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CONVEXITY OR CONCAVITY INEQUALITIES FOR HERMITIAN OPERATORS

JEAN-CHRISTOPHE BOURIN

(*communicated by F. Hansen*)

Abstract. Given a Hermitian operator, a monotone convex function f and a subspace $\mathscr{E}$, $\dim \mathscr{E} < \infty$, there exists a unitary operator U on $\mathscr{E}$ such that $f(A_{\mathscr{E}}) \leqslant Uf(A)_{\mathscr{E}}U^*$. (Here $X_{\mathscr{E}}$ denotes the compression of X onto $\mathscr{E}$). A related result is: For a monotone convex function f, $0 < \alpha, \beta < 1$, $\alpha + \beta = 1$, and Hermitian operators A, B on a finite dimensional space, there exists a unitary U such that $f(\alpha A + \beta B) \leqslant U\{\alpha f(A) + \beta f(B)\}U^*$. More general convexity results are established. Also, several old and new trace inequalities of Brown-Kosaki and Hansen-Pedersen type are derived. We study the behaviour of the map $p \longrightarrow \{(A^p)_{\mathscr{E}}\}^{1/p}$, $A \geqslant 0$, $0 < p < \infty$.

Introduction

Given an operator A on a separable Hilbert space $\mathscr{H}$ and a subspace $\mathscr{E} \subset \mathscr{H}$, we denote by $A_{\mathscr{E}}$ the compression of A onto $\mathscr{E}$, i.e. the restriction of EAE to $\mathscr{E}$, E being the projection onto $\mathscr{E}$. If $\mathscr{E}$ is a finite dimensional subspace, we show that, for any Hermitian operator A and any monotone convex function f defined on the spectrum of A, there exits a unitary operator U on $\mathscr{E}$ such that the operator inequality

$$f(A_{\mathscr{E}}) \leqslant Uf(A)_{\mathscr{E}}U^*. \qquad (*)$$

holds. Here, $f(A)_{\mathscr{E}}$ must be read as $(f(A))_{\mathscr{E}}$. This result together with the elementary method of its proof motivate the whole paper. In Section 1 we prove the above inequality and give a version. We also study the map $p \longrightarrow \{(A^p)_{\mathscr{E}}\}^{1/p}$, $0 < p < \infty$ for a positive operator A on a finite dimensional space and d-dimensional subspace $\mathscr{E}$. In general, this map converges to an operator B on $\mathscr{E}$ whose eigenvalues are the d largest eigenvalues of A.

Section 2 is concerned with eigenvalues inequalities (equivalently operator inequalities) which improve some trace inequalities of Brown-Kosaki and Hansen-Pedersen: Given a monotone convex function f defined on the real line with $f(0) \leqslant 0$, a Hermitian operator A and a contractive operator Z acting on a finite dimensional space, there exists a unitary operator U such that

$$f(Z^*AZ) \leqslant UZ^*f(A)ZU^*.$$

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In Section 3, we prove that

$$\mathrm{Tr} f(Z^*AZ) \leqslant \mathrm{Tr}\, Z^* f(A) Z$$

for every positive operator A and expansive operator Z on a finite dimensional space, and every concave function f defined on an interval $[0,b]$, $b \geqslant \|Z^*AZ\|_\infty$, with $f(0) \geqslant 0$ ($\|\cdot\|_\infty$ denotes the usual operator norm). Under the additional assumption that f is nondecreasing and nonnegative, this trace inequality entails

$$\|f(Z^*AZ)\|_\infty \leqslant \|Z^* f(A) Z\|_\infty$$

where stands for the usual operator norm. In a forthcomming project we will extend this result to the infinite dimensional setting. We will also give a version of $(*)$ for infine dimensional subspace $\mathscr{E}$, by adding a rI term in the right hand side, where I stands for the identity and $r > 0$ is arbitrarily small.

1. Compressions and convex functions

By a classical result of C. Davis [4] (see also [1, p. 117–9]), a function f on (a,b) is operator convex if and only if for every subspace $\mathscr{E}$ and every Hermitian operator A whose spectrum lies in (a,b) one has

$$f(A_{\mathscr{E}}) \leqslant f(A)_{\mathscr{E}}. \tag{1}$$

What can be said about convex, not operator convex functions ? Let g be operator convex on (a,b) and let ϕ be a nondecreasing, convex function on $g((a,b))$. Then, $f = \phi \circ g$ is convex and we say that f is *unitary* convex on (a,b). Since $t \longrightarrow -t$ is trivially operator convex, we note that *the class of unitary convex functions contains the class of monotone convex functions*. The following result holds:

