Fernandez [17].

In contrast to the case of the classical real method for couples, where K- and J-spaces coincide to within equivalence of norms (see [1] or [26]), K- and J-spaces for N-tuples ($N\geqslant 3$) do not coincide in general. We only have the continuous inclusion $\overline{A}_{(\alpha,\beta),q;J}\hookrightarrow\overline{A}_{(\alpha,\beta),q;K}$. For example, if Π is the unit square and $\ell_1(w_n)$ is the weighted ℓ_1-space with weights w_n, then (see [8], Example 2.8)

$$\left(\ell_1\Big(\frac{1}{\sqrt{n}}\Big),\ell_1\Big(\frac{1}{n}\Big),\ell_1\Big(\frac{1}{\sqrt{n}}\Big),\ell_1\Big(\frac{1}{n}\Big)\right)_{(\frac12,\frac12),1;J}=\ell_1\Big(\frac{1}{\sqrt{n}}\Big),\quad\text{but}$$
$$\left(\ell_1\Big(\frac{1}{\sqrt{n}}\Big),\ell_1\Big(\frac{1}{n}\Big),\ell_1\Big(\frac{1}{\sqrt{n}}\Big),\ell_1\Big(\frac{1}{n}\Big)\right)_{(\frac12,\frac12),1;K}=\ell_1\Big(\frac{1+\log n}{n}\Big).$$

Let $\overline{B}=\{B_1,\dots,B_N\}$ be another Banach N-tuple. We write $T\in\mathscr{L}(\overline{A},\overline{B})$ to mean that T is a linear operator from $\Sigma(\overline{A})$ to $\Sigma(\overline{B})$ whose restriction to each A_j defines a bounded operator from A_j into B_j ($j=1,\dots,N$). We put

$$\|T\|_{\overline{A},\overline{B}}=\max\{\|T\|_{A_j,B_j}:\ j=1,\dots,N\}.$$

If $T\in\mathscr{L}(\overline{A},\overline{B})$, it is easy to see that the restriction of T to $\overline{A}_{(\alpha,\beta),q;K}$ gives a bounded operator $T:\overline{A}_{(\alpha,\beta),q;K}\mapsto\overline{B}_{(\alpha,\beta),q;K}$. The same holds for J-spaces.

If the N-tuple $\overline{A}$ (respectively, $\overline{B}$) reduce to a single Banach space, that is, if $A_1=\cdots=A_N=A$ (respectively $B_1=\cdots=B_N=B$), then we write $T\in\mathscr{L}(\overline{A},B)$ (respectively, $T\in\mathscr{L}(A,\overline{B})$).

We say that a Banach space A is intermediate with respect to the Banach N-tuple $\overline{A}$ if $\Delta(\overline{A}) \hookrightarrow A \hookrightarrow \Sigma(\overline{A})$, with continuous inclusions. To work with the intermediate space A, the following functions will be useful

$$\begin{aligned} \rho_A(t,s) &= \inf\{J(t,s;a) : a \in \Delta(\overline{A}),\ \|a\|_A = 1\}, \\ \psi_A(t,s) &= \sup\{K(t,s;a) : a \in A,\ \|a\|_A = 1\}. \end{aligned}$$

These functions are versions with two parameters of the functions used in [4] (see also [13] and [22]).

We end this section by recalling that given a bounded linear operator T between the Banach spaces A and B, the (ball) measure of non-compactness of T is defined by

$$\gamma(T_{A,B}) = \inf\Big\{r > 0 :\ T(\mathscr{U}_A) \subseteq \bigcup_{j=1}^{n}\{y_j + \varepsilon\mathscr{U}_B\} \text{ for some finite set } \{y_1,\dots,y_n\} \subseteq B\Big\},$$

where $\mathscr{U}_A$ and $\mathscr{U}_B$ denote the closed unit balls of the spaces A and B respectively (see [2] or [15]).

3. Intermediate spaces and compactness

We start by establishing compactness results of Lions-Peetre type for general intermediate spaces with respect to a given N-tuple.

THEOREM 3.1. *Let $\Pi = \overline{P_1 \cdots P_N}$ be a convex polygon with vertices $P_j = (x_j, y_j)$, let $\overline{A} = \{A_1,\dots,A_N\}$ be a Banach N-tuple and let A be an intermediate space with respect to $\overline{A}$. For any Banach space B and any operator $T \in \mathscr{L}(B,\overline{A})$, we have*

$$\gamma(T_{B,A}) \leqslant 2 \inf_{t,s>0} \max_{1\leqslant j\leqslant N} \Big\{\frac{t^{x_j}s^{y_j}}{\rho_A(t,s)}\gamma(T_{B,A_j})\Big\}. \tag{1}$$

Proof. For $j = 1,\dots,N$, take any $k_j > \gamma(T_{B,A_j})$ and let $\{a_1^j,\dots,a_{n_j}^j\}$ be a finite set in A_j so that

$$T(\mathscr{U}_B) \subseteq \bigcup_{r=1}^{n_j}\{a_r^j + k_j\mathscr{U}_{A_j}\}. \tag{2}$$

Consider the intersections $\bigcap_{j=1}^{N}\{a_{r_j}^j + k_j\mathscr{U}_{A_j}\}$ and take one element $w(r_1,\dots,r_N)$ in $\bigcap_{j=1}^{N}\{a_{r_j}^j + k_j\mathscr{U}_{A_j}\}$ if the last set is not empty. Let W be the set formed by all those $w(r_1,\dots,r_N)$. The set W is contained in $\Delta(\overline{A})$, it is finite and not empty because, by (2),

$$T(\mathscr{U}_B) \subseteq \bigcup_{\substack{1\leqslant r_1\leqslant n_1\\ \cdots \\ 1\leqslant r_N\leqslant n_N}} \bigcap_{j=1}^{N}\{a_{r_j}^j + k_j\mathscr{U}_{A_j}\}.$$

Given any $b \in \mathscr{U}_B$ we can find $w \in W$ such that $\|Tb - w\|_{A_j} \leqslant 2k_j$, for $j = 1, \dots, N$. Whence, for any $t, s > 0$, we get

$$\|Tb - w\|_A \leqslant \frac{J(t,s;Tb-w)}{\rho_A(t,s)} = \max_{1\leqslant j\leqslant N}\left\{\frac{t^{x_j}s^{y_j}}{\rho_A(t,s)}\|Ta - w\|_{A_j}\right\} \leqslant 2\max_{1\leqslant j\leqslant N}\left\{\frac{t^{x_j}s^{y_j}}{\rho_A(t,s)}k_j\right\}.$$

This implies that

$$\gamma(T_{B,A}) \leqslant 2\inf_{t,s>0}\max_{1\leqslant j\leqslant N}\left\{\frac{t^{x_j}s^{y_j}}{\rho_A(t,s)}\gamma(T_{B,A_j})\right\}.$$

□

When the N-tuple is in the domain of the operator, the corresponding result reads as follows.

