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CONTINUOUS SHARPENING OF HÖLDER'S AND MINKOWSKI'S INEQUALITIES

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Dedicated to the memory of prof. Mladen Alić

(*communicated by N. Elezović*)

Abstract. Some properties of functions which in special cases lead to sharpening of Hölder's and other interesting inequalities are proved. Results analogue to theorems leading to reverse Hölder's inequality are presented.

1. Introduction

For numbers $x_{ij} \in \mathbf{R}^+$ $(i = 1, \ldots, m;\ j = 1, \ldots, n)$ and real numbers $0 < p_j < 1$ with $\sum_{j=1}^n p_j = 1$ denote $y_i = \prod_{j=1}^n x_{ij}^{p_j}$ $(i = 1, \ldots, m)$ and define the function h by

$$h(t) = \prod_{j=1}^n \left[\sum_{i=1}^m y_i \left(\frac{x_{ij}}{y_i} \right)^t \right]^{p_j} \qquad (0 \leqslant t \leqslant 1). \tag{1}$$

In [4] Yang proved that h is increasing on $[0,1]$. As

$$h(0) = \sum_{i=1}^m \prod_{j=1}^n x_{ij}^{p_j} \qquad \text{and} \qquad h(1) = \prod_{j=1}^n \left(\sum_{i=1}^m x_{ij} \right)^{p_j},$$

this result may be regarded as a generalization of Hölder's inequality

$$\sum_{i=1}^m \prod_{j=1}^n x_{ij}^{p_j} \leqslant \prod_{j=1}^n \left(\sum_{i=1}^m x_{ij} \right)^{p_j}.$$

But this result is a special case of an older result by J. Pečarić and P. Beesack [2] which we give here in somewhat more generalized and therefore abbreviated form:

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THEOREM A. *Let A be an isotonic linear functional on a linear class L of real-valued functions defined on E. For fixed $r \in \mathbf{R}$, $f_i \in L$ and $0 < q_i \in \mathbf{R}$ $(i = 1, \dots, n)$ with $\sum_{i=1}^{n} \frac{1}{q_i} = 1$ define the function g by*

$$g(x) = \prod_{j=1}^{n} A\left(f_j^{q_j x} \prod_{k=1}^{n} f_k^{r-x}\right)^{\frac{1}{q_j}} \qquad (x \in \mathbf{R}).$$

Then

$$g(x) \leqslant g(y) \tag{2}$$

for all $x, y \in \mathbf{R}$ with $|x| \leqslant |y|$ and $xy > 0$.

Proof. The proof of inequality (2) is based on using Hölder's inequality and a suitable substitutions as follows: Let $s_i^j \in [0,1]$ with $\sum_{i=1}^{n} s_i^j = 1$ $(j = 1, \dots, n)$. Using Hölder's inequality n-times we get

$$\prod_{j=1}^{n} A\left(a_1^{s_1^j} \cdots a_n^{s_n^j}\right)^{\frac{1}{q_j}} \leqslant \prod_{j=1}^{n} \left(A(a_1)^{s_1^j} \cdots A(a_n)^{s_n^j}\right)^{\frac{1}{q_j}} = \prod_{i=1}^{n} A(a_i)^{\sum_{j=1}^{n} \frac{s_i^j}{q_j}}.$$

By the substitutions

$$a_i(y) = f_i^{q_i y} \cdot (f_1 \cdots f_n)^{r-y}, \quad s_i^j = \frac{1}{q_i}\left(1 - \frac{x}{y}\right) \ (i \neq j), \quad s_i^i = \frac{1}{q_i}\left(1 - \frac{x}{y}\right) + \frac{x}{y}$$

we get (2).

If A is a functional defined as

$$A(f) = \sum_{i=1}^{m} f(i) \qquad (f : E = \{1, 2, \dots, m\} \to \mathbf{R}),$$

we obtain Yang's result. Another functional A often used is defined as $A(f) = \int_a^b f(x)\,dx$ for all integrable functions $f : E = [a, b] \to \mathbf{R}$. For some other examples of isotonic linear functionals see [3].