THEOREM 1.1. *Let f be a monotone convex, or more generally unitary convex, function on (a,b) and let A be a Hermitian operator whose spectrum lies in (a,b). If $\mathscr{E}$ is a finite dimensional subspace, then there exists a unitary operator U on $\mathscr{E}$ such that*

$$f(A_{\mathscr{E}}) \leqslant U f(A)_{\mathscr{E}} U^*.$$

Proof. We begin by assuming that f is monotone. Let $d = \dim \mathscr{E}$ and let $\{\lambda_k(X)\}_{k=1}^d$ denote the eigenvalues of the Hermitian operator X on $\mathscr{E}$, arranged in decreasing order and counted with their multiplicities. Let k be an integer, $1 \leqslant k \leqslant d$. There exists a spectral subspace $\mathscr{F} \subset \mathscr{E}$ for $A_{\mathscr{E}}$ (hence for $f(A_{\mathscr{E}})$), $\dim \mathscr{F} = k$, such that

$$\begin{aligned}
\lambda_k[f(A_{\mathscr{E}})] &= \min_{h\in\mathscr{F};\ \|h\|=1} \langle h, f(A_{\mathscr{F}})h\rangle \\
&= \min\{f(\lambda_1(A_{\mathscr{F}}))\,;\, f(\lambda_k(A_{\mathscr{F}}))\} \\
&= \min_{h\in\mathscr{F};\ \|h\|=1} f(\langle h, A_{\mathscr{F}}h\rangle) \\
&= \min_{h\in\mathscr{F};\ \|h\|=1} f(\langle h, Ah\rangle)
\end{aligned}$$

where at the second and third steps we use the monotony of f. The convexity of f implies

$$f(\langle h, Ah\rangle) \leqslant \langle h, f(A)h\rangle$$

for all normalized vectors h. Therefore, by the minmax principle,

$$\begin{aligned}\lambda_k[f(A_{\mathscr{E}})] &\leqslant \min_{h\in\mathscr{F};\ \|h\|=1} \langle h, f(A)h\rangle \\ &\leqslant \lambda_k[f(A)_{\mathscr{E}}].\end{aligned}$$

This statement is equivalent to the existence of a unitary operator U on $\mathscr{E}$ satisfying the conclusion of the theorem.

If f is unitary convex, $f = \phi \circ g$ with g operator convex and ϕ nondecreasing convex; inequality (1) applied to g combined with the fact that ϕ is nondecreasing yield a unitary operator V on $\mathscr{E}$ for which

$$\phi \circ g(A_{\mathscr{E}}) \leqslant V\phi[g(A)_{\mathscr{E}}]V^*.$$

Applying the first part of the proof to ϕ gives a unitary operator W on $\mathscr{E}$ such that

$$\phi[g(A)_{\mathscr{E}}] \leqslant W[\phi \circ g(A)]_{\mathscr{E}}W^*.$$

We then get the result by letting $U = VW$. $\square$

Later, we will see that Theorem 1.1 can *not* be extended to all convex functions f (Example 2.4). Of course Theorem 1.1 holds with a reverse inequality for monotone concave functions f (or $f = \phi \circ g$, g operator convex and ϕ decreasing concave).

Given a positive operator A on a finite dimensional space and a subspace $\mathscr{E}$, it is natural to study the behaviour of the map

$$p \longrightarrow \{(A^p)_{\mathscr{E}}\}^{1/p}$$

on $(0,\infty)$. The notation $A = \sum_k \lambda_k(A) f_k \otimes f_k$ means that f_k is a norm one eigenvector associated to $\lambda_k(A)$ and $f_k \otimes f_k$ is the corresponding norm one projection.

THEOREM 1.2. *Let $A = \sum_k \lambda_k(A) f_k \otimes f_k$ be a positive operator on a finite dimensional space and let $\mathscr{E}$ be a subspace. Assume $\mathscr{E} \cap \operatorname{span}\{f_j : j > d\} = 0$. Then, for every integer $k \leqslant \dim \mathscr{E}$, the map $p \longrightarrow \lambda_k(\{(A^p)_{\mathscr{E}}\}^{1/p})$ increases on $(0,\infty)$ and*

$$\lim_{p\to\infty} \lambda_k(\{(A^p)_{\mathscr{E}}\}^{1/p}) = \lambda_k(A).$$

Moreover the family $\{(A^p)_{\mathscr{E}}\}^{1/p}$ converges in norm when $p \to \infty$ and the map $p \longrightarrow \{(A^p)_{\mathscr{E}}\}^{1/p}$ is increasing for the Loewner order on $[1,\infty)$.

Proof. Let $p > 0$ and $r > 1$. By Theorem 1.1, there exists a unitary $U : \mathscr{E} \longrightarrow \mathscr{E}$ such that

$$\{(A^p)_{\mathscr{E}}\}^r \leqslant U(A^{pr})_{\mathscr{E}}U^*,$$

hence, for all k,

$$\lambda_k^r((A^p)_{\mathscr{E}}) \leqslant \lambda_k((A^{pr})_{\mathscr{E}}),$$

so,

$$\lambda_k(\{(A^p)_{\mathscr{E}}\}^{1/p}) \leqslant \lambda_k(\{(A^{pr})_{\mathscr{E}}\}^{1/pr}),$$

that is, the map $p \longrightarrow \lambda_k(\{(A^p)_{\mathscr{E}}\}^{1/p})$ increases on $(0,\infty)$. In order to study its convergence when $p \to \infty$, we first show that

$$\lim_{p\to\infty} \lambda_1((EA^pE)^{1/p}) = \lambda_1(A) \tag{2}$$

where E denotes the projection onto $\mathscr{E}$. We note that

$$\lim_{p\to\infty} \lambda_1((EA^pE)^{1/p}) \leqslant \lambda_1(A). \tag{3}$$

Recall that $A = \sum_k \lambda_k(A) f_k \otimes f_k$. Since by assumption $f_1 \notin \mathscr{E}^\perp$, there exists a normalized vector g in $\mathscr{E}$ such that $\langle g, f_1\rangle \neq 0$. Setting $G = g\otimes g$, we have

$$\lambda_1((GA^pG)^{1/p}) = \langle g, A^p g\rangle^{1/p} = \Big(\sum_k \lambda_k^p(A)|\langle g, f_k\rangle|^2\Big)^{1/p}.$$