THEOREM 3.2. *Let $\Pi = \overline{P_1 \cdots P_N}$ be a convex polygon with vertices $P_j = (x_j, y_j)$, let $\overline{A} = \{A_1, \dots, A_N\}$ be a Banach N-tuple and let A be an intermediate space with respect to $\overline{A}$. For any Banach space B and any operator $T \in \mathscr{L}(\overline{A}, B)$, we have*

$$\gamma(T_{A,B}) \leqslant N\inf_{t,s>0}\max_{1\leqslant j\leqslant N}\left\{\frac{\psi_A(t,s)}{t^{x_j}s^{y_j}}\gamma(T_{A_j,B})\right\}. \tag{3}$$

Proof. For $j = 1, \dots, N$, take any $k_j > \gamma(T_{A_j,B})$ and let $\{b_1^j, \dots, b_{n_j}^j\}$ be a finite set in B such that

$$T(\mathscr{U}_{A_j}) \subseteq \bigcup_{r=1}^{n_j}\{b_r^j + k_j\mathscr{U}_B\}. \tag{4}$$

Let $t, s > 0$. Given any $a \in \mathscr{U}_A$ and any $\varepsilon > 0$, there is a representation $a = \sum_{j=1}^N a_j$ of a with $a_j \in A_j$ and

$$\sum_{j=1}^N t^{x_j}s^{y_j}\|a_j\|_{A_j} \leqslant K(t,s;a) + \varepsilon \leqslant \psi_A(t,s) + \varepsilon.$$

It follows that $a_j \in t^{-x_j}s^{-y_j}(\psi_A(t,s)+\varepsilon)\mathscr{U}_{A_j}$, $j = 1, \dots, N$, and so, using (4), we can find $1 \leqslant r_j \leqslant n_j$ such that

$$\left\|Ta_j - \frac{\psi_A(t,s)+\varepsilon}{t^{x_j}s^{y_j}}b_{r_j}^j\right\|_B \leqslant \frac{\psi_A(t,s)+\varepsilon}{t^{x_j}s^{y_j}}k_j.$$

Consequently,

$$\left\|Ta - \sum_{j=1}^N \frac{\psi_A(t,s)+\varepsilon}{t^{x_j}s^{y_j}}b_{r_j}^j\right\|_B \leqslant \sum_{j=1}^N\left\{\frac{\psi_A(t,s)+\varepsilon}{t^{x_j}s^{y_j}}k_j\right\} \leqslant N\max_{1\leqslant j\leqslant N}\left\{\frac{\psi_A(t,s)+\varepsilon}{t^{x_j}s^{y_j}}k_j\right\}.$$

This yields

$$\gamma(T_{A,B}) \leqslant N\inf_{t,s>0}\max_{1\leqslant j\leqslant N}\left\{\frac{\psi_A(t,s)}{t^{x_j}s^{y_j}}\gamma(T_{A_j,B})\right\}.$$

□

Assume that for some $(\alpha,\beta)\in \operatorname{Int}\Pi$ and some $M>0$ the intermediate space A satisfies that

$$\psi_A(t,s)\leqslant Mt^{\alpha}s^{\beta} \quad \text{for all } t,s>0, \tag{5}$$

$$(\text{respectively}, Mt^{\alpha}s^{\beta}\leqslant \rho_A(t,s) \quad \text{for all } t,s>0.) \tag{6}$$

In this case, using [8], Thm. 1.9, we can rewrite inequality (3) as

$$\gamma(T_{A,B})\leqslant NM\max\left\{\gamma(T_{A_i,B})^{c_i}\gamma(T_{A_j,B})^{c_j}\gamma(T_{A_k,B})^{c_k} : \{i,j,k\}\in \mathscr{P}_{\alpha,\beta}\right\}. \tag{7}$$

Here $\mathscr{P}_{\alpha,\beta}$ is the collection of all triples $\{i,j,k\}$ such that (α,β) belongs to the triangle with vertices P_i,P_j,P_k and c_i,c_j,c_k are the barycentric coordinates of (α,β) with respect to P_i,P_j,P_k. Similarly, if (6) holds, then (1) says

$$\gamma(T_{B,A})\leqslant \frac{2}{M}\max\left\{\gamma(T_{B,A_i})^{c_i}\gamma(T_{B,A_j})^{c_j}\gamma(T_{B,A_k})^{c_k} : \{i,j,k\}\in \mathscr{P}_{\alpha,\beta}\right\}. \tag{8}$$

It is easy to check that condition (5) is equivalent to the continuous embedding $A\hookrightarrow \overline{A}_{(\alpha,\beta),\infty;K}$. On the other hand, using the discrete description of the space $\overline{A}_{(\alpha,\beta),1;J}$, it can be verified that (6) is equivalent to the continuous embedding $\overline{A}_{(\alpha,\beta),1;J}\hookrightarrow A$. In the terminology of Nikolova [21], condition (5) means that the space A is of the class $K_{\alpha,\beta}(\overline{A})$, and condition (6) means that A is of the class $J_{\alpha,\beta}(\overline{A})$.

Formulae (7) and (8) shows that Theorems 3.1 and 3.2 comprise Theorem 1.1. These results apply, in particular, to Sparr spaces and to Fernandez spaces. But they also apply to extensions of the complex method. For example, they work for the extension given by Favini [16] to triples of Banach spaces $[A_1,A_2,A_3]_{\alpha,\beta}$, and for the extension given by Fernandez [18] and Dore, Guidetti and Venni [14] to 4-tuples of Banach spaces $[A_1,A_2,A_3,A_4]_{[(\alpha,\beta),q]}$. Indeed, if $C_{\alpha,\beta}$ denotes any of these extensions of the complex method, one can check that

$$\overline{A}_{(\alpha,\beta),1;J}\hookrightarrow C_{\alpha,\beta}(\overline{A})\hookrightarrow \overline{A}_{(\alpha,\beta),\infty;K}.$$

Next we want to characterize those arbitrary intermediate spaces A that satisfy the conclusion of Theorem 1.1 for *all* operators T and *all* Banach spaces B. We first establish two auxiliary results.