We will discuss here instead of h as defined in [4] a more general function, and as a special case we get again Hölder's inequality.

In the sequel we need the following notations:

DEFINITION 1. Let f_j $(j = 1, \dots, n-1)$ be positive real-valued functions on $\mathbf{R}^+$ and let $\mathbf{a} = (a_1, \dots, a_n) \in \mathbf{R}^n$ with $a_j > 0$ $(j = 1, \dots, n)$. For $r, s \in \{1, 2, \dots, n-1\}$, $r \leqslant s$ we denote

$$g_{r,s}(a_r, a_{r+1}, \dots, a_{s+1}) = a_r f_r\left(\frac{a_{r+1}}{a_r} f_{r+1}\left(\frac{a_{r+2}}{a_{r+1}} \dots f_s\left(\frac{a_{s+1}}{a_s}\right)\right)\right),$$

$$g_{s+1,s}(a_{s+1}) = a_{s+1}$$

and

$$g(\mathbf{a}) = g_{1,n-1}(\mathbf{a}) = a_1 f_1\left(\frac{a_2}{a_1} f_2\left(\frac{a_3}{a_2} \dots f_{n-1}\left(\frac{a_n}{a_{n-1}}\right)\right)\right).$$

Similarly, for positive real-valued functions $r, F_j \ (j = 1, \dots, n-1)$ on $\mathbf{R}^+$ denote

$$R(a,b) = ar\Big(\frac{b}{a}\Big) \qquad \big((a,b) \in \mathbf{R}^2\big),$$
$$G_{r,s}(a_r, a_{r+1}, \dots, a_{s+1}) = a_r F_r\Big(\frac{a_{r+1}}{a_r} F_{r+1}\Big(\frac{a_{r+2}}{a_{r+1}} \dots F_s\Big(\frac{a_{s+1}}{a_s}\Big)\Big)\Big),$$
$$G_{s+1,s}(a_{s+1}) = a_{s+1}$$

and

$$G(\mathbf{a}) = G_{1,n-1}(\mathbf{a}) = a_1 F_1\Big(\frac{a_2}{a_1} F_2\Big(\frac{a_3}{a_2}, \dots, F_{n-1}\Big(\frac{a_n}{a_{n-1}}\Big)\Big)\Big).$$

This definition leads to the following lemma which can be easily proved (see also [1: Theorem 1] where this lemma is included).

LEMMA 1. *Let* $f_j \ (j = 1, \dots, n-1)$ *be positive increasing concave functions defined on* $\mathbf{R}^+$*, let* $g_{r,s}$ *be as defined above and* $\mathbf{x}_i = (x_{i1}, x_{i2}, \dots, x_{in}) \in \mathbf{R}^n \ (i = 1, \dots, m)$ *with* $x_{ij} > 0 \ (j = 1, \dots, n)$*. Then the inequalities*

$$\begin{aligned}
&\sum_{i=1}^m g_{1,n-1}(\mathbf{x}_i) \\
&\quad \leqslant g_{1,1}\Big(\sum_{i=1}^m x_{i1}, \sum_{i=1}^m g_{2,n-1}(x_{i2}, \dots, x_{in})\Big) \\
&\quad \vdots \\
&\quad \leqslant g_{1,j}\Big(\sum_{i=1}^m x_{i1}, \sum_{i=1}^m x_{i2}, \dots, \sum_{i=1}^m x_{ij}, \sum_{i=1}^m g_{j+1,n-1}(x_{i,j+1}, \dots, x_{in})\Big) \\
&\quad \leqslant g_{1,n-1}\Big(\sum_{i=1}^m x_{i1}, \sum_{i=1}^m x_{i2}, \dots, \sum_{i=1}^m x_{in}\Big)
\end{aligned} \tag{3}$$

hold. Moreover, if all f_j *are nonlinear on any subinterval of* $\mathbf{R}^+$*, then equality holds in*