The above expression is a weighted l^p-norm of the sequence $\{\lambda_k(A)\}$. When $p \to \infty$, this tends towards the l^∞-norm which is $\lambda_1(A)$. Since

$$\lambda_1((GA^pG)^{1/p}) \leqslant \lambda_1((EA^pE)^{1/p})$$

we then deduce with (3) that (2) holds.

In order to prove the general limit assertion, we consider antisymmetric tensor products. Let F be the projection onto $\mathscr{F} = \operatorname{span}\{f_j : j \leqslant \dim \mathscr{E}\}$. By assumption F maps $\mathscr{E}$ onto $\mathscr{F}$. Therefore $\wedge^k(F)$ maps $\wedge^k(\mathscr{E})$ onto $\wedge^k(\mathscr{F})$ and we may find a norm one tensor $\gamma \in \wedge^k(\mathscr{E})$ such that $\langle \gamma, f_1 \wedge \cdots \wedge f_k\rangle \neq 0$. Hence, with $\wedge^k E$ and $\wedge^k A$ in place of E and A, $1 \leqslant k \leqslant \dim \mathscr{E}$, we may apply (2) to obtain

$$\lim_{p\to\infty} \lambda_1(\wedge^k (EA^pE)^{1/p}) = \lambda_1(\wedge^k A)$$

meaning that

$$\lim_{p\to\infty} \prod_{1\leqslant j\leqslant k} \lambda_j((EA^pE)^{1/p}) = \prod_{1\leqslant j\leqslant k} \lambda_j(A).$$

From these relations we infer that, for every $k \leqslant \dim \mathscr{E}$, we have

$$\lim_{p\to\infty} \lambda_k((EA^pE)^{1/p}) = \lambda_k(A)$$

proving the main assertion of the theorem.

For $p, r \geqslant 1$ we have

$$(EA^{pr}E)^{1/r} \geqslant EA^pE$$

by Hansen's inequality [6]. Since $t \longrightarrow t^{1/p}$ is operator monotone by the Loewner theorem [9, p. 2], we have

$$(EA^{pr}E)^{1/pr} \geqslant (EA^pE)^{1/p}.$$

Thus $p \longrightarrow (EA^pE)^{1/p}$ increases on $[1,\infty)$. Since this map is bounded, it converges in norm. □

The author is indebted to a referee for having pointed out a misconception in the initial statement and proof of Theorem 1.2.

2. Contractions and convex functions

In [6] and [7], the authors show that inequality (1) is equivalent to the following statement.

THEOREM 2.1. (Hansen-Pedersen) *Let A and $\{A_i\}_{i=1}^m$ be Hermitian operators and let f be an operator convex function defined on an interval $[a,b]$ containing the spectra of A and A_i, $i=1,\dots m$.*

(1) *If Z is a contraction, $0\in[a,b]$ and $f(0)\leqslant 0$,*

$$f(Z^*AZ)\leqslant Z^*f(A)Z.$$

(2) *If $\{Z_i\}_{i=1}^m$ is an isometric column,*

$$f(\sum_i Z_i^*A_iZ_i)\leqslant \sum_i Z_i^*f(A_i)Z_i.$$

Here, an isometric column $\{Z_i\}_{i=1}^m$ means that $\sum_{i=1}^m Z_i^*Z_i=I$.

In a similar way, Theorem 1.1 is equivalent to the next one. We state it in the finite dimensional setting, but an analogous version exists in the infinite dimensional setting by adding a rI term in the right hand side of the inequalities.

THEOREM 2.2. *Let A and $\{A_i\}_{i=1}^m$ be Hermitian operators on a finite dimensional space and let f be a monotone, or more generally unitary, convex function defined on an interval $[a,b]$ containing the spectra of A and A_i, $i=1,\dots m$.*

(1) *If Z is a contraction, $0\in[a,b]$ and $f(0)\leqslant 0$, then there exists a unitary operator U such that*

$$f(Z^*AZ)\leqslant UZ^*f(A)ZU^*.$$

(2) *If $\{Z_i\}_{i=1}^m$ is an isometric column, then there exists a unitary operator U such that*

$$f(\sum_i Z_i^*A_iZ_i)\leqslant U\{\sum_i Z_i^*f(A_i)Z_i\}U^*.$$

Here, we give a first proof based on Theorem 1.1. A more direct proof is given at the end of the section.

Proof. Theorem 2.2 and Theorem 1.1 are equivalent. Indeed, to prove Theorem 1.1 we may first assume, by a limit argument, that f is defined on the whole real line. Then, we may assume that $f(0)=0$ so that Theorem 1.1 follows from Theorem 2.2 by taking Z as the projection onto $\mathscr{E}$.

