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NORM INEQUALITIES INVOLVING MATRIX MONOTONE FUNCTIONS

M. SINGH AND H. L. VASUDEVA

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Abstract. Let A,B,X be complex matrices with A,B Hermitian positive definite and let $f:(0,\infty)\to(0,\infty)$ be matrix monotone increasing. We prove

$$(2+t)\ |||\ A^{\frac{1}{2}}(f(A)Xf^{\perp}(B)+f^{\perp}(A)Xf(B))B^{\frac{1}{2}}\ |||\leqslant 2\ \ |||\ A^2X+tAXB+XB^2\ |||$$

and

$$(2+t)\ |||\ f(A)X+Xf(B)\ |||\leqslant 2\,\frac{f(\lambda)}{\lambda}\ |||\ A^{\frac{3}{2}}XB^{-\frac{1}{2}}+tA^{\frac{1}{2}}XB^{\frac{1}{2}}+A^{-\frac{1}{2}}XB^{\frac{3}{2}}\ |||$$

where $f^{\perp}(x)=x(f(x))^{-1}$, $t\in[-2,2]$ and $\lambda=\min\{\sigma(A),\sigma(B)\}$; $\sigma(A),\sigma(B)$ being the spectrum of A,B respectively and $|||\ .\ |||$ any unitarily invariant norm. These inequalities generalize Zhan's inequalities.

1. Introduction

Let $M_{m,n}$ be the space of complex matrices and $M_n=M_{n,n}$. The Hadamard product of two matrices $A=(a_{ij})_{i,j}$ and $B=(b_{ij})_{i,j}$ in M_n is defined to be the matrix $A\circ B$ whose $i,j-$entry is $a_{ij}b_{ij}$. By $\sigma(A)$ and $\sigma(B)$, we denote the spectrum of A and B. The symbol $|||\ .\ |||$ will denote a unitarily invariant norm whereas $||\ .\ ||$ will denote the spectral norm throughout this paper. In [4] Bhatia and Kittaneh proved an arithmetic geometric mean inequality for arbitrary matrices in M_n. This says that for all $A,B\in M_n$ and all unitarily invariant norms $|||\cdot|||$

$$2\ |||\ A^*B\ |||\leqslant|||\ AA^*+BB^*\ |||\ .$$

In [3] and later in [8] this was strengthened to

$$2\ |||\ A^*XB\ |||\leqslant|||\ AA^*X+XBB^*\ ||| \tag{1}$$

for all $A,B,X\in M_n$. The insertion of X is no idle generalization. A judicious choice can lead to powerful perturbation theorems. This has been demonstrated in [7]. In [11] Zhan observed that the converse of (1) is also true: that is, if (1) holds for all $A,B,X\in M_n$ then $|||\cdot|||$ must be unitarily invariant norm. In [11] Zhan generalized

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the inequality (1) by introducing two parameters r and t. He showed that for complex matrices A,B,X with A,B Hermitian positive semidefinite

$$(2+t)\ |||\ A^rXB^{2-r}+A^{2-r}XB^r\ |||\leqslant 2\ |||\ A^2X+tAXB+XB^2\ ||| \tag{2}$$

for r,t real numbers satisfying $1\leqslant 2r\leqslant 3,\ \ -2<t\leqslant 2.$ The case $r=1,\ t=0$ reduces to (1).

For Hermitian matrices A and B, we write $A\geqslant B$ to mean $A-B$ is positive semidefinite. A real-valued function f is said to be matrix monotone increasing on a real interval J if for all Hermitian matrices A and B of all orders whose eigenvalues lie in J

$$A\geqslant B \text{ implies } f(A)\geqslant f(B).$$

It is matrix monotone decreasing if the inequality is reversed after the application of f. A matrix monotone increasing function $f:(0,\infty)\to(0,\infty)$ can be represented as follows

$$f(x)=\alpha+\beta x+\int_0^\infty \frac{x}{x+s}d\mu(s), \tag{3}$$

where $\alpha,\ \beta\geqslant 0$ and μ is a positive measure (see [5] p. 226 and [10] p. 133).

The functions x^p $(0\leqslant p\leqslant 1)$, $\log x$ and $\log(1+x)$ are some examples of monotone matrix functions on $(0,\infty)$.