$$\sum_{i=1}^m g_{1,n-1}(\mathbf{x}_i) \leqslant g_{1,n-1}\Big(\sum_{i=1}^m x_{i1}, \sum_{i=1}^m x_{i2}, \dots, \sum_{i=1}^m x_{in}\Big) \tag{4}$$

if and only if all $\mathbf{x}_i \ (i = 1, \dots, m)$ *are proportional.*

In other words, (4) shows that $g_{1,n-1}$ is a superadditive function which means that

$$\sum_{i=1}^m g_{1,n-1}(\mathbf{x}_i) \leqslant g_{1,n-1}\Big(\sum_{i=1}^m \mathbf{x}_i\Big)$$

for $\mathbf{x}_i = (x_{i1}, x_{i2}, \dots, x_{in}) \in \mathbf{R}^n$ with $x_{ij} > 0 \ (i = 1, \dots, m, j = 1, \dots, n)$ or, more explicitly,

$$\sum_{i=1}^{m} x_{i1} f_1\left(\frac{x_{i2}}{x_{i1}} f_2\left(\frac{x_{i3}}{x_i} \cdots f_{n-1}\left(\frac{x_{in}}{x_{i,n-1}}\right)\right)\right)$$
$$\leqslant \left(\sum_{i=1}^{m} x_{i1}\right) f_1\left(\frac{\sum_{i=1}^m x_{i2}}{\sum_{i=1}^m x_{i1}} f_2\left(\frac{\sum_{i=1}^m x_{i3}}{\sum_{i=1}^m x_{i2}} \cdots f_{n-1}\left(\frac{\sum_{i=1}^m x_{in}}{\sum_{i=1}^m x_{i,n-1}}\right)\right)\right).$$

Using Lemma 1 we prove the following

THEOREM 1. *Let* r, f_j, F_j $(j = 1, \ldots, n-1)$ *be positive concave functions on* $\mathbf{R}^+$ *and for* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n$ *with* $x_i > 0$ $(i = 1, \ldots, n)$ *denote by* g, G, R *the functions accordingly to Definition* 1. *Then the function* $A = A(\mathbf{x})$ *defined for* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n$ *with* $x_i > 0$ $(i = 1, \ldots, n)$ *by*

$$A(\mathbf{x}) = G\Big(R(g(\mathbf{x}), x_1), R(g(\mathbf{x}), x_2), \ldots, R(g(\mathbf{x}), x_n)\Big)$$

is superadditive.

Proof. As $f_j(x) > 0$ and therefore $x f_j\left(\frac{1}{x}\right)$ $(j = 1, \ldots, n-1)$ are increasing for $x > 0$, it follows that the function g is increasing, too. The same holds for the function G and R. According to Lemma 1, g, G and R are superadditive functions as r, f_j, F_j are concave on $\mathbf{R}^+$. Hence, for $\mathbf{x}_i = (x_{i1}, \ldots, x_{in}) \in \mathbf{R}^n$ with $x_{ij} > 0$ $(i = 1, \ldots, m, j = 1, \ldots, n)$,

$$\begin{aligned}
\sum_{i=1}^{m} A(\mathbf{x}_i) &= \sum_{i=1}^{m} G\big(R(g(\mathbf{x}_i), x_{i1}), \ldots, R(g(\mathbf{x}_i), x_{in})\big) \\
&\leqslant G\Big(\sum_{i=1}^{m} R(g(\mathbf{x}_i), x_{i1}), \ldots, \sum_{i=1}^{m} R(g(\mathbf{x}_i), x_{in})\Big) \\
&\leqslant G\Big(R\Big(\sum_{i=1}^{m} g(\mathbf{x}_i), \sum_{i=1}^{m} x_{i1}\Big), \ldots, R\Big(\sum_{i=1}^{m} g(\mathbf{x}_i), \sum_{i=1}^{m} x_{in}\Big)\Big) \\
&\leqslant G\Big(R\Big(g\Big(\sum_{i=1}^{m} \mathbf{x}_i\Big), \sum_{i=1}^{m} x_{i1}\Big), \ldots, R\Big(g\Big(\sum_{i=1}^{m} \mathbf{x}_i\Big), \sum_{i=1}^{m} x_{in}\Big)\Big) \\
&= A\Big(\sum_{i=1}^{m} \mathbf{x}_i\Big).
\end{aligned}$$