Theorem 1.1 entails Theorem 2.2(1): to see that, we introduce the partial isometry V and the operator $\tilde{A}$ on $\mathscr{H}\oplus\mathscr{H}$ defined by

$$V=\begin{pmatrix} Z & 0\\ (I-|Z|^2)^{1/2} & 0\end{pmatrix},\quad \tilde{A}=\begin{pmatrix} A & 0\\ 0 & 0\end{pmatrix}.$$

Denoting by $\mathscr{H}$ the first summand of the direct sum $\mathscr{H}\oplus\mathscr{H}$, we observe that

$$f(Z^*AZ)=f(V^*\tilde{A}V):\mathscr{H}=V^*f(\tilde{A}_{V(\mathscr{H})})V:\mathscr{H}.$$

Applying Theorem 1.1 with $\mathscr{E} = V(\mathscr{H})$, we get a unitary operator W on $V(\mathscr{H})$ such that

$$f(Z^*AZ) \leqslant V^*Wf(\tilde{A})_{V(\mathscr{H})}W^*V : \mathscr{H}.$$

Equivalently, there exists a unitary operator U on $\mathscr{H}$ such that

$$\begin{aligned} f(Z^*AZ) &\leqslant UV^*f(\tilde{A})_{V(\mathscr{H})}(V : \mathscr{H})U^* \\ &= UV^* \begin{pmatrix} f(A) & 0 \\ 0 & f(0) \end{pmatrix} (V : \mathscr{H})U^* \\ &= U\{Z^*f(A)Z + (I-|Z|^2)^{1/2}f(0)(I-|Z|^2)^{1/2}\}U^*. \end{aligned}$$

Using $f(0) \leqslant 0$ we obtain the first claim of Theorem 2.2.

Similarly, Theorem 1.1 implies Theorem 2.2(2) (we may assume $f(0) = 0$) by considering the partial isometry and the operator on $\oplus^m \mathscr{H}$,

$$\begin{pmatrix} Z_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ Z_m & 0 & \cdots & 0 \end{pmatrix}, \qquad \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_m \end{pmatrix}.$$

□

We note that Theorem 2.2 strengthens some well-known trace inequalities:

COROLLARY 2.3.. *Let A and $\{A_i\}_{i=1}^m$ be Hermitian operators on a finite dimensional space and let f be a convex function defined on an interval $[a,b]$ containing the spectra of A and A_i, $i = 1,\dots m$.*

(1) (Brown-Kosaki [2]) *If Z is a contraction, $0 \in [a,b]$ and $f(0) \leqslant 0$, then*

$$\mathrm{Tr} f(Z^*AZ) \leqslant \mathrm{Tr}\, Z^*f(A)Z.$$

(2) (Hansen-Pedersen [7]) *If $\{Z_i\}_{i=1}^m$ is an isometric column, then*

$$\mathrm{Tr} f\Big(\sum_i Z_i^*A_iZ_i\Big) \leqslant \mathrm{Tr}\Big\{\sum_i Z_i^*f(A_i)Z_i\Big\}.$$

Proof. By a limit argument, we may assume that f is defined on the whole real line and can be written as $f(x) = g(x) - \lambda x$ for some convex monotone function g and some scalar λ. We then apply Theorem 2.2 to g. □

A very special case of Theorem 2.2(2) is: *Given two Hermitian operators A, B and a monotone convex or unitary convex function f on a suitable interval, there exists a unitary operator U such that*

$$f\left(\frac{A+B}{2}\right) \leqslant U\frac{f(A)+f(B)}{2}U^*.$$

This shows that Theorem 2.2, and consequently Theorem 1.1, can not be valid for all convex functions:

EXAMPLE 2.4. Theorems 1.1 and 2.2 are not valid for a simple convex function such as $t \longrightarrow |t|$. Indeed, it is well-known that the inequality

$$|A+B| \leqslant U(|A|+|B|)U^* \tag{5}$$

is not always true, even for Hermitians A, B. We reproduce the counterexample [8, p. 1]: Take

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}.$$

Then, as the two eigenvalues of $|A+B|$ equal to $\sqrt{2}$ while $|A|+|B|$ has an eigenvalue equal to $2-\sqrt{2}$, inequality (5) can not hold.

In connection with Example 2.4, a famous result (e.g., [1, p. 74]) states the existence, for any operators A, B on a finite dimensional space, of unitary operators U, V such that

$$|A+B| \leqslant U|A|U^* + V|B|V^*. \tag{6}$$

In the case of Hermitians A, B, the above inequality has the following generalization:

PROPOSITION 2.5. *Let A, B be hermitian operators on a finite dimensional space and let f be an even convex function on the real line. Then, there exist unitary operators U, V such that*

$$f\left(\frac{A+B}{2}\right) \leqslant \frac{Uf(A)U^* + Vf(B)V^*}{2}.$$

Proof. Since $f(X) = f(|X|)$, inequality (6) and the fact that f is increasing on $[0,\infty)$ give unitary operators U_0, V_0 such that

$$f\left(\frac{A+B}{2}\right) \leqslant f\left(\frac{U_0|A|U_0^* + V_0|B|V_0^*}{2}\right).$$

Since f is monotone convex on $[0,\infty)$, Theorem 2.2 completes the proof. □

QUESTION 2.6. Does Proposition 2.5 hold for all convex functions defined on the whole real line?