LEMMA 3.3. *Let $\Pi=\overline{P_1\cdots P_N}$ be a convex polygon with vertices $P_j=(x_j,y_j)$, and let P_k,P_{k+1} be two fixed adjacent vertices of Π. Then there exist sequences of positive numbers $\{t_n\},\{s_n\}$ such that for all $1\leqslant j\leqslant N$ with $j\neq k,k+1$, we have*

$$\lim_{n\to\infty} t_n^{x_k-x_j}s_n^{y_k-y_j}=0=\lim_{n\to\infty} t_n^{x_{k+1}-x_j}s_n^{y_{k+1}-y_j}$$

and

$$t_n^{x_k-x_{k+1}}s_n^{y_k-y_{k+1}}=1 \textit{ for all } n\in\mathbb{N}.$$

Proof. Let $ax+by=c$ be the equation of the line through P_k and P_{k+1}. Since the vertices P_k,P_{k+1} are adjacent, the polygon lies in a side of the line, say in $ax+by>c$ (see Figure 1).

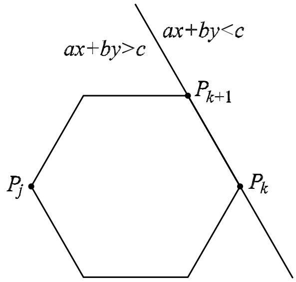

Figure 1.

For any $1 \leqslant j \leqslant N$ with $j \neq k, k+1$, it follows that $a(x_k - x_j) + b(y_k - y_j) < 0$ and $a(x_{k+1} - x_j) + b(y_{k+1} - y_j) < 0$. Put $\{t_n\} = \{e^{na}\}, \{s_n\} = \{e^{nb}\}$. Then

$$\begin{aligned} t_n^{x_k - x_{k+1}} s_n^{y_k - y_{k+1}} &= e^{n(a(x_k - x_{k+1}) + b(y_k - y_{k+1}))} = e^0 = 1, \\ t_n^{x_k - x_j} s_n^{y_k - y_j} &= e^{n(a(x_k - x_j) + b(y_k - y_j))} \to 0 \quad \text{as } n \to \infty \end{aligned}$$

and

$$t_n^{x_{k+1} - x_j} s_n^{y_{k+1} - y_j} = e^{n(a(x_{k+1} - x_j) + b(y_{k+1} - y_j))} \to 0 \quad \text{as } n \to \infty.$$

□

LEMMA 3.4. *Let $\Pi = \overline{P_1 \cdots P_N}$ be a convex polygon with vertices $P_j = (x_j, y_j)$, and let P_k, P_{k+1} be two fixed adjacent vertices of Π. Assume that $\overline{A} = \{A_1, \ldots, A_N\}$ is a Banach N-tuple and that A is an intermediate space with respect to $\overline{A}$. Let $a \in \Sigma(\overline{A})$. Then the following conditions are equivalent.*

(i) $\inf_{t,s>0} \max\left\{\dfrac{K(t,s;a)}{t^{x_k} s^{y_k}}, \dfrac{K(t,s;a)}{t^{x_{k+1}} s^{y_{k+1}}}\right\} = 0.$

(ii) *The element a belongs to $\overline{\sum_{\substack{1 \leqslant j \leqslant N \\ j \neq k,k+1}} A_j}^{\Sigma(\overline{A})}$, the closure of $W = \sum_{\substack{1 \leqslant j \leqslant N \\ j \neq k,k+1}} A_j$ in $\Sigma(\overline{A})$.*

Proof. For every $t, s > 0$, we have

$$\max\left\{\frac{K(t,s;a)}{t^{x_k} s^{y_k}}, \frac{K(t,s;a)}{t^{x_{k+1}} s^{y_{k+1}}}\right\} \geqslant \frac{\inf\limits_{a = \sum_{j=1}^N a_j} \{t^{x_k} s^{y_k} \|a_k\|_{A_k} + t^{x_{k+1}} s^{y_{k+1}} \|a_{k+1}\|_{A_{k+1}}\}}{\min\{t^{x_k} s^{y_k}, t^{x_{k+1}} s^{y_{k+1}}\}}$$

$$\geqslant \inf_{a = \sum_{j=1}^N a_j} \{\|a_k\|_{\Sigma(\overline{A})} + \|a_{k+1}\|_{\Sigma(\overline{A})}\} \geqslant \inf_{a' \in W} \{\|a - a'\|_{\Sigma(\overline{A})}\}.$$

This shows that (i) implies (ii).

Assume now that (ii) holds. Let $\{t_n\},\{s_n\}$ be the sequences constructed in Lemma 3.3, and put $u_n = 1/t_n, v_n = 1/s_n$. Then

$$\lim_{n\to\infty} u_n^{x_j-x_k} v_n^{y_j-y_k} = 0 = \lim_{n\to\infty} u_n^{x_j-x_{k+1}} v_n^{y_j-y_{k+1}}. \tag{9}$$

Given any $\varepsilon > 0$, we can find $a' = \sum_{\substack{1\leqslant j\leqslant N\\ j\neq k,k+1}} a'_j \in W$ such that $\|a-a'\|_{\Sigma(\overline{A})} \leqslant \varepsilon$. Let $M_{a'} = \max\{\|a'_j\|_{A_j} : 1\leqslant j\leqslant N, j\neq k,k+1\}$. We have

$$\begin{aligned}\frac{K(u_n,v_n;a)}{u_n^{x_k}v_n^{y_k}} &\leqslant \frac{K(u_n,v_n;a-a')}{u_n^{x_k}v_n^{y_k}} + \sum_{\substack{1\leqslant j\leqslant N\\ j\neq k,k+1}} u_n^{x_j-x_k} v_n^{y_j-y_k}\|a'_j\|_{A_j} \\ &\leqslant \max_{1\leqslant j\leqslant N}\{u_n^{x_j-x_k} v_n^{y_j-y_k}\}\|a-a'\|_{\Sigma(\overline{A})} + M_{a'} \sum_{\substack{1\leqslant j\leqslant N\\ j\neq k,k+1}} u_n^{x_j-x_k} v_n^{y_j-y_k} \\ &\leqslant \max_{\substack{1\leqslant j\leqslant N\\ j\neq k,k+1}}\{1, u_n^{x_j-x_k} v_n^{y_j-y_k}\}\varepsilon + M_{a'} \sum_{\substack{1\leqslant j\leqslant N\\ j\neq k,k+1}} u_n^{x_j-x_k} v_n^{y_j-y_k}.\end{aligned}$$