In this sequence of inequalities, the first is a result of the superadditivity of G, the second is a consequence of the superadditivity of R and the monotonicity of G, and the third follows from the superadditivity of g and the monotonicity of R and G. Moreover, if additionally r, f_j, F_j $(j = 1, \ldots, n-1)$ are nonlinear on any subinterval of $\mathbf{R}^+$, equality holds in $\sum_{i=1}^m A(\mathbf{x}_i) \leqslant A\big(\sum_{i=1}^m \mathbf{x}_i\big)$ only if all $\mathbf{x}_i$ are proportional.

EXAMPLE 1. Let the conditions of Theorem 1 be fulfilled. Then if $r(x) = 1$ $(x > 0)$, we get the inequality

$$\sum_{i=1}^{m} A(\mathbf{x}_i) = \sum_{i=1}^{m} G\big(g(\mathbf{x}_i), \ldots, g(\mathbf{x}_i)\big)$$
$$= \sum_{i=1}^{m} g(\mathbf{x_i}) G(1, \ldots, 1)$$
$$\leqslant g\Big(\sum_{i=1}^{m} \mathbf{x}_i\Big) G(1, \ldots, 1)$$
$$= A\Big(\sum_{i=1}^{m} \mathbf{x}_i\Big)$$

which in the case of $f_j(x) = x^{q_j}$ with $0 < q_j < 1$ $(j = 1, \ldots, n-1)$ is Hölder's inequality. If $r(x) = x$ $(x > 0)$, then we get the inequality

$$\sum_{i=1}^{m} A(\mathbf{x}_i) = \sum_{i=1}^{m} G(\mathbf{x}_i) \leqslant G\Big(\sum_{i=1}^{m} x_{i1}, \ldots, \sum_{i=1}^{m} x_{in}\Big) = A\Big(\sum_{i=1}^{m} \mathbf{x}_i\Big)$$

which in the case of $F_j(x) = x^{q_j}$ $(x > 0)$ with $0 < q_j < 1$ $(j = 1, \ldots, n)$ is Hölder's inequality, too.

2. Main results

In the following, the functions $g = g(\mathbf{a})$ and $G = G(\mathbf{a})$ are defined by functions f_j and F_j as in Definition 1, for elements $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbf{R}^n$ with $a_i > 0$ $(i = 1, \ldots, n)$. However, the function r is now dependent on two variables $r = r(x,t)$ and the function R generated by it is now in analogy given by

$$R(a,b,t) = ar\Big(\frac{b}{a}, t\Big) \qquad \big((a,b) \in \mathbf{R}^2, t_0 \leqslant t \leqslant t_1\big).$$

We will discuss conditions on r, f_j, F_j $(j = 1, \ldots, n-1)$ for which the function $H = H(t)$ defined by

$$H(t) = G(\mathbf{z}(t)) = G\Big(\sum_{i=1}^{m} R(g(\mathbf{x}_i), x_{i1}, t), \ldots, \sum_{i=1}^{m} R(g(\mathbf{x}_i), x_{in}, t)\Big) \tag{5}$$

is monotone on $[t_0, t_1]$, where $\mathbf{z} = (z_1, \ldots, z_n)$ with z_j $(j = 1, \ldots, n)$ given by

$$z_j(t) = \sum_{i=1}^{m} R(g(\mathbf{x}_i), x_{ij}, t) \qquad (t_0 \leqslant t \leqslant t_1) \tag{6}$$

and as before $\mathbf{x}_i = (x_{i1}, \ldots, x_{in}) \in \mathbf{R}^n$ with $x_{ij} > 0$.

