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LOCAL DIAMETERS OF COMPACT SETS

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Abstract. Given, in a normed space, a compact set K and $P \in K$, let $r(K,P) = \max_{R \in K} ||P - R||$. For $P_1 \in K$ we consider sequences $P_i, i = 1, 2, ...,$ such that $||P_{i+1} - P_i|| = r(K, P_i)$. The behaviour of such sequences for K contained in the Euclidean plane, and their limits were studied by Alarcon and Stolarsky in [1]. Here we try to sharpen some of their results and to extend them to a more general setting.

1. Introduction

Let $(X, ||.||)$ be a normed space. For $P \in X$ and $r \geqslant 0$ we indicate by $B(P,r)$ the closed ball centered at P, with radius r. For P, Q in X, we denote by PQ the segment joining P and Q.

Given a compact set K, let

$$\mathrm{diam\,(K)} = \max\{||\mathrm{P} - \mathrm{Q}||;\ \mathrm{P}, \mathrm{Q} \in \mathrm{K}\} \qquad (\text{diameter of } K).$$

We say that (P,Q) is a *diametral pair* if $||P - Q|| = \mathrm{diam\,(K)}$. Also, for $P \in K$, we set

$$r(K,P) = \max\{||P - R||; \quad R \in K\}.$$

Given $P \in K$, we say that (P,Q) is a *first order maximal chord from* P if $||P - Q|| = r(K,P)$. Starting from $P_1 \in K$ we can define recursively a sequence $(P_i)_{i=1,2,...}$ so that (P_i, P_{i+1}) is a first order maximal chord from P_i: (P_i, P_{i+1}) will be also called a i*-th order maximal chord from* P_1, and $(P_i)_{i=1,2,...}$ a *maximal chord chain.*

Also, we say that (P,Q) is a *local diameter* if $||P - Q|| = r(K,P) = r(K,Q)$.

The following estimate was given in [1] for X the Euclidean plane.

THEOREM 1.1. (see [1, Theorem 4.1]). *If $(P_j)_{j=1,2,...}$ is a maximal chord chain for a compact set K in the Euclidean plane, then*

(i) $$||P_1 - P_2|| \geqslant \frac{\mathrm{diam\,(K)}}{2}$$

(ii) $$||P_i - P_{i+1}|| \geqslant \frac{\mathrm{diam\,(K)}}{\sqrt{3}},\ i \geqslant 2.$$

The constants $\frac{1}{2}, \frac{1}{\sqrt{3}}$ are the best possible.

Here we strengthen and extend the above estimate (ii) to uniformly convex spaces.

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2. New estimates

We recall the following definition.

We say that $(X, ||.||)$ is *uniformly convex* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that:

$$||x|| \leqslant 1, \quad ||y|| \leqslant 1, \quad ||x-y|| \geqslant \varepsilon \quad \text{imply} \quad \left|\left|\frac{x+y}{2}\right|\right| \leqslant 1-\delta. \tag{1}$$

The function

$$\delta(\varepsilon) = \inf\left\{1 - \left|\left|\frac{x+y}{2}\right|\right|;\ ||x|| \leqslant 1,\ ||y|| \leqslant 1,\ ||x-y|| \geqslant \varepsilon\right\}, \quad 0 \leqslant \varepsilon \leqslant 2,$$

is called the *modulus of uniform convexity* of X. It is simple to see that for X uniformly convex, we have (for any $r > 0$):

$$||x|| \leqslant r, \quad ||y|| \leqslant r, \quad ||x-y|| \geqslant \varepsilon \quad \text{imply} \quad \left|\left|\frac{x+y}{2}\right|\right| \leqslant r\Big(1-\delta\Big(\frac{\varepsilon}{r}\Big)\Big). \tag{2}$$

Recall that, in any space, we have $(0 \leqslant \varepsilon \leqslant 2)$:

$$\delta(\varepsilon) \leqslant 1 - \sqrt{1-\frac{\varepsilon^2}{4}} \tag{3}$$

while equality in (3) characterizes inner product spaces (see [3], pp. 58–59).

Now we shall prove the main result.