We close this section by giving a direct and proof of Theorem 2.2, which is a simple adaptation of the proof of Theorem 1.1.

Proof. We restrict ourselves to the case when f is monotone. We will use the following observation which follows from the standard Jensen's inequality: for any vector u of norm less than or equal to one, since f is convex and $f(0) \leqslant 0$,

$$f(\langle u, Au \rangle) \leqslant \langle u, f(A)u \rangle.$$

We begin by proving assertion (1). We have, for each integer k less than or equal to the dimension of the space, a subspace $\mathscr{F}$ of dimension k such that

$$\begin{aligned}\lambda_k[f(Z^*AZ)] &= \min_{h\in\mathscr{F};\ \|h\|=1} \langle h, f(Z^*AZ)h\rangle \\ &= \min_{h\in\mathscr{F};\ \|h\|=1} f(\langle h, Z^*AZh\rangle) \\ &= \min_{h\in\mathscr{F};\ \|h\|=1} f(\langle Zh, AZh\rangle).\end{aligned}$$

where we have used the monotony of f. Then, using the above observation and the minmax principle,

$$\begin{aligned}\lambda_k[f(Z^*AZ)] &\leqslant \min_{h\in\mathscr{F};\ \|h\|=1} \langle Zh, f(A)Zh\rangle \\ &\leqslant \lambda_k[Z^*f(A)Z].\end{aligned}$$

We turn to assertion (2). For any integer k less than or equal to the dimension of the space, we have a subspace $\mathscr{F}$ of dimension k such that

$$\begin{aligned}\lambda_k[f(\sum Z_i^*A_iZ_i)] &= \min_{h\in\mathscr{F};\ \|h\|=1} \langle h, f(\sum Z_i^*A_iZ_i)h\rangle \\ &= \min_{h\in\mathscr{F};\ \|h\|=1} f(\langle h, \sum Z_i^*A_iZ_ih\rangle) \\ &= \min_{h\in\mathscr{F};\ \|h\|=1} f(\sum \|Z_ih\|^2(\langle Z_ih, A_iZ_ih\rangle/\|Z_ih\|^2)) \\ &\leqslant \min_{h\in\mathscr{F};\ \|h\|=1} \sum \|Z_ih\|^2 f(\langle Z_ih, A_iZ_ih\rangle/\|Z_ih\|^2) \qquad (7) \\ &\leqslant \min_{h\in\mathscr{F};\ \|h\|=1} \sum \langle Z_ih, f(A_i)Z_ih\rangle) \qquad (8) \\ &\leqslant \min_{h\in\mathscr{F};\ \|h\|=1} \langle h, \sum Z_i^*f(A_i)Z_ih\rangle) \\ &\leqslant \lambda_k[\sum Z_i^*f(A_i)Z_i]\end{aligned}$$

where we have used in (7) and (8) the convexity of f. □

3. Inequalities involving expansive operators

In this section we are in the finite dimensional setting.

For two reals a, z, with $z > 1$, we have $f(za) \geqslant zf(a)$ for every convex function f with $f(0) \leqslant 0$. In view of Theorem 2.2, one might expect the following result: If Z is an expansive operator (i.e. $Z^*Z \geqslant I$), A is a Hermitian operator and f is a convex function with $f(0) \leqslant 0$, then there exists a unitary operator U such that

$$f(Z^*AZ) \geqslant UZ^*f(A)ZU^*. \qquad (*)$$

But, as we shall see, this is not always true, even for $A \geqslant 0$ and f nonnegative with $f(0) = 0$. Let us first note the following remark:

REMARK 3.1. Let $f : [0,\infty) \longrightarrow [0,\infty)$ be a continuous function with $f(0) = 0$. If

$$\operatorname{Tr} f(Z^*AZ) \leqslant \operatorname{Tr} Z^* f(A) Z$$

for every positive operator A and every contraction Z, then f is convex.

To check this, it suffices to consider:

$$A = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \end{pmatrix}$$

where x, y are arbitrary nonnegative scalars. Indeed, $\operatorname{Tr} f(Z^*AZ) = f((x+y)/2)$ and $\operatorname{Tr} Z^* f(A) Z = (f(x)+f(y))/2$.

We may now state

PROPOSITION 3.2. *Let $f : [0,\infty) \longrightarrow [0,\infty)$ be a continuous one to one function with $f(0) = 0$ and $f(\infty) = \infty$. Then, the following conditions are equivalent:*

(1) *The function $g(t) = 1/f(1/t)$ is convex on $[0,\infty)$.*

(2) *For every positive operator A and every expansive operator Z, there exists a unitary operator U such that*

$$Z^* f(A) Z \leqslant U f(Z^*AZ) U^*.$$

Proof. We may assume that A is invertible. If g is convex, (note that g is also nondecreasing) then Theorem 2.2 entails that

$$g(Z^{-1}A^{-1}Z^{-1*}) \leqslant U^* Z^{-1} g(A^{-1}) Z^{-1*} U$$

for some unitary operator U. Taking the inverses, since $t \longrightarrow t^{-1}$ is operator decreasing on $(0,\infty)$, this is the same as saying

$$Z^* f(A) Z \leqslant U f(Z^*AZ) U^*.$$

The converse direction follows, again by taking the inverses, from the above remark. □

It is not difficult to find convex functions $f : [0,\infty) \longrightarrow [0,\infty)$, with $f(0) = 0$ which do not satisfy to the conditions of Proposition 3.2. So, in general, $(*)$ can not hold. Let us give an explicit simple example.