Our aim here is to obtain a stronger version of the above inequality (2) in the same spirit. We will prove

THEOREM 1.1. *Let $A\in M_m$, $B\in M_n$ be Hermitian positive definite and $X\in M_{m,n}$ be arbitrary. Then for matrix monotone increasing $f:(0,\infty)\to(0,\infty)$ and $t\in[-2,2]$, the inequality*

$$(2+t)\ |||\ A^{\frac12}(f(A)Xf^{\perp}(B)+f^{\perp}(A)Xf(B))B^{\frac12}\ |||\leqslant 2\ |||\ A^2X+tAXB+XB^2\ |||$$

holds, where $f^{\perp}(x)=x(f(x))^{-1}$ and $A^{\frac12}$ is the unique positive definite square root of A.

Following the techniques employed in the proof of Theorem 1.1, we obtain the following generalization of a theorem of Zhan (Theorem 9, [11])

$$(2+t)\ |||\ A^{\frac12}[\log(I+A)X+X\log(I+B)]B^{\frac12}\ |||\leqslant\ 2\ |||\ A^2X+tAXB+XB^2\ |||\ .$$

Its proof being no different from that of Theorem 1.1 is only sketched.

THEOREM 1.2. *Let $A\in M_m$, $B\in M_n$ be Hermitian positive definite and $X\in M_{m,n}$ be arbitrary. Then for matrix monotone increasing function $f:(0,\infty)\to(0,\infty)$, $t\in[-2,2]$ and $\lambda=\min\{\sigma(A),\sigma(B)\}$, the inequality*

$$(2+t)\ |||\ f(A)X+Xf(B)\ |||\leqslant\ 2\,\frac{f(\lambda)}{\lambda}\ \ |||\ A^{\frac32}XB^{-\frac12}+tA^{\frac12}XB^{\frac12}+A^{-\frac12}XB^{\frac32}\ |||$$

holds.

2. Norm inequalities

Let $A \in M_n$ be fixed. Define the linear map $\mathscr{T}_A : M_n \to M_n$ by $\mathscr{T}_A(X) = A \circ X$, for $X \in M_n$ (also called Hadamard multiplier). Let $\| \mathscr{T}_A \|$ denote the induced Hadamard multiplier spectral norm of $\mathscr{T}_A$:

$$\| \mathscr{T}_A \| = \max\{\| A \circ X \|, X \in M_n : \| X \| \leqslant 1\}.$$

It is not easy to compute $\| \mathscr{T}_A \|$ for a general matrix A, but in the special case when $A = (a_{ij})_{i,j}$ is Hermitian positive semidefinite it is known that (see [1])

$$\| \mathscr{T}_A \| = \max\{a_{ii} : i = 1, 2, \cdots, n\}.$$

The following lemma, a proof of which can be found in [1] (also see [11]) will be the main tool in proving our results.

LEMMA 2.1. *For $A, B \in M_n$, the inequality*

$$||| A \circ B ||| \leqslant \| \mathscr{T}_A \| \; ||| B |||$$

holds.

The following result due to Kwong [10] will be needed in the sequel, also see [9].

LEMMA 2.2. *Let the function $g(x) = f(x) + h(x)$ with f positive matrix monotone increasing and h positive matrix monotone decreasing. Then for any Hermitian positive definite $A \in M_n$ and Hermitian positive semidefinite $P \in M_n$ the solution X of the matrix equation*

$$A^2X + tAXA + XA^2 = g(A)P + Pg(A),$$

is Hermitian positive semidefinite for all $t \in (-2, 2]$.

Our statement of Lemma 2.2 given above is a slight variation on Theorem 10, [10], which results using continuity arguement.

Now, taking P, the matrix whose all entries are 1, $A = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$, $\lambda_i > 0$, and $g(x) = f(x)$, a matrix monotone increasing or matrix monotone decreasing function, we obtain

$$A^2X + tAXA + XA^2 = ((\lambda_i^2 + t\lambda_i\lambda_j + \lambda_j^2)x_{ij})_{i,j}$$

and

$$g(A)P + Pg(A) = (f(\lambda_i) + f(\lambda_j))_{i,j}.$$

Thus from Lemma 2.2, we obtain

$$X = (x_{ij})_{i,j} = \left(\frac{f(\lambda_i) + f(\lambda_j)}{\lambda_i^2 + t\lambda_i\lambda_j + \lambda_j^2} \right)_{i,j} \tag{4}$$

is Hermitian positive semidefinite.

Bhatia and Parthasarthy [5] also gave the proof of (4) in case $f(x) = x^r$, $r \in [-1, 1]$ and showed that the interval $(-2, 2]$ of t is the largest possible for the conclusion to hold.