We need the following lemmas.

LEMMA 2. *Let $F \geqslant 0$ be a concave function on $\mathbf{R}^+$. Then F is increasing. If F is also differentiable, then $0 \leqslant x\frac{F'(x)}{F(x)} \leqslant 1$ on $\mathbf{R}^+$.*

LEMMA 3. *Let the functions F_j generating by Definition 1 the function G and the functions z_j in (6) be differentiable and denote*

$$F_j'(D_j) = \frac{d}{dD_j}F_j(D_j)$$

and

$$\begin{aligned} D_j &= \frac{1}{z_j(t)}G_{j+1,n-1}\big(z_{j+1}(t),\ldots,z_n(t)\big) \\ &= \frac{z_{j+1}(t)}{z_j(t)}F_{j+1}\Big(\frac{z_{j+2}(t)}{z_{j+1}(t)}F_{j+2}\Big(\frac{z_{j+3}(t)}{z_{j+2}(t)}\ldots F_{n-1}\Big(\frac{z_n(t)}{z_{n-1}(t)}\Big)\Big)\Big). \end{aligned}$$

for $j = 1,\ldots,n$. Then

$$\begin{aligned} \frac{G_t'(\mathbf{z}(t))}{G(\mathbf{z}(t))} &= \sum_{j=1}^{n-1}\frac{z_j'(t)}{z_j(t)}\Big(1-\frac{F_j'(D_j)}{F_j(D_j)}D_j\Big)\prod_{k=1}^{j-1}\frac{F_k'(D_k)}{F_k(D_k)}D_k + \frac{z_n'(t)}{z_n(t)}\prod_{k=1}^{n-1}\frac{F_k'(D_k)}{F_k(D_k)}D_k \\ &= \sum_{j=1}^{n}\frac{z_j'(t)}{z_j(t)}P_j \end{aligned}$$

where

$$\begin{aligned} P_j &= \Big(1-\frac{F_j'(D_j)}{F_j(D_j)}D_j\Big)\prod_{k=1}^{j-1}\frac{F_k'(D_k)}{F_k(D_k)}D_k \quad (1 \leqslant j \leqslant n-1) \qquad (7) \\ P_n &= \prod_{k=1}^{n-1}\frac{F_k'(D_k)}{F_k(D_k)}D_k \quad \text{and} \quad \sum_{j=1}^{n}P_j = 1. \end{aligned}$$

LEMMA 4. *Let the functions $F_j = F_j(x)$ generating by Definition 1 the function G and the function $r = r(x,t)$ be differentiable in x and t, respectively, set $V = \frac{r_t'}{r}$, let P_j given by (7) and denote $w_{ij} = \frac{x_{ij}}{g(\mathbf{x}_i)}$. Then the function H given by (5) is differentiable,*

$$\begin{aligned} \frac{H_t'(t)}{H(t)} &= \sum_{j=1}^{n}P_j\frac{\sum_{i=1}^m g(\mathbf{x}_i)r(w_{ij},t)V(w_{ij},t)}{\sum_{i=1}^m g(\mathbf{x}_i)r(w_{ij},t)} \\ &= \sum_{j=1}^{n}P_j\Big[\frac{\sum_{i=1}^m g(\mathbf{x}_i)r(w_{ij},t)V(w_{ij},t)}{\sum_{i=1}^m g(\mathbf{x}_i)r(w_{ij},t)} - \frac{\sum_{i=1}^m g(\mathbf{x}_i)V(w_{ij},t)}{\sum_{i=1}^m g(\mathbf{x}_i)}\Big] \\ &\quad + \sum_{j=1}^{n}\Big(\frac{\sum_{i=1}^m g(\mathbf{x}_i)V(w_{ij},t)}{\sum_{i=1}^m g(\mathbf{x}_i)}\Big)P_j. \end{aligned}$$