Using (9), we get that $K(u_n,v_n;a)/u_n^{x_k}v_n^{y_k} \leqslant 2\varepsilon$ for n big enough. Whence, we have that $\lim_{n\to\infty}(K(u_n,v_n;a)/u_n^{x_k}v_n^{y_k}) = 0$. A similar argument yields that

$$\lim_{n\to\infty}(K(u_n,v_n;a)/u_n^{x_{k+1}}v_n^{y_{k+1}}) = 0.$$

Consequently, (i) is satisfied. □

We can now characterize those intermediate spaces that satisfy the conclusion of Theorem 1.1/(ii) for all Banach spaces B and all operators T.

THEOREM 3.5. *Let $\Pi = \overline{P_1\cdots P_N}$ be a convex polygon with vertices $P_j = (x_j,y_j)$, and let P_k, P_{k+1} be two fixed adjacent vertices of Π. Assume that $\overline{A} = \{A_1,\dots,A_N\}$ is a Banach N-tuple and that A is an intermediate space with respect to $\overline{A}$. Then the following are equivalent.*

(i) $\inf_{t,s>0} \max\left\{\frac{t^{x_k}s^{y_k}}{\rho_A(t,s)}, \frac{t^{x_{k+1}}s^{y_{k+1}}}{\rho_A(t,s)}\right\} = 0.$

(ii) *For every Banach space B, if $T \in \mathscr{L}(B,\overline{A})$ is such that $T : B \mapsto A_j$ is compact for all $1\leqslant j\leqslant N$ with $j\neq k,k+1$, then $T : B\mapsto A$ is compact.*

(iii) *If $T\in\mathscr{L}(\ell_1,\overline{A})$ is such that $T:\ell_1\mapsto A_j$ is compact for all $1\leqslant j\leqslant N$ with $j\neq k,k+1$, then $T:\ell_1\mapsto A$ is compact.*

Proof. By Theorem 3.1 we get that (i) implies (ii). Clearly, (ii) implies (iii). To show that (iii) implies (i), let us suppose that (i) does not hold, that is,

$$\inf_{t,s>0} \max\left\{\frac{t^{x_k}s^{y_k}}{\rho_A(t,s)}, \frac{t^{x_{k+1}}s^{y_{k+1}}}{\rho_A(t,s)}\right\} > \delta > 0.$$

Then, taking $M = 1/\delta$, for all $t,s > 0$ we can find $a_{t,s} \in \Delta(\overline{A})$ with $\|a_{t,s}\|_A = 1$ and $J(t,s;a_{t,s}) \leqslant M \max\{t^{x_k}s^{y_k}, t^{x_{k+1}}s^{y_{k+1}}\}$. It follows that

$$\|a_{t,s}\|_{A_j} \leqslant M \max\{t^{x_k-x_j}s^{y_k-y_j}, t^{x_{k+1}-x_j}s^{y_{k+1}-y_j}\}, \quad \text{for } j = 1,\dots,N.$$

Let $\{t_n\},\{s_n\}$ be the sequences given by Lemma 3.3 and put $a_n = a_{t_n,s_n}$. We have $\|a_n\|_A = 1$, $\|a_n\|_{A_k} \leqslant M$, $\|a_n\|_{A_{k+1}} \leqslant M$ and

$$\lim_{n\to\infty} \|a_n\|_{A_j} = 0 \quad \text{for all } 1 \leqslant j \leqslant N \text{ with } j \neq k, k+1. \tag{10}$$

Consider the operator $T \in \mathscr{L}(\ell_1, \overline{A})$ defined by $T\{\lambda_n\} = \sum_{n=1}^{\infty} \lambda_n a_n$. This operator satisfies that $T : \ell_1 \mapsto A_j$ is compact for all $1 \leqslant j \leqslant N$ with $j \neq k, k+1$, because $T : \ell_1 \mapsto A_j$ is the limit of the sequence of finite-rank operators $R_m\{\lambda_n\} = \sum_{j=1}^{m} \lambda_j a_j$. However, the image by T of the sequence of unit vectors $\{e_n\}$ does not have any convergent subsequence in A. Indeed, if this were the case, since $\{Te_n\} = \{a_n\}$, we could find $a \in A$ and a subsequence $\{a_{n'}\}$ of $\{a_n\}$ such that $\{a_{n'}\} \to a$ in A. Hence $\|a\|_A = 1$. On the other hand, (10) implies that $\{a_{n'}\} \to 0$ in $\Sigma(\overline{A})$. Since $A \hookrightarrow \Sigma(\overline{A})$, it follows that $a = 0$, which contradicts $\|a\|_A = 1$. So, $T : \ell_1 \mapsto A$ is not compact. This shows that (iii) does not hold and establishes that (iii) implies (i). □

Next we turn our attention to the case when the N-tuple $\overline{A}$ is in the domain of the operators. We need to impose a mild additional assumption on $\overline{A}$ and the intermediate space A.

THEOREM 3.6. *Let $\Pi = \overline{P_1 \cdots P_N}$ be a convex polygon with vertices $P_j = (x_j, y_j)$, and let P_k, P_{k+1} be two fixed adjacent vertices of Π. Assume that $\overline{A} = \{A_1,\dots,A_N\}$ is a Banach N-tuple and that A is an intermediate space with respect to $\overline{A}$ such that*

$$\inf_{t,s>0} \max\left\{\frac{K(t,s;a)}{t^{x_k}s^{y_k}}, \frac{K(t,s;a)}{t^{x_{k+1}}s^{y_{k+1}}}\right\} = 0, \quad \text{for all } a \in A. \tag{11}$$

Then the following are equivalent.