EXAMPLE 3.3. Let $f(t) = t + (t-1)_+$ and

$$A = \begin{pmatrix} 3/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \qquad Z = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then $\lambda_2(f(ZAZ)) = 0.728\ldots < 0.767\ldots = \lambda_2(Zf(A)Z)$. So, $(*)$ does not hold.

In spite of the previous example, we have the following positive result:

LEMMA 3.4. *Let A be a positive operator, let Z be an expansive operator and β be a nonnegative scalar. Then, there exists a unitary operator U such that*

$$Z^*(A-\beta I)_+Z \leqslant U(Z^*AZ-\beta I)_+U^*.$$

Proof. We will use the following simple fact: If B is a positive operator with $\mathrm{Sp}B \subset \{0\}\cup(x,\infty)$, then we also have $\mathrm{Sp}Z^*BZ \subset \{0\}\cup(x,\infty)$. Indeed Z^*BZ and $B^{1/2}ZZ^*B^{1/2}$ (which is greater than B) have the same spectrum.

Let P be the spectral projection of A corresponding to the eigenvalues strictly greater than β and let $A_\beta = AP$. Since $t \longrightarrow t_+$ is nondecreasing, there exists a unitary operator V such that

$$(Z^*AZ-\beta I)_+ \geqslant V(Z^*A_\beta Z-\beta I)_+V^*.$$

Since $Z^*(A-\beta I)_+Z = Z^*(A_\beta-\beta I)_+Z$ we may then assume that $A = A_\beta$. Now, the above simple fact implies

$$(Z^*A_\beta Z-\beta I)_+ = Z^*A_\beta Z-\beta Q$$

where $Q = \mathrm{supp}Z^*A_\beta Z$ is the support projection of $Z^*A_\beta Z$. Hence, it suffices to show the existence of a unitary operator W such that

$$Z^*A_\beta Z-\beta Q \geqslant WZ^*(A_\beta-\beta P)ZW^* = WZ^*A_\beta ZW^*-\beta WZ^*PZW^*.$$

But, here we can take $W = I$. Indeed, we have

$$\mathrm{supp}Z^*PZ = Q \quad (*) \quad \text{and} \quad \mathrm{Sp}Z^*PZ \subset \{0\}\cup[1,\infty) \quad (**)$$

where $(**)$ follows from the above simple fact and the identity $(*)$ from the observation below with $X = P$ and $Y = A_\beta$.

Observation. If X, Y are two positive operators with $\mathrm{supp}X = \mathrm{supp}Y$, then for every operator Z we also have $\mathrm{supp}Z^*XZ = \mathrm{supp}Z^*YZ$.

To check this, we establish the corresponding equality for the kernels,

$$\ker Z^*XZ = \{h\ :\ Zh \in \ker X^{1/2}\} = \{h\ :\ Zh \in \ker Y^{1/2}\} = \ker Z^*YZ. \qquad \square$$

THEOREM 3.5. *Let A be a positive operator and Z be an expansive operator. Assume that f is a continuous function defined on $[0,b]$, $b \geqslant \|Z^*AZ\|_\infty$. Then,*

(1) *If f is concave and $f(0) \geqslant 0$,*

$$\mathrm{Tr}\, f(Z^*AZ) \leqslant \mathrm{Tr}\, Z^*f(A)Z.$$

(2) *If f is convex and $f(0) \leqslant 0$,*

$$\mathrm{Tr}\, f(Z^*AZ) \geqslant \mathrm{Tr}\, Z^*f(A)Z.$$

EXAMPLE 3.6. Here, contrary to the Brown-Kosaki trace inequalities (Corollary 2.3(1)), the assumption $A \geqslant 0$ is essential. For instance, in the convex case, consider $f(t) = t_+$,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad Z = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then, we have $\mathrm{Tr} f(Z^*AZ) = 3 < 5 = \mathrm{Tr}\, Z^* f(A) Z$. Of course, the assumption $A \geqslant 0$ is also essential in Lemma 3.4.

We turn to the proof of Theorem 3.5.

Proof. Of course, assertions (1) and (2) are equivalent. Let us prove (2). Since Z is expansive we may assume that $f(0) = 0$. By a limit argument we may then assume that

$$f(t) = \lambda t + \sum_{i=1}^{m} \alpha_i (t - \beta_i)_+$$

for a real λ and some nonnegative reals $\{\alpha_i\}_{i=1}^m$ and $\{\beta_i\}_{i=1}^m$. The result then follows from the linearity of the trace and Lemma 3.4. □

In order to extend Theorem 3.5(2) to all unitarily invariant norms, i.e. those norms $\|\cdot\|$ such that $\|UXV\| = \|X\|$ for all operators X and all unitaries U and V, we need a simple lemma. A family of positive operators $\{A_i\}_{i=1}^m$ is said to be *monotone* if there exists a positive operator Z and a family of nondecreasing nonnegative functions $\{f_i\}_{i=1}^m$ such that $f_i(Z) = A_i$, $i = 1, \dots m$.