THEOREM 2.1. *Let* $(X, ||.||)$ *be uniformly convex and let* $\delta(\varepsilon)$ *be its modulus of uniform convexity. Let* K *be a compact set with diameter* diam (K). *Given* $P_1 \in K$, *set* $r = r(K, P_1)$ *and let* (P_1, P_2) *be a first order maximal chord from* P_1, *of length* $r(K, P_2) = \alpha$. *Then we have*

$$\delta(\text{diam}\,(\text{K})) \leqslant \alpha\Big(1-\delta\Big(\frac{\text{diam}\,(\text{K})}{\alpha}\Big)\Big). \tag{4}$$

Proof. By rescaling we may assume that $||P_1 - P_2|| = r(K, P_1) = 1$. Let (P_2, P_3) be a second order maximal chord from P_1, and let $||P_2 - P_3|| = \alpha = r(K, P_2)$. Set $diam(K) = 2d$; then $1 \leqslant \alpha \leqslant 2d \leqslant 2$.

Let (P, Q) be a diametral pair of K: it is clear that PQ is contained in $B(P_1, 1) \cap B(P_2, \alpha)$. Set $M = \frac{P+Q}{2}$; we obtain:

$$1 = ||P_1 - P_2|| \leqslant ||P_1 - M|| + ||M - P_2||. \tag{5}$$

By using (1) (after a translation of the origin to P_1) we have $||P_1 - P|| \leqslant 1$, $||P_1 - Q|| \leqslant 1$ and $||P - Q|| = 2d$. Thus $||P_1 - M|| \leqslant 1 - \delta(2d)$. Also, by using (2) (after a translation of the origin to P_2) we have $||P_2 - P|| \leqslant \alpha$, $||P_2 - Q|| \leqslant \alpha$ and $||P - Q|| = 2d$. Therefore $||P_2 - M|| \leqslant \alpha(1 - \delta(\frac{2d}{\alpha}))$. Thus by (5) we have

$$1 \leqslant 1 - \delta(2d) + \alpha\Big(1-\delta\Big(\frac{2d}{\alpha}\Big)\Big) \tag{5'}$$

and the result (4) follows. □

REMARK. Since the function δ is non decreasing, (5') also implies

$$1 \leqslant (1+\alpha)\Big(1-\delta\Big(\frac{2d}{\alpha}\Big)\Big) \iff \delta\Big(\frac{\mathrm{diam}\,(\mathrm{K})}{\alpha}\Big)(1+\alpha) \leqslant \alpha. \tag{5''}$$

COROLLARY 2.2. *Let X be an inner product space and K a compact set. Let P_1, P_2 and α be as in Theorem 2.1 with $||P_1 - P_2|| = 1$. (Here α is the length of a second order maximal chord). Then*

$$\alpha \geqslant \big(2 - \sqrt{4 - \mathrm{diam}^2(\mathrm{K})}\big)^{1/2}. \tag{6}$$

Proof. From (3) we have

$$\delta(2d) = 1 - \sqrt{1-d^2} \quad \text{and} \quad \delta\Big(\frac{2d}{\alpha}\Big) = 1 - \sqrt{1 - \frac{d^2}{\alpha^2}}.$$

So by (4) we have

$$1 - \sqrt{1-d^2} \leqslant \alpha\left(1 - \left(1 - \sqrt{1-\frac{d^2}{\alpha^2}}\right)\right)$$

which simplifies to

$$1 \leqslant \sqrt{1-d^2} + \sqrt{\alpha^2 - d^2} \quad (d \leqslant 1 \leqslant \alpha).$$

This is equivalent to

$$0 \leqslant -d^2 + \alpha^2 - d^2 + 2\sqrt{\alpha^2 - d^2 - \alpha^2 d^2 + d^4}.$$

If $d \leqslant \frac{\alpha}{\sqrt{2}}$ this is certainly true. Otherwise, if $\alpha < d\sqrt{2} = \frac{\mathrm{diam}\,(\mathrm{K})}{2}$, it is equivalent to $4(\alpha^2 - d^2 - \alpha^2 d^2 + d^4) \geqslant (2d^2 - \alpha^2)^2 \iff \alpha^4 - 4\alpha^2 + 4d^2 \leqslant 0$.

We have $\Delta = 16 - 16d^2 \geqslant 0$, so the above inequality is equivalent to $2 - \sqrt{4 - \mathrm{diam}^2(\mathrm{K})} \leqslant \alpha^2 \leqslant 2 + \sqrt{4 - \mathrm{diam}^2(\mathrm{K})}$.