(i) $\inf_{t,s>0} \max\left\{\frac{\psi_A(t,s)}{t^{x_k}s^{y_k}}, \frac{\psi_A(t,s)}{t^{x_{k+1}}s^{y_{k+1}}}\right\} = 0.$

(ii) *For every Banach space B, if $T \in \mathscr{L}(\overline{A}, B)$ is such that $T : A_j \mapsto B$ is compact for all $1 \leqslant j \leqslant N$ with $j \neq k, k+1$, then $T : A \mapsto B$ is compact.*

(iii) *If $T \in \mathscr{L}(\overline{A}, \ell_\infty)$ is such that $T : A_j \mapsto \ell_\infty$ is compact for all $1 \leqslant j \leqslant N$ with $j \neq k, k+1$, then $T : A \mapsto \ell_\infty$ is compact.*

Proof. According to Theorem 3.2, condition (i) implies (ii). Obviously, (ii) implies (iii). It remains to show that (iii) implies (i). Let us suppose that (i) does not hold, that is,

$$\inf_{t,s>0} \max\left\{\frac{\psi_A(t,s)}{t^{x_k}s^{y_k}}, \frac{\psi_A(t,s)}{t^{x_{k+1}}s^{y_{k+1}}}\right\} > \delta > 0.$$

Then, for all $t,s>0$, there exists $a_{t,s}\in A$ with $\|a_{t,s}\|_A=1$ and

$$\max\left\{\frac{K(t,s;a_{t,s})}{t^{x_k}s^{y_k}},\frac{K(t,s;a_{t,s})}{t^{x_{k+1}}s^{y_{k+1}}}\right\}>\delta. \tag{12}$$

Let $\{t_n\},\{s_n\}$ be the sequences constructed in Lemma 3.3 and put $u_n=1/t_n,\ v_n=1/s_n$. It follows from the second part of the proof of Lemma 3.4 that, for any $a\in A$,

$$\lim_{n\to\infty}\frac{K(u_n,v_n;a)}{u_n^{x_k}v_n^{y_k}}=0=\lim_{n\to\infty}\frac{K(u_n,v_n;a)}{u_n^{x_{k+1}}v_n^{y_{k+1}}}. \tag{13}$$

Put $a_n=a_{u_n,v_n}$ and consider the sequence of norms on $\Sigma(\overline{A})$ defined by

$$\|a\|_n=\max\left\{\frac{K(u_n,v_n;a)}{u_n^{x_k}v_n^{y_k}},\frac{K(u_n,v_n;a)}{u_n^{x_{k+1}}v_n^{y_{k+1}}}\right\}.$$

According to (12), we have

$$\|a_n\|_A=1 \quad\text{and}\quad \|a_n\|_n>\delta \quad \text{for all } n\in\mathbb{N}, \tag{14}$$

and, by (13),

$$\lim_{n\to\infty}\|a_m\|_n=0 \quad \text{for every } m\in\mathbb{N}. \tag{15}$$

Using the Hahn-Banach theorem, for each $n\in\mathbb{N}$, there is a linear functional f_n on $\Sigma(\overline{A})$ such that $f_n(a_n)=\|a_n\|_n$ and $|f_n(a)|\leqslant\|a\|_n$ for all $a\in\Sigma(\overline{A})$. Let $T\in\mathscr{L}(\overline{A},\ell_\infty)$ be the operator defined by $Ta=\{f_n(a)\}$. By the definition of $\|\cdot\|_n$, it is clear that

$$\|f_n\|_{A_j^*}\leqslant\max\{u_n^{x_j-x_k}v_n^{y_j-y_k},u_n^{x_j-x_{k+1}}v_n^{y_j-y_{k+1}}\}.$$

Hence, $\|f_n\|_{A_k^*}\leqslant 1, \|f_n\|_{A_{k+1}^*}\leqslant 1$ and $\lim_{n\to\infty}\|f_n\|_{A_j^*}=0$ for all $1\leqslant j\leqslant N$ with $j\neq k,k+1$. It follows that $T:A_j\mapsto\ell_\infty$ is compact for all $1\leqslant j\leqslant N$ with $j\neq k,k+1$, because it is the limit of a sequence of finite-rank operators. However, $T:A\mapsto\ell_\infty$ is not compact. Indeed, if T were compact, the sequence $\{Ta_n\}$ would have a convergent subsequence in ℓ_∞, say $\{Ta_{n'}\}$. Using (14), for n' and m' big enough, we get

$$\begin{aligned}\frac{\delta}{2}&\geqslant\|Ta_{n'}-Ta_{m'}\|_{\ell_\infty}\geqslant|f_{n'}(a_{n'}-a_{m'})|\geqslant|f_{n'}(a_{n'})|-|f_{n'}(a_{m'})|\\&\geqslant\|a_{n'}\|_{n'}-\|a_{m'}\|_{n'}>\delta-\|a_{m'}\|_{n'}.\end{aligned}$$

Whence, $\|a_{m'}\|_{n'}>\delta/2$ for every m',n' sufficiently big, which contradicts (15). Consequently, (iii) does not hold.

The proof is complete. □

Lemma 3.4 explains the meaning of (11). Note that condition (11) holds if $\Delta(\overline{A})$ is dense in A, or if $\Delta(\overline{A})$ is dense in A_k+A_{k+1}. Theorem 3.6 is not valid in general without condition (11) as the following example shows.

COUNTEREXAMPLE 3.7. Let Π be the simplex $\{(0,0),(1,0),(0,1)\}$, let $\overline{A} = \{X,\ell_1,\ell_2\}$, where X is the subspace of ℓ_∞ formed by all the sequences having the first coordinate equal to 0, and let $A = c_0$. For every Banach space B, if $T \in \mathscr{L}(\overline{A},B)$ satisfies that $T : X \mapsto B$ is compact, then $T : c_0 \mapsto B$ is compact. Indeed, let $P \in \mathscr{L}(c_0,X)$ be the projection given by $P\{\xi_n\} = \{0,\xi_2,\xi_3,\ldots\}$, let $e_1 = \{1,0,0,\ldots\}$, and consider the operators $R,S \in \mathscr{L}(c_0,B)$ defined by $R\{\xi_n\} = \xi_1 Te_1$, $S\{\xi_n\} = T(P\{\xi_n\})$. Compactness of $T : X \mapsto B$ implies that $S : c_0 \mapsto B$ is compact. Moreover, the operator R is also compact because its rank is 1. Since $T = R+S$, we derive that $T : c_0 \mapsto B$ is compact. Consequently, statement (ii) in Theorem 3.6 is satisfied. However, condition (11) fails, and therefore statement (i) is not verified. Indeed, given any $t,s>0$, we have

$$\max\left\{\frac{\psi_A(t,s)}{t},\frac{\psi_A(t,s)}{s}\right\} \geqslant \max\left\{\frac{K(t,s;e_1)}{t},\frac{K(t,s;e_1)}{s}\right\}$$
$$\geqslant \max\left\{\inf_{\lambda+\mu=1}\left\{|\lambda|+\frac{s}{t}|\mu|\right\},\inf_{\lambda+\mu=1}\left\{\frac{t}{s}|\lambda|+|\mu|\right\}\right\} \geqslant 1.$$

