Mathematical
Inequalities
& Applications
Volume 8, Number 1 (2005), 169–172

INEQUALITIES FOR THE DERIVATIVES AND STABLE DIFFERENTIATION OF PIECEWISE–SMOOTH DISCONTINUOUS FUNCTIONS

A. G. RAMM

(*communicated by J. Pečarić*)

Abstract. Formulas for stable differentiation of piecewise-smooth functions are given. The data are noisy values of these functions. The locations of discontinuity points and the sizes of the jumps across these points are not assumed known, but found stably from the noisy data. The results generalize earlier results of the author obtained for smooth functions. The obtained formulas are useful in applications.

1. Introduction

Let f be a piecewise $C^2[0,1]$ function, $0 < x_1 < x_2 < \cdots < x_J, 1 \leqslant j \leqslant J$, are discontinuity points of f. We do not assume their locations x_j and their number J known a priori. We assume that the limits $f(x_j \pm 0)$ exist, and

$$\sup_{x \neq x_j, 1 \leqslant j \leqslant J} |f^{(m)}(x)| \leqslant M_m, \quad m = 0, 1, 2. \tag{1.1}$$

Assume that f_δ is given, $\|f - f_\delta\| := \sup_{x \neq x_j, 1 \leqslant j \leqslant J} |f - f_\delta| \leqslant \delta$, where $f_\delta \in L^\infty(0,1)$ are the noisy data.

The problem is: given $\{f_\delta, \delta\}$, *where* $\delta \in (0, \delta_0)$ *and* $\delta_0 > 0$ *is a small number, estimate stably* f', *find the locations of discontinuity points* x_j *of* f *and their number* J, *and estimate the jumps* $p_j := f(x_j + 0) - f(x_j - 0)$ *of* f *across* x_j, $1 \leqslant j \leqslant J$.

There is a large literature on stable differentiation of noisy smooth functions (e.g., see references in [5],[7]), but the problem stated above was not solved for piecewise-smooth functions by the method given below. A statistical estimation of the location of discontinuity points from noisy discrete data is given in [1].

In [2] the following formula was proposed originally for stable estimation of $f'(x)$, assuming $f \in C^2([0,1])$ and given noisy data f_δ:

$$R_\delta f_\delta := \frac{f_\delta(x + h(\delta)) - f_\delta(x - h(\delta))}{2h(\delta)}, \quad h(\delta) := \left(\frac{2\delta}{M_2}\right)^{\frac{1}{2}}, \quad h(\delta) \leqslant x \leqslant 1 - h(\delta), \tag{1.2}$$

Mathematics subject classification (2000): 65D35, 65D05.

Key words and phrases: stable differentiation, noisy data, discontinuities, jumps, signal processing, edge detection.

Paper MIA-08-16

and

$$\|R_\delta f_\delta - f'\| \leqslant \sqrt{2M_2\delta}, \tag{1.3}$$

where the norm in (1.3) in [2] was $L^\infty(0,1)-$ norm. Moreover, (cf [6]),

$$\inf_T \sup_{f\in K(M_2,\delta)} \|Tf_\delta - f'\| \geqslant \varepsilon(\delta) := \sqrt{2M_2\delta}, \tag{1.4}$$

where $T : L^\infty(0,1) \to L^\infty(0,1)$ runs through the set of all bounded operators, $K(M_2,\delta) := \{f : \|f''\| \leqslant M_2, \ \|f - f_\delta\| \leqslant \delta\}$. Therefore estimate (1.2) is the best possible estimate of f', given noisy data f_δ, and assuming $f \in K(M_2,\delta)$.

In [5] this result was generalized to the case $f \in K(M_a,\delta)$, $\|f^{(a)}\| \leqslant M_a$, $1 < a \leqslant 2$, where $\|f^{(a)}\| := \|f\| + \|f'\| + \sup_{x,x'} \frac{|f'(x)-f'(x')|}{|x-x'|^{a-1}}$, $1 < a \leqslant 2$, and $f^{(a)}$ is the fractional-order derivative of f.

The aim of this paper is to extend the above results to the case of piecewise-smooth functions. In Section 2 the results are formulated, and proofs are given. In Section 3 the case of continuous piecewise-smooth functions is treated.