Note that since $\alpha < \frac{\mathrm{diam}\,(\mathrm{K})}{\sqrt{2}} \leqslant \sqrt{2}$, the right inequality is trivially satisfied. Thus the left inequality is true. Next, easy computations show that

$$(\mathrm{diam}^2(\mathrm{K}))/2 \geqslant 2 - \sqrt{4 - \mathrm{diam}^2(\mathrm{K})}$$

so

$$\alpha \geqslant \big(2 - \sqrt{4 - \mathrm{diam}^2(\mathrm{K})}\big)^{1/2}$$

and this proves (6). □

REMARK 1. Both the estimate (6) and (see (ii) in section 1)

$$\alpha \geqslant \frac{\mathrm{diam}\,(\mathrm{K})}{\sqrt{3}} \tag{6'}$$

are meaningful for diam(K) $> \sqrt{3}$ (since $\alpha \geqslant 1$ is trivial). But it can be seen that (under this assumption) (6) is better than (6').

REMARK 2. We can look at (6) in another way. Let $\alpha < \sqrt{2}$. Then $4 - \text{diam}^2(K) \geqslant (2-\alpha^2)^2$, so

$$d \leqslant \alpha\sqrt{1-\frac{\alpha^2}{4}}. \tag{6''}$$

In Theorem 2.3 we shall give a direct proof of a more precise result, by using properties of inner product spaces.

REMARK 3. (6) becomes an equality when $B(P_1,1) \cap B(P_2,\alpha)$ (see the proof of Theorem 2.1) contains a diametral pair. Since, unlike inner product spaces, uniformly convex spaces can have different shapes in different directions, equality in (4) only holds when the direction given by P_1P_2 is one of those determining the value of the convexity modulus of X.

We conclude with an application to sums of distances between points in inner product spaces.

Let $K = \{x_1, ..., x_n\}$ be a finite set. Set

$$\sum(K) = \sum_{1 \leqslant i,j \leqslant n} ||x_i - x_j||. \tag{7}$$

This number, together with some related ones (for K finite) received some attention several years ago: see e.g. [2] and the references therein.

The next result indicates how $\sum(K)$ can be estimated by using the inequalities stated in the previous section.

For simplicity we shall only consider inner product spaces: in fact, the inequalities we could obtain in general, by using (4) and the modulus of uniform convexity, are complicated.

Also, better estimates could be obtained for example by using estimates concerning third order maximal chords.

THEOREM 2.3. *Let $K = \{x_1, ..., x_n\}$ be a finite set in an inner product space; $r(K,x_1) = 1 = ||x_1 - x_2||$; $r(K,x_2) = \alpha \leqslant \sqrt{2}$. Then we have:*

$$\sum(K) \leqslant n-1+(n-2)\alpha + \left[\frac{n(n-1)}{2} - (n-1+n-2)\right]\alpha\sqrt{4-\alpha^2}. \tag{8}$$

Proof. Let $\alpha \leqslant \sqrt{2}$; according to (6 ''), we have diam(K) $\leqslant \alpha\sqrt{4-\alpha^2}$.

From $r(K,x_1) = 1$ we obtain $||x_1 - x_j|| \leqslant 1$ for $j = 2, ..., n$. Also, $r(K,x_2) = \alpha$ implies $||x_2 - x_j|| \leqslant \alpha$ for $j = 3, ..., n$. Moreover, for $i,j \geqslant 3$: $||x_i - x_j|| \leqslant$ diam(K); since the sum defining $\sum(K)$ contains $\frac{n(n-1)}{2}$ terms, this gives (8).

REMARK. Inequality (8) can also be written as:

$$\sum(K) \leqslant n-1+(n-2)\alpha\left[1+\frac{n-3}{2}\sqrt{4-\alpha^2}\right]. \tag{8'}$$

Also, if $\alpha > \sqrt{2}$, then by simply using $\operatorname{diam}(\mathrm{K}) \leqslant 2$, we obtain instead of (8) the general estimate

$$\sum(K) \leqslant n-1+(n-2)\alpha+\frac{(n-2)(n-3)}{2}2 = n-1+(n-2)(\alpha+n-3). \quad (8")$$

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