LEMMA 3.7. *Let $\{A_i\}_{i=1}^m$ be a monotone family of positive operators and let $\{U_i\}_{i=1}^m$ be a family of unitary operators. Then, for every unitarily invariant norm $\|\cdot\|$, we have*

$$\|\sum_i U_i A_i U_i^*\| \leqslant \|\sum_i A_i\|.$$

Proof. By the Ky Fan dominance principle, it suffices to consider the Ky Fan k-norms $\|\cdot\|_{(k)}$ [1, pp. 92–3]. There exists a rank k projection E such that

$$\|\sum_i U_i A_i U_i^*\|_{(k)} = \sum_i \mathrm{Tr}\, U_i A_i U_i^* E \leqslant \sum_i \|A_i\|_{(k)} = \|\sum_i A_i\|_{(k)}$$

where the inequality comes from the maximal characterization of the Ky Fan norms and the last equality from the monotony of the family $\{A_i\}$. □

PROPOSITION 3.8. *Let A be a positive operator and Z be an expansive operator. Assume that f is a nonnegative convex function defined on $[0,b]$, $b \geqslant \|Z^*AZ\|_\infty$. Assume also that $f(0) = 0$. Then, for every unitarily invariant norm $\|\cdot\|$,*

$$\|f(Z^*AZ)\| \geqslant \|Z^* f(A) Z\|.$$

Proof. It suffices to consider the case when

$$f(t) = \lambda t + \sum_{i=1}^{m} \alpha_i (t - \beta_i)_+$$

for some nonnegative reals λ, $\{\alpha_i\}_{i=1}^m$ and $\{\beta_i\}_{i=1}^m$. By Lemma 3.4, we have

$$\begin{aligned} Z^* f(A) Z &= \lambda Z^*AZ + \sum_i Z^* \alpha_i (A - \beta_i I)_+ Z \\ &\leqslant \lambda Z^*AZ + \sum_i U_i \alpha_i (Z^*AZ - \beta_i I)_+ U_i^* \end{aligned}$$

for some unitary operators $\{U_i\}_{i=1}^m$. Since λZ^*AZ and $\{\alpha_i(Z^*AZ-\beta_i I)_+\}_{i=1}^m$ form a monotone family, Lemma 3.7 completes the proof. □

THEOREM 3.9. *Let A be a positive operator, let Z be an expansive operator and let $f:[0,\infty)\longrightarrow[0,\infty)$ be a nondecreasing concave function. Then,*

$$\|f(Z^*AZ)\|_\infty \leqslant \|Z^*f(A)Z\|_\infty.$$

Proof. Here, we assume that we are in the finite dimensional setting.

Since Z is expansive we may assume $f(0)=0$. By a continuity argument we may assume that f is onto. Let g be the reciprocal function. Note that g is convex and $g(0)=0$. By Proposition 3.8,

$$\|g(Z^*AZ)\|_\infty \geqslant \|Z^*g(A)Z\|_\infty.$$

Hence

$$f(\|g(Z^*AZ)\|_\infty) \geqslant f(\|Z^*g(A)Z\|_\infty).$$

Equivalently,

$$\|Z^*AZ\|_\infty \geqslant \|f(Z^*g(A)Z)\|_\infty,$$

so, letting $B=g(A)$,

$$\|Z^*f(B)Z\|_\infty \geqslant \|f(Z^*BZ)\|_\infty,$$

proving the result because $A\longrightarrow g(A)$ is onto. □

Our next result is a straightforward application of Theorem 2.2.

PROPOSITION 3.10. *Let A be a positive operator and Z be an expansive operator. Assume that f is a nonnegative function defined on $[0,b]$, $b\geqslant\|Z^*AZ\|_\infty$. Then:*

(1) *If f is concave nondecreasing,*

$$\det f(Z^*AZ) \leqslant \det Z^*f(A)Z.$$

(2) *If f is convex increasing and $f(0)=0$,*

$$\det f(Z^*AZ) \geqslant \det Z^*f(A)Z.$$

Proof. For instance, consider the concave case. By Theorem 2.2, there exists a unitary operator U such that $Z^{*-1}f(Z^*AZ)Z^{-1}\leqslant Uf(A)U^*$; hence the result follows. □

We note the following fact about operator convex functions:

PROPOSITION 3.11. *Let $f:[0,\infty)\longrightarrow[0,\infty)$ be a one to one continuous function with $f(0)=0$ and $f(\infty)=\infty$. The following statements are equivalent:*