2. Formulation of the result

THEOREM 1. *Formula (1.2) gives stable estimate of f' on the set $S_\delta := [h(\delta), 1-h(\delta)] \setminus \bigcup_{j=1}^J (x_j - h(\delta), x_j + h(\delta))$, and (1.3) holds with the norm $\|\cdot\|$ taken on the set S_δ. Computing the quantities $f_j := \frac{f_\delta(jh+h)-f_\delta(jh-h)}{2h}$, where $h := h(\delta) := \left(\frac{2\delta}{M_2}\right)^{\frac{1}{2}}$, $1 \leqslant j < [\frac{1}{h}]$, for sufficiently small δ, one finds the location of discontinuity points of f with accuracy $2h$, and their number J. Here $[\frac{1}{h}]$ is the integer smaller than $\frac{1}{h}$ and closest to $\frac{1}{h}$. The discontinuity points of f are located on the intervals $(jh-h, jh+h)$ such that $|f_j| > 6\varepsilon(\delta)$ for sufficiently small δ, where $\varepsilon(\delta)$ is defined in (1.4).*

Let us assume that $\min_j p_j := p \gg h(\delta)$, where $\gg$ means “much greater than.” Then x_j is located on the j-th interval $[jh-h, jh+h]$, $h := h(\delta)$, such that

$$|f_j| := \left|\frac{f_\delta(jh+h) - f_\delta(jh-h)}{2h}\right| \gg 1, \tag{2.1}$$

so that x_j is localized with the accuracy $2h(\delta)$. The jump p_j can be estimated by the formula: $p_j \approx f_\delta(jh+h) - f_\delta(jh-h)$, and the error estimate of this formula can be given:

$$|p_j - [f_\delta(jh+h) - f_\delta(jh-h)]| \leqslant 2\delta + 4M_1 h = 2\delta + 4M_1\sqrt{\frac{2\delta}{M_2}}. \tag{2.2}$$

Thus, the error of the calculation of p_j is $O(\delta^{\frac{1}{2}})$ as $\delta \to 0$.

Proof of Theorem 1. If $x \in S_\delta$, then using Taylor's formula one gets:

$$|(R_\delta f_\delta)(x) - f'(x)| \leqslant \frac{\delta}{h} + \frac{M_2 h}{2}. \tag{2.3}$$

Minimizing the right-hand side of (2.3) with respect to h yields formula (1.2) for the minimizer $h = h(\delta)$ defined in (1.2), and estimate (1.3) for the minimum of the right-hand side of (2.3). If $p \gg h(\delta)$, and (2.1) holds, then the discontinuity points are located with the accuracy $2h(\delta)$. Estimate (2.2) can be obtained as follows. For $jh - h \leqslant x_j \leqslant jh + h$, one has

$$\begin{aligned}&|f(x_j+0) - f(x_j-0) - f_\delta(jh+h) + f_\delta(jh-h)| \\ &\quad \leqslant 2\delta + |f(x_j+0) - f(jh+h)| + |f(x_j-0) - f(jh-h)| \\ &\quad \leqslant 2\delta + M_1 2h + M_1 2h, \quad h = h(\delta).\end{aligned}$$

This yields formula (2.2). Computing the quantities f_j for $1 \leqslant j < [\frac{1}{h}]$, and finding the intervals on which (2.1) holds for sufficiently small δ, one finds the location of discontinuity points of f with accuracy $2h$, and the number J of these points. Theorem 1 is proved. □

REMARK. Similar results can be derived if $\|f^{(a)}\|_{S_\delta} \leqslant M_a$, $1 < a \leqslant 2$. In this case $h = h(\delta) = c_a \delta^{\frac{1}{a}}$, where $c_a = \left[\frac{2}{M_a(a-1)}\right]^{\frac{1}{a}}$, $R_\delta f_\delta$ is defined in (1.2), and the error of the estimate is:

$$\|R_\delta f_\delta - f'\|_{S_\delta} \leqslant a M_a^{\frac{1}{a}} \left(\frac{2}{a-1}\right)^{\frac{a-1}{a}} \delta^{\frac{a-1}{a}}.$$

The proof is similar to that given in Section 3. It is proved in [5] that for C^a-functions given with noise it is possible to construct stable differentiation formulas if $a > 1$ and it is impossible to construct such formulas if $a \leqslant 1$. The obtained formulas are useful in applications [3]. One can also use L^p-norm on S_δ in the estimate $\|f^{(a)}\|_{S_\delta} \leqslant M_a$ (cf. [5]).