LEMMA 2.3. *Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be positive real numbers and $t \in (-2,2]$. Let $f:(0,\infty) \to (0,\infty)$ be a matrix monotone increasing function. The following $n \times n$ matrix is Hermitian positive semidefinite:*

$$\left(\frac{f(\lambda_i)^2\lambda_i^{-1} + f(\lambda_j)^2\lambda_j^{-1}}{\lambda_i^2 + t\lambda_i\lambda_j + \lambda_j^2}\right)_{i,j}. \tag{5}$$

Proof. Using (3) and Fubini's Theorem, we obtain $f(x)^2 = \alpha^2 + 2\alpha\beta x + \beta^2 x^2 + 2\alpha \int\limits_0^\infty \frac{x}{x+s} d\mu(s) + 2\beta \int\limits_0^\infty \frac{x^2}{x+s} d\mu(s) + \int\limits_0^\infty \int\limits_0^\infty \frac{x^2}{(x+s_1)(x+s_2)} d(\mu \times \mu)(s_1,s_2)$.

Now, on replacing $f(x)^2$ by 1, x, x^2, $\frac{x}{x+s}$ and $\frac{x^2}{x+s}$; $s \geqslant 0$, in (5) and using (4), it follows that, the matrix $\left(\frac{f(\lambda_i)^2\lambda_i^{-1}+f(\lambda_j)^2\lambda_j^{-1}}{\lambda_i^2+t\lambda_i\lambda_j+\lambda_j^2}\right)_{i,j}$ is Hermitian positive semidefinite in each of the above cases. Finally it remains to check the positive semidefinitness of $\left(\frac{f(\lambda_i)^2\lambda_i^{-1}+f(\lambda_j)^2\lambda_j^{-1}}{\lambda_i^2+t\lambda_i\lambda_j+\lambda_j^2}\right)_{i,j}$ for $f(x)^2 = \frac{x^2}{(x+s_1)(x+s_2)}$; $s_1, s_2 \geqslant 0$. To see this, note that

$$\left(\frac{f(\lambda_i)^2\lambda_i^{-1}+f(\lambda_j)^2\lambda_j^{-1}}{\lambda_i^2+t\lambda_i\lambda_j+\lambda_j^2}\right)_{i,j} = \left(\frac{\lambda_i\lambda_j(\lambda_i+\lambda_j)+2\lambda_i\lambda_j(s_1+s_2)+s_1s_2(\lambda_i+\lambda_j)}{(\lambda_i+s_1)(\lambda_i+s_2)[\lambda_i^2+t\lambda_i\lambda_j+\lambda_j^2](\lambda_j+s_1)(\lambda_j+s_2)}\right)_{i,j}$$
$$= \left(\frac{\lambda_i}{(\lambda_i+s_1)(\lambda_i+s_2)}\left(\frac{(\lambda_i+\lambda_j)+2(s_1+s_2)+s_1s_2(\lambda_i^{-1}+\lambda_j^{-1})}{\lambda_i^2+t\lambda_i\lambda_j+\lambda_j^2}\right)\frac{\lambda_j}{(\lambda_j+s_1)(\lambda_j+s_2)}\right)_{i,j}$$

is Hermitian positive semidefinite if and only if the matrix

$$\left(\frac{(\lambda_i+\lambda_j)+2(s_1+s_2)+s_1s_2(\lambda_i^{-1}+\lambda_j^{-1})}{\lambda_i^2+t\lambda_i\lambda_j+\lambda_j^2}\right)_{i,j} \tag{6}$$

is Hermitian positive semidefinite. Now once again in view of (4), the matrix (6), being the sum of three Hermitian positive semidefinite matrices $\left(\frac{\lambda_i+\lambda_j}{\lambda_i^2+t\lambda_i\lambda_j+\lambda_j^2}\right)_{i,j}$, $\left(\frac{2(s_1+s_2)}{\lambda_i^2+t\lambda_i\lambda_j+\lambda_j^2}\right)_{i,j}$ and $\left(\frac{\lambda_i^{-1}+\lambda_j^{-1}}{\lambda_i^2+t\lambda_i\lambda_j+\lambda_j^2}\right)_{i,j}$, is Hermitian positive semidefinite. This completes the proof. □