(i) *$f(t)$ is operator convex.*

(ii) *$1/f(1/t)$ is operator convex.*

Proof. Since the map $f(t)\longrightarrow\Psi(f)(t)=1/f(1/t)$ is an involution on the set of all one to one continuous functions f on $[0,\infty)$ with $f(0)=0$ and $f(\infty)=\infty$, it suffices to check that (i) $\Rightarrow$ (ii). But, by the Hansen-Pedersen inequality [6], (i) is equivalent to

$$f(Z^*AZ) \leqslant Z^*f(A)Z \tag{9}$$

for all $A \geqslant 0$ and all contractions Z. By a limit argument, it suffices to require (9) when both A and Z are invertible. Then, as $t \longrightarrow t^{-1}$ is operator decreasing, (9) can be written

$$f^{-1}(Z^*AZ) \geqslant Z^{-1}f^{-1}(A)Z^{*-1},$$

or

$$f^{-1}(A) \leqslant Zf^{-1}(Z^*AZ)Z^*,$$

but this is the same as saying that (9) holds for $\Psi(f)$, therefore $\Psi(f)$ is operator convex. $\square$

We wish to sketch another proof of Proposition 3.10. By a result of Hansen and Pedersen [6], for a continuous function f on $[0,\infty)$, the following conditions are equivalent:

(i) $f(0) \leqslant 0$ and f is operator convex.

(ii) $t \longrightarrow f(t)/t$ is operator monotone on $(0,\infty)$.

Using the operator monotony of $t \longrightarrow 1/t$ on $(0,\infty)$, we note that if $f(t)$ satisfies to (ii), then so does $1/f(1/t)$. This proves Proposition 3.10.

QUESTION 3.12. Does Theorem 3.9 extend to all nonnegative concave functions on $[0,b]$ and/or to all unitarily invariant norms?

4. Addendum

There exist several inequalities involving $f(A+B)$ and $f(A)+f(B)$ where A, B are Hermitians and f is a function with special properties. We wish to state and prove one of the most basic results in this direction which can be derived from a more general result due to Rotfel'd (see [1, p. 97]). The simple proof given here is inspired by that of Theorem 3.5.

PROPOSITION 4.1. (Rotfel'd) *Let A, B be positive operators.*

(1) *If f is a convex nonnegative function on $[0,\infty)$ with $f(0) \leqslant 0$, then*

$$\mathrm{Tr} f(A+B) \geqslant \mathrm{Tr} f(A) + \mathrm{Tr} f(B).$$

(2) *If f is a concave nonnegative function on $[0,\infty)$, then*

$$\mathrm{Tr}\, g(A+B) \leqslant \mathrm{Tr}\, g(A) + \mathrm{Tr}\, g(B).$$

Proof. By limit arguments, we may assume that we are in the finite dimensional setting. Since, on any compact interval $[a,b]$, $a > 0$, we may write $g(x) = \lambda x - f(x) + \mu$ for some scalar $\lambda, \mu \geqslant 0$ and some convex function f with $f(0) = 0$, it suffices to consider the convex case. Clearly we may assume $f(0) = 0$. Then, f can be uniformly approximated, on any compact interval, by a positive combination of functions $f_\alpha(x) = \max\{0, x-\alpha\}$, $\alpha > 0$.

Therefore, still using the notation S_+ for the positive part of the Hermitian operator S, we need only to show that

$$\operatorname{Tr}(A+B-\alpha)_+ \geqslant \operatorname{Tr}(A-\alpha)_+ + \operatorname{Tr}(B-\alpha)_+.$$

To this end, consider an orthonormal basis $\{e_i\}_{i=1}^n$ of eigenvectors for $A+B$. We note that:

(a) If $\langle e_i,(A+B-\alpha)_+e_i\rangle = 0$, then $\langle e_i,(A+B-\alpha)_+e_i\rangle \leqslant \alpha$ so that we also have $\langle e_i,(A-\alpha)_+e_i\rangle = \langle e_i,(B-\alpha)_+e_i\rangle = 0$.

(b) If $\langle e_i,(A+B-\alpha)_+e_i\rangle > 0$, then we may write

$$\langle e_i,(A+B-\alpha)_+e_i\rangle = \langle e_i,Ae_i\rangle - \theta\alpha + \langle e_i,Be_i\rangle - (1-\theta)\alpha$$

for some $0 \leqslant \theta \leqslant 1$ chosen in such a way that $\langle e_i,Ae_i\rangle - \theta\alpha \geqslant 0$ and $\langle e_i,Be_i\rangle - (1-\theta)\alpha \geqslant 0$. Hence, we have

$$\langle e_i,Ae_i\rangle - \theta\alpha = \langle e_i,(A-\theta\alpha)_+e_i\rangle \geqslant \langle e_i,(A-\alpha)_+e_i\rangle$$

and

$$\langle e_i,Be_i\rangle - (1-\theta)\alpha = \langle e_i,(B-(1-\theta)\alpha)_+e_i\rangle \geqslant \langle e_i,(B-\alpha)_+e_i\rangle$$

by using the simple fact that for two commuting Hermitian operators S,T, $S \leqslant T \Rightarrow S_+ \leqslant T_+$.

From (a) and (b) we derive the desired trace inequality by summing over $i = 1,\dots n$. □

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Jean-Christophe Bourin
Les Coteaux
8 rue Henri Durel 78510 Triel
France
e-mail: bourinjc@club-internet.fr