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ON EXTENDED SINGULAR SET OF POTENTIALS

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Dedicated to the memory
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Abstract. We describe a class of potentials $v = G * f$, such that if x_0 is from extended singular set of v, that is, $r^{-N} \int_{B_r(x_0)} v(x)\,dx \to +\infty$ for some sequence $r \to 0$, then necessarily $v(x_0) = \infty$. This class includes Bessel potentials and Riesz potentials. The result was exploited in our previous paper in order to show that singular dimension of the Bessel potential space $L^{\alpha,p}(\mathbf{R}^N)$ (that is, the supremum of Hausdorff's dimension of extended singular sets, taken over all functions from the space) is equal to $N - \alpha p$, provided $\alpha p \leqslant N$.

Assume that $u : \mathbf{R}^N \to \mathbf{R}$ is a Lebesgue measurable function. In [5] we have introduced extended singular set of u defined by

$$\text{e-Sing}\, u = \{x_0 \in \mathbf{R}^N : \limsup_{r \to 0} \frac{1}{r^N} \int_{B_r(x_0)} u(x)\,dx = +\infty\}. \tag{1}$$

As we see, e-Sing u is contained in the complement of the set of Lebesgue points of u. Furthermore, it is easy to verify that e-Sing u contains the singular set Sing u (that is, the set of points $x_0 \in \mathbf{R}^N$ such that there exist positive numbers C, γ and r satisfying $u(x) \geqslant C|x - x_0|^{-\gamma}$ a.e. in the open ball $B_r(x_0)$), and also all iterated logarithmic singularities of u (i.e. points x_0 such that $u(x) \geqslant C \log \ldots \log |x - x_0|^{-\gamma}$ a.e. in a ball around x_0).

The main result of this paper is formulated in Theorem 1, which has been announced without proof in [5, Theorem 4]. It was important in proving that singular dimension s-dim $L^{\alpha,p}(\mathbf{R}^N)$ of the Bessel potential space $L^{\alpha,p}(\mathbf{R}^N)$ (see [1] or [4]), where $1 < p < \infty$ and $0 < \alpha \leqslant N/p$, is equal to $N - \alpha p$, that is,

$$\text{s-dim}\, L^{\alpha,p}(\mathbf{R}^N) := \sup\{\dim_H(\text{e-Sing}\, u) : u \in L^{\alpha,p}(\mathbf{R}^N)\} = N - \alpha p.$$

Here $\dim_H$ denotes Hausdorff's dimension. In particular, singular dimension of the Sobolev space $W^{k,p}(\mathbf{R}^N)$, where $1 < p < \infty$ and $kp \leqslant N$, is equal to $N - kp$.

See [5] for the proof of these results and more general definition of upper and lower singular dimensions of spaces and sets of functions from $\mathbf{R}^N$ to $\mathbf{R}$. In our forthcoming

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paper it will be shown that the supremum is actually achieved, and moreover, that there exists a constructive Sobolev function u in $W^{k,p}(\mathbf{R}^N)$, $kp \leqslant N$ (or more generally, in $L^{\alpha,p}(\mathbf{R}^N)$, $\alpha p \leqslant N$) such that $\dim_H(\operatorname{Sing} u) = N - kp$ (or $N - \alpha p$ respectively). This result seems to be new even for $k = 0$, that is, for the Lebesgue space $L^p(\mathbf{R}^N)$, see [6].

Among early results related to the question of size of singular sets of Sobolev functions we cite Reshetnyak [3, Theorem 1.8]: if $f \in L^p(\mathbf{R}^N)$, $f \geqslant 0$, and G_α is the Bessel potential kernel (see [1] or [4]), then the set of $x \in \mathbf{R}^N$ for which $(G_\alpha * f)(x) = \infty$, has (α, p)-Bessel capacity equal to zero (and hence its Hausdorff's dimension is $\leqslant N - \alpha p$). It is therefore of interest to know for what points x the condition $(G_\alpha * f)(x) = \infty$ can be fulfilled. The answer is given by Theorem 1.

Furthermore, for any given compact (fractal) set A in $\mathbf{R}^N$ having its upper Minkowski-Bouligand dimension (also known as the upper box dimension, see [2]) arbitrarily close and less than $N - \alpha p$, it is possible to construct a function $f \in L^p(\mathbf{R}^N)$ such that $(G_\alpha * f)(x) = \infty$ on A, and moreover, $A \subseteq \operatorname{Sing}(G_\alpha * f)$, see [5, Theorem 2].

Theorem 1 implies seemingly obvious inclusion $\operatorname{Sing} v \subseteq \{v = +\infty\}$ for a class of Bessel potentials v (and also for Riesz potentials, provided $\alpha p < N$).