3. Continuous piecewise-smooth functions

Suppose now that $\xi \in (mh - h, mh + h)$, where $m > 0$ is an integer, and ξ *is a point at which* f *is continuous but* $f'(\xi)$ *does not exist*. Thus, the jump of f across ξ is zero, but ξ is not a point of smoothness of f. How does one locate the point ξ?

The algorithm we propose consists of the following. Calculate the numbers $f_j := \frac{f_\delta(jh+h) - f_\delta(jh-h)}{2h}$, $j = 1, 2, \dots$, $h = h(\delta) = \sqrt{\frac{2\delta}{M_2}}$, and find those j for which

$$f_j f_{j+1} < 0. \tag{3.1}$$

If

$$\min(|f_{j+1}|, |f_j|) > \varepsilon(\delta) := \sqrt{2M_2\delta}, \tag{3.2}$$

then the interval $(jh - h, jh + 2h)$ contains a critical point of f or a point ξ at which f' does not exist. To determine which one of these two cases holds, one calculates

$$\begin{aligned}|f_{j+1} - f_j| &\geqslant \left|\frac{f(jh+2h) - f(jh) - f(jh+h) + f(jh-h)}{2h}\right| - \frac{4\delta}{2h} \\ &\geqslant |f'(jh+2h) - f'(jh-h)| - \frac{2M_2(2h)^2}{2 \cdot 2h} - \frac{4\delta}{2h}.\end{aligned}$$

Here $\frac{4\delta}{2h} = \varepsilon(\delta)$ and $\frac{2M_2(2h)^2}{2\cdot 2h} = 2\varepsilon(\delta)$. Thus,

$$|f_{j+1} - f_j| \geqslant |f'(jh+2h) - f'(jh-h)| - 3\varepsilon(\delta). \tag{3.3}$$

We have assumed that ξ is the point of continuity of f. If ξ is the point of discontinuity of f', i.e., $|f'(\xi+0)-f'(\xi-0)| := P > 0$, then, as $\delta \to 0$, one has $|f_{j+1}-f_j| > 6\varepsilon(\delta)$.

If, however, $\xi \in S_\delta$, then $|f'(jh+2h) - f'(jh-h)| \leqslant 3hM_2 = 3\varepsilon(\delta)$, so an argument similar to the one used in the derivation of (3.3) yields:

$$|f_{j+1} - f_j| \leqslant 6\varepsilon(\delta) \quad \text{on the set} \quad S_\delta. \tag{3.4}$$

Thus, (3.1) and (3.4) for sufficiently small δ imply that ξ is not a point of discontinuity of f'.

These arguments prove the following theorem:

THEOREM 2. *Assume that (3.1) and (3.2) hold. If $|f_{j+1} - f_j| > 6\varepsilon(\delta)$ for sufficiently small $\delta > 0$, then $\xi \in (jh-h, jh+2h)$ is a discontinuity point of f'. If $|f_{j+1} - f_j| < 6\varepsilon(\delta)$ for sufficiently small $\delta > 0$, then there is no discontinuity point of f' in the interval $(jh-h, jh+2h)$, but there is a critical point of f in this interval.*

REFERENCES

[1] A. I. KATSEVICH, A. G. RAMM, *Nonparametric estimation of the singularities of a signal from noisy measurements*, Proc. AMS, 120, N8, (1994), 1121–1134.
[2] A. G. RAMM, *On numerical differentiation*, Mathem., Izvestija vuzov, **11** (1968), 131–135. (in Russian)
[3] A. G. RAMM, *Simplified optimal differentiators*, Radiotech. i Electron., **17** (1972), 1325–1328.
[4] A. G. RAMM, *Stable solutions of some ill-posed problems*, Math. Meth. in the appl. Sci., **3** (1981), 336–363.
[5] A. G. RAMM, A. SMIRNOVA, *Stable numerical differentiation: when is it possible?* Jour. Korean SIAM, **7** N1, (2003), 47–61.
[6] A. G. RAMM, *Scattering by obstacles*, Reidel, Dordrecht, 1986, Appendix 4.
[7] A. G. RAMM, *Inverse problems*, Springer, New York, 2004.

(Received June 1, 2004)

A. G. Ramm
Mathematics Department
Kansas State University
Manhattan, KS 66506-2602
USA
e-mail: ramm@math.ksu.edu