Theorems 3.5 and 3.6 extend [4], Thms. 3.15 and 3.17, to the setting of interpolation using polygons. Since the mentioned results of [4] are closely related to [10], Thms. 3.7 and 3.8, one might think that if $T \in \mathscr{L}(B,\overline{A})$ with $T : B \mapsto \Sigma(\overline{A})$ compactly, and A is an intermediate space with respect to $\overline{A}$ with

$$\inf_{t,s>0}\frac{t^{x_j}s^{y_j}}{\rho_A(t,s)} = 0 \quad \text{for } j = 1,\ldots,N, \tag{16}$$

then $T : B \mapsto A$ should be compact. Similarly, one might believe that if $T \in \mathscr{L}(\overline{A},B)$ with $T : \Delta(\overline{A}) \mapsto B$ compactly and

$$\inf_{t,s>0}\frac{\psi_A(t,s)}{t^{x_j}s^{y_j}} = 0 \quad \text{for } j = 1,\ldots,N, \tag{17}$$

then $T : A \mapsto B$ should be also compact. But this is not the case, as the following examples show. We work with weighted sequence spaces.

COUNTEREXAMPLE 3.8. Let Π be the unit square $\{(0,0),(1,0),(1,1),(0,1)\}$ and let $(\alpha,\beta) \in \operatorname{Int}\Pi$ with $\beta \geqslant 1-\alpha$. Take $\overline{A} = \{\ell_1,\ell_1(n),\ell_1(n),\ell_1(n)\}$ and let $A = \overline{A}_{(\alpha,\beta),1;J}$. Then $\rho_A(t,s) \geqslant Mt^\alpha s^\beta$ and so we have for $j = 1,2,3,4$

$$\inf_{t,s>0}\frac{t^{x_j}s^{y_j}}{\rho_A(t,s)} \leqslant \frac{1}{M}\inf_{t,s>0} t^{x_j-\alpha}s^{y_j-\beta}.$$

The last infimum is equal to 0 as it can be checked with a similar argument as in Lemma 3.3. Whence, condition (16) is satisfied.

Take $B = \ell_1(n)$ and choose T as the identity operator $T\{\xi_n\} = \{\xi_n\}$. Since $\Sigma(\overline{A}) = \ell_1$, we have that $T : B \mapsto \Sigma(\overline{A})$ is compact. However, by [8], Thm. 2.5, $A = \ell_1(n)$ and it is obvious that $T : A \mapsto B$ is not compact.

COUNTEREXAMPLE 3.9. Choose Π and (α,β) as in Counterexample 3.8. Take $\overline{A} = \{\ell_\infty(n), \ell_\infty, \ell_\infty, \ell_\infty\}$, $A = \overline{A}_{(\alpha,\beta),\infty;K}$, $B = \ell_\infty$ and choose again T as the identity operator. Condition (17) holds because $\psi_A(t,s) \leqslant Mt^\alpha s^\beta$. Moreover, since $\Delta(\overline{A}) = \ell_\infty(n)$, we have that $T : \Delta(\overline{A}) \mapsto B$ is compact. But, according to [8], Thm. 2.3, $A = \ell_\infty$ and so $T : A \mapsto B$ fails to be compact.

4. Rank-one interpolation spaces

Recall that an intermediate space A with respect to an N-tuple $\overline{A}$ is said to be an *interpolation space* if, for any $T \in \mathscr{L}(\overline{A},\overline{A})$, the restriction of T to A gives a bounded operator from A into itself. It is a consequence of the closed graph theorem that then there exists a constant $C = C(A,\overline{A})$ such that

$$\|T\|_{A,A} \leqslant C\|T\|_{\overline{A},\overline{A}} \tag{18}$$

for all operators $T \in \mathscr{L}(\overline{A},\overline{A})$.

We say that the intermediate space A is a *rank-one interpolation space*, or a partly interpolation space, if (18) holds for all operator T of the special form $Tx = f(x)a$ where $f \in \Sigma(\overline{A})^*$ and $a \in \Delta(\overline{A})$.

EXAMPLE 4.1. Let G be the symmetric function space constructed in [19] p. 122. Then, for any $1 < p < \infty$, G is an intermediate space with respect to the triple (L_1, L_p, L_∞), but G is not an interpolation space (see [19] Thm. II.5.11). However, G is a rank-one interpolation space with respect to (L_1, L_p, L_∞) because, by results of Dmitriev [13] and Pustylnik [23], any space lying between the Lorentz and the Marcinkiewicz space with the same fundamental function is a rank-one interpolation space with respect to (L_1, L_∞) and, therefore, with respect to (L_1, L_p, L_∞).

Working with Banach couples, inequality $\psi_A(t) \leqslant C\rho_A(t)$, $t > 0$, characterizes rank-one interpolation spaces (see [13] and [23]). Next we show that working with N-tuples $(N \geqslant 3)$, the corresponding inequality is a characterization of rank-one interpolation spaces only if $N = 3$.

THEOREM 4.2. *Let $\Pi = \overline{P_1 \cdots P_N}$ be a convex polygon with vertices $P_j = (x_j, y_j)$, let $\overline{A} = \{A_1, \dots, A_N\}$ be a Banach N-tuple and let A be an intermediate space with respect to $\overline{A}$.*

If A is a rank-one interpolation space, then there is a constant $C = C(A,\overline{A})$ such that

$$\psi_A(t,s) \leqslant C\rho_A(t,s), \quad \textit{for all } t,s > 0. \tag{19}$$

Moreover, if $N = 3$ and (19) holds, then A is a rank-one interpolation space.