Proof of Theorem 1.1. It is sufficient to prove the result for $t \in (-2,2]$, since the case $t=-2$ is trivial. We shall also confine ourself to the case $m=n$, since the non square matrices can be augmented to the square matrices with zero blocks and this change does not effect their unitarily invariant norms.
We first prove the result for $B=A$, i.e.,

$$(2+t)\ |||\ A^{\frac{1}{2}}(f(A)Xf^{\perp}(A)+f^{\perp}(A)Xf(A))A^{\frac{1}{2}}\ ||| \leqslant 2\ |||\ A^2X+tAXA+XA^2\ |||. \tag{7}$$

Let $A = \operatorname{diag}(\lambda_1,\lambda_2,\cdots,\lambda_n)$; $\lambda_i > 0$. (For non-diagonal Hermitian positive definite A, write $A = U^\star \operatorname{diag} A U$ with U a unitary matrix. Since $|||\ .\ |||$ is unitarily invariant,

$X \in M_n$ in (7) gets replaced by $Y \in M_n$, $Y = U^\star XU$. This neither affects the inequality nor does it effect its proof). Then

$$\begin{aligned}
&A^{\frac{1}{2}}(f(A)Xf^{\perp}(A) + f^{\perp}(A)Xf(A))A^{\frac{1}{2}} \\
&\quad = \left((\lambda_i^{\frac{1}{2}}(f(\lambda_i)f^{\perp}(\lambda_j) + f^{\perp}(\lambda_i)f(\lambda_j))\lambda_j^{\frac{1}{2}})x_{ij}\right)_{i,j} \\
&\quad = \left(\frac{\lambda_i^{\frac{1}{2}}(f(\lambda_i)f^{\perp}(\lambda_j) + f^{\perp}(\lambda_i)f(\lambda_j))\lambda_j^{\frac{1}{2}}}{\lambda_i^2 + t\lambda_i\lambda_j + \lambda_j^2}\right)_{i,j} \circ (A^2X + tAXA + XA^2) \\
&\quad = Z \circ (A^2X + tAXA + XA^2), \qquad (8)
\end{aligned}$$

where $Z = (z_{ij})_{i,j} = \left(\frac{\lambda_i^{\frac{1}{2}}(f(\lambda_i)f^{\perp}(\lambda_j)+f^{\perp}(\lambda_i)f(\lambda_j))\lambda_j^{\frac{1}{2}}}{\lambda_i^2+t\lambda_i\lambda_j+\lambda_j^2}\right)_{i,j}$ and is Hermitian positive semidefinite. Indeed $f^{\perp}(x) = x(f(x))^{-1}$ and

$$\begin{aligned}
&\left(\frac{\lambda_i^{\frac{1}{2}}(f(\lambda_i)f^{\perp}(\lambda_j)+f^{\perp}(\lambda_i)f(\lambda_j))\lambda_j^{\frac{1}{2}}}{\lambda_i^2+t\lambda_i\lambda_j+\lambda_j^2}\right)_{i,j} \\
&\quad = \left(\lambda_i^{\frac{3}{2}}(f(\lambda_i))^{-1}\left(\frac{f(\lambda_i)^2\lambda_i^{-1} + f(\lambda_j)^2\lambda_j^{-1}}{\lambda_i^2 + t\lambda_i\lambda_j + \lambda_j^2}\right)\lambda_j^{\frac{3}{2}}(f(\lambda_j))^{-1}\right)_{i,j}
\end{aligned}$$

is Hermitian positive semidefinite if and only if the matrix

$$\left(\frac{f(\lambda_i)^2\lambda_i^{-1} + f(\lambda_j)^2\lambda_j^{-1}}{\lambda_i^2 + t\lambda_i\lambda_j + \lambda_j^2}\right)_{i,j} \qquad (9)$$

is Hermitian positive semidefinite, which is so, by Lemma 2.3. Moreover, each diagonal entry of Z is $\frac{2}{2+t}$. Hence we obtain the desired result from (8) on using Lemma 2.1 for the case $B = A$.

General case follows on replacing A by $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and X by $\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ in (7). □

Proof of Theorem 1.2. As in Theorem 1.1, we need only to prove the result for the case $m = n$, $t \in (-2, 2]$ and $B = A$ i.e.,

$$(2+t)\, |||\, f(A)X + Xf(A)\, ||| \leqslant 2\frac{f(\lambda)}{\lambda}\, |||\, A^{\frac{3}{2}}XA^{-\frac{1}{2}} + tA^{\frac{1}{2}}XA^{\frac{1}{2}} + A^{-\frac{1}{2}}XA^{\frac{3}{2}}\, |||,$$