THEOREM 1. *Assume that* $1 < p < \infty$, $0 < \alpha < N$, *and let* $G : \mathbf{R}^N \times \mathbf{R}^N \to \overline{\mathbf{R}}$ *be a nonnegative potential kernel, such that* $G(x,y)$ *is lower semicontinuous in* x *for a.e.* y, *and measurable in* y *for all* x. *We assume that there exist positive numbers* C_1, C_2, *and* R *such that for any* x,

$$\frac{C_1}{|x-y|^{N-\alpha}} \leqslant G(x,y) \leqslant \frac{C_2}{|x-y|^{N-\alpha}}, \quad \textit{for a.e. } y \in B_R(x), \tag{2}$$

and there exists a bounded, nonnegative, nonincreasing function $g : (R, \infty) \to \mathbf{R}$, *contained in the weighted Lebesgue space* $L^{p'}((R,\infty); r^{N-1})$, *such that for all* x *we have* $G(x,y) \leqslant g(|x-y|)$ *for a.e.* $y \in \mathbf{R}^N \setminus B_R(x)$. *Let* $v = G * f$, *where*

$$(G * f)(x) := \int_{\mathbf{R}^N} G(x,y) f(y)\, dy,$$

and $f \in L^p(\mathbf{R}^N)$, $f \geqslant 0$. *Then* $\text{e-Sing}\, v \subseteq \{v = \infty\}$.

Proof. We argue by contradiction: assume that there exists $x_0 \in \text{e-Sing}\, v$ such that $v(x_0) < \infty$. With fixed x_0, for any $r > 0$ we denote $B_r = B_r(x_0)$, and for $\rho \in (r, \infty]$, we define $B_{r,\rho} = B_\rho \setminus B_r$.

(a) Using $v(x_0) < \infty$ and the left-hand side inequality in (2) we obtain that

$$M_R := \int_{B_R} \frac{f(y)}{|x_0 - y|^{N-\alpha}}\, dy < \infty. \tag{3}$$

Since $|x_0 - y| \leqslant r$ when $y \in B_r$, we have

$$\int_{B_r} f(y)\, dy \leqslant r^{N-\alpha} M_r, \quad \lim_{r \to 0} M_r = 0. \tag{4}$$

In the sequel we fix $r > 0$ small enough.

(b) Since $v(x) = \int_{B_r} G(x,y)f(y) + \int_{B_{r,\infty}} G(x,y)f(y)\,dy$, using the right-hand side inequality in (2) we have

$$C_2 \int_{Br} \frac{f(y)}{|x-y|^{N-\alpha}}\,dy + \int_{B_{r,\infty}} G(x,y)f(y)\,dy \geqslant v(x), \quad \forall x \in B_r.$$

Integrating over $B_{r/2}$, we obtain

$$I_1 + I_2 \geqslant \int_{B_{r/2}} v(x)\,dx, \tag{5}$$

where

$$\begin{aligned} I_1 &= C_2 \int_{B_{r/2}} dx \int_{B_r} \frac{f(y)}{|x-y|^{N-\alpha}}\,dy, \\ I_2 &= \int_{B_{r/2}} dx \int_{B_{r,\infty}} G(x,y)f(y)\,dy. \end{aligned} \tag{6}$$

It is easy to see that $B_{r/2} = B_{r/2}(x_0) \subseteq B_{3r/2}(y)$. Hence, using Fubini's theorem we have

$$\begin{aligned} I_1 &\leqslant C_2 \int_{B_r} f(y)\,dy \leqslant \int_{B_{r/2}} \frac{dx}{|x-y|^{N-\alpha}} \\ &\leqslant C_2 \int_{B_r} f(y)\,dy \leqslant \int_{B_{3r/2}(y)} \frac{dx}{|x-y|^{N-\alpha}} \\ &\leqslant C \cdot r^\alpha \int_{B_r} f(y)\,dy. \end{aligned} \tag{7}$$

In order to estimate I_2, let us fix $x \in B_{r/2}$. First we extend $g(r)$ from (R,∞) to $(0,\infty)$ by defining

$$\overline{g}(r) = \begin{cases} \frac{C_2}{r^{N-\alpha}}, & 0 < r < R, \\ g(r), & r \geqslant R. \end{cases}$$