Proof. Assume that A is a rank-one interpolation space and let $C > 0$ be the constant in (18) for rank-one operators. To establish (19) it suffices to show that for any $a \in A$ and any $b \in \Delta(\overline{A})$ it holds

$$K(t,s;a)\,\|b\|_A \leqslant C\,J(t,s;b)\,\|a\|_A, \quad \text{for all } t,s > 0. \tag{20}$$

This can be done using ideas of [23].

Given $a \in A$, by the Hahn-Banach theorem, there is $f \in \Sigma(\overline{A})^*$ such that $f(a) = K(t,s;a)$ and $|f(x)| \leqslant K(t,s;x)$ for any $x \in \Sigma(\overline{A})$. We have that $\|f\|_{A_j^*} \leqslant t^{x_j}s^{y_j}$ for $j = 1,\dots,N$. Take any $b \in \Delta(\overline{A})$ and consider the operator $Tx = f(x)b$. Since

$$\|T\|_{A_j,A_j} = \|f\|_{A_j^*}\,\|b\|_{A_j} \leqslant t^{x_j}s^{y_j}\,\|b\|_{A_j}$$

and A is a rank-one interpolation space, we get that

$$\|Ta\|_A \leqslant C\,J(t,s;b)\,\|a\|_A.$$

Then (20) follows by observing that

$$\|Ta\|_A = |f(a)|\,\|b\|_A = K(t,s;a)\,\|b\|_A.$$

Suppose now that (19) holds and that $N = 3$. Take any $f \in \Sigma(\overline{A})^*$ and any $b \in \Delta(\overline{A})$, and put $Tx = f(x)b$. Using (19), for any $a \in A$ we have

$$\begin{aligned}\|Ta\|_A = |f(a)|\,\|b\|_A &\leqslant |f(a)|\frac{J(t,s;b)}{\rho_A(t,s)} \leqslant C|f(a)|\frac{J(t,s;b)}{\psi_A(t,s)}\\ &\leqslant C\frac{|f(a)|}{K(t,s;a)}J(t,s;b)\|a\|_A.\end{aligned}$$

Choose a decomposition $a = \sum_{j=1}^{3} a_j$ with $a_j \in A_j$ and $\sum_{j=1}^{3} t^{x_j}s^{y_j}\|a_j\|_{A_j} \leqslant 2K(t,s;a)$. Then

$$\begin{aligned}\|Ta\|_A &\leqslant 2\,C\frac{\sum_{j=1}^{3}|f(a_j)|}{\sum_{j=1}^{3} t^{x_j}s^{y_j}\|a_j\|_{A_j}}J(t,s;b)\|a\|_A\\ &\leqslant 6\,C\max\left\{\frac{|f(a_1)|}{t^{x_1}s^{y_1}\|a_1\|_{A_1}},\frac{|f(a_2)|}{t^{x_2}s^{y_2}\|a_2\|_{A_2}},\frac{|f(a_3)|}{t^{x_3}s^{y_3}\|a_3\|_{A_3}}\right\}J(t,s;b)\|a\|_A\\ &\leqslant 6\,C\max\{t^{-x_1}s^{-y_1}\|f\|_{A_1^*},t^{-x_2}s^{-y_2}\|f\|_{A_2^*},t^{-x_3}s^{-y_3}\|f\|_{A_3^*}\}J(t,s;b)\|a\|_A.\end{aligned}$$

Here t,s are positive numbers at our disposal. By [8], Lemma 1.7, we can choose them such that

$$t^{-x_1}s^{-y_1}\|f\|_{A_1^*} = t^{-x_2}s^{-y_2}\|f\|_{A_2^*} = t^{-x_3}s^{-y_3}\|f\|_{A_3^*}.$$

Call M this common value. It follows that

$$\begin{aligned}\|Ta\|_A &\leqslant 6\,C\,MJ(t,s;b)\,\|a\|_A\\ &= 6\,C\max\{Mt^{x_1}s^{y_1}\|b\|_{A_1},Mt^{x_2}s^{y_2}\|b\|_{A_2},Mt^{x_3}s^{y_3}\|b\|_{A_3}\}\|a\|_A\\ &= 6\,C\max\{\|f\|_{A_1^*}\|b\|_{A_1},\|f\|_{A_2^*}\|b\|_{A_2},\|f\|_{A_3^*}\|b\|_{A_3}\}\,\|a\|_A\\ &= 6\,C\,\|T\|_{\overline{A},\overline{A}}\,\|a\|_A.\end{aligned}$$

□

We end the paper with an example that shows that if $N > 3$ then (19) is not sufficient to guarantee that A is a rank-one interpolation space.

COUNTEREXAMPLE 4.3. Let Π be the unit square $\{(0,0),(1,0),(1,1),(0,1)\}$ and let $\overline{A}$ be the *diagonally equal* 4-tuple $(L_1[0,1], L_\infty[0,1], L_1[0,1], L_\infty[0,1])$. By [12], Example 1.25,

$$\overline{A}_{(\frac{1}{2},\frac{1}{2}),1;J} = L_\infty[0,1], \quad \overline{A}_{(\frac{1}{2},\frac{1}{2}),\infty;K} = L_1[0,1].$$

Hence, any intermediate space A with respect to $\overline{A}$ satisfies that $\overline{A}_{(\frac{1}{2},\frac{1}{2}),1;J} \hookrightarrow A \hookrightarrow \overline{A}_{(\frac{1}{2},\frac{1}{2}),\infty;K}$. This means (see Section 3.) that there are constants $M_1, M_2 > 0$ such that

$$\psi_A(t,s) \leqslant M_1\, t^{\frac{1}{2}} s^{\frac{1}{2}} \leqslant M_2\, \rho_A(t,s), \quad \text{for all } t,s > 0.$$

In other words, any intermediate space A with respect to $\overline{A}$ satisfies (19).