Once again for the same reason as in Theorem 1.1 we assume here that $A = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$, $\lambda_i > 0$. Then

$$\begin{aligned}
f(A)X \;+\; Xf(A) &= ((f(\lambda_i) + f(\lambda_j))x_{ij})_{i,j} \\
&= \left(\frac{f(\lambda_i) + f(\lambda_j)}{\lambda_i^{\frac{3}{2}}\lambda_j^{-\frac{1}{2}} + t\lambda_i^{\frac{1}{2}}\lambda_j^{\frac{1}{2}} + \lambda_i^{-\frac{1}{2}}\lambda_j^{\frac{3}{2}}}\right)_{i,j} \circ (A^{\frac{3}{2}}XA^{-\frac{1}{2}} + tA^{\frac{1}{2}}XA^{\frac{1}{2}} + A^{-\frac{1}{2}}XA^{\frac{3}{2}}) \\
&= Z \circ (A^{\frac{3}{2}}XA^{-\frac{1}{2}} + tA^{\frac{1}{2}}XA^{\frac{1}{2}} + A^{-\frac{1}{2}}XA^{\frac{3}{2}}), \qquad (10)
\end{aligned}$$

where,

$$
\begin{aligned}
Z = (z_{ij})_{i,j} &= \left(\frac{f(\lambda_i) + f(\lambda_j)}{\lambda_i^{\frac{3}{2}} \lambda_j^{-\frac{1}{2}} + t\lambda_i^{\frac{1}{2}} \lambda_j^{\frac{1}{2}} + \lambda_i^{-\frac{1}{2}} \lambda_j^{\frac{3}{2}}} \right)_{i,j} \\
&= \left(\lambda_i^{\frac{1}{2}} \left(\frac{f(\lambda_i) + f(\lambda_j)}{\lambda_i^2 + t\lambda_i\lambda_j + \lambda_j^2} \right) \lambda_j^{\frac{1}{2}} \right)_{i,j},
\end{aligned}
$$

is Hermitian positive semidefinite if and only if the matrix $\left(\frac{f(\lambda_i)+f(\lambda_j)}{\lambda_i^2+t\lambda_i\lambda_j+\lambda_j^2} \right)_{i,j}$ is positive semidefinite, which is so by (4). Moreover the diagonal entries of Z are $\left(\frac{2}{2+t}\right) \lambda_i^{-1} f(\lambda_i)$. Since $x^{-1} f(x)$ is matrix monotone decreasing (see Theorem 2, [10]) and so $\lambda_i^{-1} f(\lambda_i) \leqslant \lambda^{-1} f(\lambda)$ for $\lambda = \min \sigma(A)$.

Hence the desired result follows from (10) on using Lemma 2.1. □

COROLLARY 2.4. *If* $0 \leqslant r \leqslant 1$, $A, B \in M_n$ *Hermitian positive definite and* $\lambda = \min\{\sigma(A), \sigma(B)\}$, *then*

$$
(2+t) \, ||| \, A^r X + X B^r \, ||| \leqslant 2\, \lambda^{r-1} \, ||| \, A^{\frac{3}{2}} X B^{-\frac{1}{2}} + t A^{\frac{1}{2}} X B^{\frac{1}{2}} + A^{-\frac{1}{2}} X B^{\frac{3}{2}} \, ||| \, .
$$

Proof. Replace $f(x)$ by x^r in Theorem 1.2, the desired result follows. □

COROLLARY 2.5. *If* $A, B \in M_n$ *Hermitian positive definite and* $\lambda = \min\{\sigma(A), \sigma(B)\}$, *then*

$$
(2+t) \, ||| \, A^{\frac{1}{2}} [\log(I+A)X + X\log(I+B)] B^{\frac{1}{2}} \, ||| \leqslant 2 \, \frac{\log(1+\lambda)}{\lambda} \, ||| \, A^2 X + tAXB + XB^2 \, ||| \, . \tag{11}
$$

Proof. Replace X by $A^{\frac{1}{2}} X B^{\frac{1}{2}}$ and $f(x)$ by $\log(1+x)$ in Theorem 1.2, we get the desired result.

Since $\frac{\log(1+\lambda)}{\lambda} < 1$, so (11) is more precise estimate than Theorem 9 [11]. This was indicated by one of the referees.

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Mandeep Singh
Department of Mathematics
S.H.S.L.C.I.E.T. Longowal-148 106
India e-mail: msrawla@yahoo.com

Harkrishan Lal Vasudeva
Department of Mathematics
Panjab University, Chandigarh-160 014
India