Since $g(r)$ is bounded, we can achieve that $\overline{g}(r)$ is nonincreasing for all $r > 0$, by taking C_2 large enough. Now we have that $G(x,y) \leqslant \overline{g}(|x-y|)$ for a.e. $y \in \mathbf{R}^N$. Taking any $y \in B_{r,\infty}$ and $x \in B_{r/2}$, we have $|x-y| \geqslant |y-x_0| - r/2$, and from this $\frac{|x-y|}{|y-x_0|} \geqslant 1 - \frac{r}{2|y-x_0|} \geqslant 1/2$. Thus,

$$\begin{aligned} \int_{B_{r,\infty}} G(x,y)f(y)\,dy &\leqslant \int_{B_{r,\infty}} \overline{g}(|x-y|)f(y)\,dy \\ &\leqslant \int_{B_{r,\infty}} \overline{g}(\frac{1}{2}|x_0-y|)f(y)\,dy \\ &\leqslant \int_{\mathbf{R}^N} \overline{g}(\frac{1}{2}|x_0-y|)f(y)\,dy \\ &\leqslant J_1 + J_2 + J_3, \end{aligned}$$

where J_1, J_2 and J_3 are the corresponding integrals of $y \mapsto \overline{g}(\frac{1}{2}|x_0 - y|)f(y)$ over B_R, $B_{R,2R}$ and $B_{2R,\infty}$ respectively. We have

$$\begin{aligned} J_1 &\leqslant \int_{B_R} \frac{C_2 f(y)}{\frac{1}{2}|x_0 - y|)^{N-\alpha}}\, dy \leqslant C \cdot M_R, \\ J_2 &\leqslant \frac{C_2}{(R/2)^{N-\alpha}} \int_{B_{R,2R}} f(y)\, dy \leqslant C_R \|f\|_{L^p(\mathbf{R}^N)}, \\ J_3 &\leqslant \|f\|_{L^p(\mathbf{R}^N)} \cdot \left(\int_{B_{2R,\infty}} g(\frac{1}{2}|y - x_0|)^{p'} dy \right)^{1/p'}, \end{aligned}$$

where $p' = p/(p-1)$ is the conjugate exponent of p. Since

$$\int_{B_{2R,\infty}} g(\frac{1}{2}|y - x_0|)^{p'} dy = 2^N \int_{B_{R,\infty}(0)} g(|z|)^{p'} dz = 2^N \int_R^\infty g(\rho)^{p'} \rho^{N-1} d\rho,$$

we have

$$\begin{aligned} \int_{B_{r,\infty}} G(x,y) f(y)\, dy \;&\leqslant\; C \cdot M_R + C_R \cdot \|f\|_{L^p(\mathbf{R}^N)} + \\ &\quad + D_R \cdot \|f\|_{L^p(\mathbf{R}^N)} \cdot \|g\|_{L^{p'}(R,\infty;\rho^{N-1})} \end{aligned}$$

Hence,

$$I_2 \leqslant D \cdot r^N, \tag{8}$$

where D is a constant depending on R, $\|f\|_{L^p(\mathbf{R}^N)}$ and $\|g\|_{L^{p'}(R,\infty;\rho^{N-1})}$, but not on r. Using (7) and (8), from (5) we obtain

$$\int_{B_r} f(y)\, dy \geqslant r^{-\alpha} \int_{B_{r/2}} v(x)\, dx - Dr^{N-\alpha}. \tag{9}$$

(c) Combining (9) and (4) we arrive to

$$M_r \geqslant r^{-N} \int_{B_{r/2}} v(x)\, dx - D. \tag{10}$$

Now we take the limit over the sequence $r = 2r_k$, with r_k chosen so that

$$\lim_{k \to \infty} \frac{1}{r_k^N} \int_{B_{r_k(x_0)}} v(x)\, dx = \infty,$$

which is possible due to $x_0 \in$ e-Sing v, see (1). From (10) we obtain the desired contradiction: $0 \geqslant \infty$. □

COROLLARY 1. *Assume that* $1 < p < \infty$ *and* $f \in L^p(\mathbf{R}^N)$, $f \geqslant 0$ *a.e.*

(a) *If* $0 < \alpha < N$, *and* $v = G_\alpha * f$, *where* G_α *is the Bessel potential kernel (see [1] or [4]), then* e-Sing $v \subseteq \{v = \infty\}$.

(b) *If* $0 < \alpha < N/p$, *and* $v = I_\alpha * f$, *where* $I_\alpha = 1/|x|^{N-\alpha}$ *is the Riesz potential kernel, then* e-Sing $v \subseteq \{v = \infty\}$.

Proof.

(a) Use Theorem 1 with $G(x,y) = G_\alpha(x-y)$ and $g(r) = C\cdot e^{-r/2}$ for $r \in (R,\infty)$, with $R > 0$ fixed.

(b) Use Theorem 1 with $G(x,y) = I_\alpha(x-y)$ and $g(r) = 1/r^{N-\alpha}$ for $r \in (R,\infty)$ with $R > 0$ fixed. Note that the condition $g \in L^{p'}((R,\infty);\rho^{N-1})$ is equivalent with $\alpha p < N$. □

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