Take now A as the subspace of $L_1[0,1]$ formed by all those functions f having a finite norm

$$\|f\|_A = \|f\chi_{(0,1/2)}\|_{L_\infty} + \|f\chi_{(1/2,1)}\|_{L_1}.$$

It is easy to check that A is an intermediate space with respect to $\overline{A}$. We claim that A is not a rank-one interpolation space. Indeed, for $n \geqslant 2$, put

$$h_n = n\chi_{(1-1/n^2,1)}, \quad g_n = n\chi_{(0,1/n^2)},$$

and consider the sequence of rank one operators $\{T_n\}$ defined by

$$T_n f = \Big(\int_0^1 f(x)h_n(x)\,dx\Big) g_n.$$

We have

$$\|T_n\|_{L_1,L_1} = \sup_{\|f\|_{L_1}\leqslant 1} \Big|\int_0^1 f(x)h_n(x)\,dx\Big| \|g_n\|_{L_1} = \|h_n\|_{L_\infty}\|g_n\|_{L_1} = n/n = 1,$$

$$\|T_n\|_{L_\infty,L_\infty} = \sup_{\|f\|_{L_\infty}\leqslant 1} \Big|\int_0^1 f(x)h_n(x)\,dx\Big| \|g_n\|_{L_\infty} = \|h_n\|_{L_1}\|g_n\|_{L_\infty} = n/n = 1.$$

Whence, $\|T_n\|_{\overline{A},\overline{A}} = 1$ for $n \geqslant 2$. But,

$$\begin{aligned}\|T_n\|_{A,A} &= \sup_{\|f\|_A\leqslant 1} \Big|\int_0^1 f(x)h_n(x)\,dx\Big|\, \|g_n\|_A \\ &= \sup_{\|f\chi_{(1/2,1)}\|_{L_1}\leqslant 1} \Big|\int_{\frac{1}{2}}^1 f(x)h_n(x)\,dx\Big|\, \|g_n\|_{L_\infty} = \|h_n\|_{L_\infty}\|g_n\|_{L_\infty} = n^2.\end{aligned}$$

Consequently, although A satisfies (19), A is not a rank-one interpolation space.

REFERENCES

[1] J. BERGH AND J. LÖFSTRÖM, *Interpolation Spaces. An Introduction*, Springer, 1976.
[2] B. CARL AND I. STEPHANI, *Entropy, Compactness and the Approximation of Operators*, Cambridge University Press, Cambridge, UK, 1990.

[3] F. COBOS, *On the optimality of compactness results for interpolation methods associated to polygons*, Indag. Math. **5** (1994) 397–401.
[4] F. COBOS, M. CWIKEL AND P. MATOS, *Best possible compactness results of Lions-Peetre type*, Proc. Edinb. Math. Soc. **44** (2001) 153–172.
[5] F. COBOS, L. M. FERNÁNDEZ-CABRERA, A. MARTÍNEZ AND E. PUSTYLNIK, *Some interpolation results that are the exclusive property of compact operators*, Proc. R. Soc. Edinb. **132A** (2002) 317–328.
[6] F. COBOS, P. FERNÁNDEZ-MARTÍNEZ AND A. MARTÍNEZ, *Interpolation of the measure of non-compactness by the real method*, Studia Math. **135** (1999) 25–38.
[7] F. COBOS, P. FERNÁNDEZ-MARTÍNEZ AND A. MARTÍNEZ, *Measure of non-compactness and interpolation methods associated to polygons*, Glasgow Math. J. **41** (1999) 65–79.
[8] F. COBOS, P. FERNÁNDEZ-MARTÍNEZ AND T. SCHONBEK, *Norm estimates for interpolation methods defined by means of polygons*, J. Approx. Theory **80** (1995) 321–351.
[9] F. COBOS, T. KÜHN AND T. SCHONBEK, *One-sided compactness results for Aronszajn-Gagliardo functors*, J. Funct. Analysis **106** (1992) 274–313.
[10] F. COBOS, A. MANZANO, A. MARTÍNEZ AND P. MATOS, *On interpolation of strictly singular operators, strictly co-singular operators and related operator ideals*, Proc. R. Soc. Edinb. **130A** (2000) 971–989.
[11] F. COBOS AND J. PEETRE, *Interpolation of compact operators: The multidimensional case*, Proc. London Math. Soc. **63** (1991) 371–400.
[12] M. CWIKEL AND S. JANSON, *Real and complex interpolation methods for finite and infinite families of Banach spaces*, Adv. in Math. **66** (1987) 234–290.
[13] A. A. DMITRIEV, *The interpolation of one-dimensional operators*, Voronež Gos. Univ. Trudy Naučn. -Issled. Inst. Mat. VGU Vyp. 11 Sb. Statej Funkcional. Anal. i Prilozen **11** (1973) 31–43 (in Russian).
[14] G. DORE, D. GUIDETTI AND A. VENNI, *Complex interpolation for* 2^n *Banach spaces*, Rend. Sem. Mat. Univ. Padova **76** (1986) 1–36.
[15] D. E. EDMUNDS AND W. D. EVANS, *Spectral Theory and Differential Operators*, Clarendon Press, Oxford, 1987.
[16] A. FAVINI, *Su una estensione del metodo d'interpolazione complesso*, Rend. Sem. Mat. Univ. Padova (1972) 243–298.
[17] D. L. FERNANDEZ, *Interpolation of* 2^n *Banach spaces*, Studia Math. **45** (1979) 175–201.
[18] D. L. FERNANDEZ, *An extension of the complex method of interpolation*, Boll. Un. Mat. Ital. (5) B-**18** (1981) 721–732.
[19] S. G. KREIN, J. I. PETUNIN AND E. M. SEMENOV, *Interpolation of Linear Operators*, Amer. Math. Soc., Providence, R.I., 1982.
[20] J. L. LIONS AND J. PEETRE, *Sur une classe d'espaces d'interpolation*, Inst. Hautes Études Sci. Publ. Math. **19** (1964) 5–68.
[21] L. I. NIKOLOVA, *Some estimates of measure of non-compactness for operators acting in interpolation spaces – the multidimensional case*, C. R. Acad. Bulg. Sci. **44** (1991) 5–8.
[22] E. PUSTYLNIK, *Minimal and maximal intermediate Banach spaces*, Ukrain. Mat. Zhurnal **29** (1977) 129–137. (English transl. in Ukrainian Math. J. **29** (1977) 102–107).
[23] E. I. PUSTYLNIK, *Embedding functions and their role in interpolation theory*, Abstract Appl. Analysis **1** (1996) 305–325.
[24] G. SPARR, *Interpolation of several Banach spaces*, Ann. Mat. Pura Appl. **99** (1974) 247–316.
[25] M. F. TEIXEIRA AND D. E. EDMUNDS, *Interpolation theory and measure of non-compactness*, Math. Nachr. **104** (1981) 129–135.
[26] H. TRIEBEL, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, 1978.

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