A simple calculation shows also that

$$\begin{aligned} \frac{H_t'(t)}{H(t)} &= \sum_{j=1}^{n}P_j\frac{\sum_{1\leqslant i\leqslant k\leqslant m} g(\mathbf{x}_i)g(\mathbf{x}_k)\big(r(w_{ij},t)-r(w_{kj},t)\big)\big(V(w_{ij},t)-V(w_{kj},t)\big)}{\sum_{i=1}^m g(\mathbf{x}_i)r(w_{ij},t)\sum_{k=1}^m g(\mathbf{x}_k)} \\ &\quad + \frac{1}{\sum_{i=1}^m g(\mathbf{x}_i)}\sum_{i=1}^{m}g(\mathbf{x}_i)\sum_{j=1}^{n}V(w_{ij},t)P_j. \end{aligned}$$

LEMMA 5. *Let* F_j $(j=1,\dots,n-1)$ *with* $F_j(x)>0$ *be differentiable concave functions on* $[0,\infty)$. *Then* P_j $(j=1,\dots,n-1)$ *defined by (7) satisfy* $0\leqslant P_j\leqslant 1$ *and* $\sum_{j=1}^n P_j=1$.

LEMMA 6. *Let* $V=V(x,t)$ *be a function given on* $\mathbf{R}^+\times[t_0,t_1]$ *such that* $V(x,t)$ *increases in* x, $V(e^x,t)$ *is convex in* x *and* $V(1,t)\geqslant 0$ *for all* t. *Further, let* $0<P_j<1$ $(j=1,\dots,n)$ *be such that* $\sum_{i=1}^n P_j=1$. *At least, let* $g=g(\mathbf{x})$ *be the function from Definition* 1 *defined for* $\mathbf{x}=(x_1,\dots,x_n)\in\mathbf{R}^n$ *with* $x_j>0$ $(j=1,\dots,n)$. *Then*

$$\frac{\prod_{j=1}^n x_j^{P_j}}{g(\mathbf{x})}\geqslant 1$$

implies

$$\sum_{j=1}^n P_jV\Big(\frac{x_j}{g(\mathbf{x})},t\Big)\geqslant V\left(\frac{\prod_{j=1}^n x_j^{P_j}}{g(\mathbf{x})},t\right)\geqslant 0$$

for all $t_0\leqslant t\leqslant t_1$.

DEFINITION 2. Functions $r(x,t)$ and $V(x,t)$ with common domain are said to be *similarly ordered* in x if, for every fixed t,

$$\big(r(x_1,t)-r(x_2,t)\big)\cdot\big(V(x_1,t)-V(x_2,t)\big)\geqslant 0$$

for all x.

THEOREM 2. *Let the functions* F_j $(j=1,\dots,n-1)$ *in Definition* 1 *generating* G *be differentiable and concave, and let the function* $r=r(x,t)$ *generating* $R=R(a,b,t)$ *be differentiable in* t *for every* $x>0$. *Further, let* P_j *be defined by (7), denote* $V=\frac{r_t'}{r}$ *and suppose* $V(x,t)\geqslant 0$ *for* $x\geqslant 1$, $V(e^x,t)$ *to be convex in* x, *and* $V(x,t)$ *and* $r(x,t)$ *to be similarly ordered in* x, *for every* t. *At least, let* $g=g(\mathbf{x})$ *be the function from Definition* 1.*Then, for* $\mathbf{x}_i=(x_{i1},\dots,x_{in})\in\mathbf{R}^n$ *with* $x_{ij}>0$ $(i=1,\dots,m, j=1,\dots,n)$,

$$\frac{\prod_{j=1}^n x_{ij}^{P_j}}{g(\mathbf{x}_i)}\geqslant 1\qquad(i=1,\dots,m)$$

implies that the function H *given by (5) is increasing on* $[t_0,t_1]$.

Proof. From Lemma 4 and also from the discrete Chebyshev inequality [3: p. 198] we get that if $r(x,t)$ and $V(x,t)$ are similarly ordered in x for each $t_0\leqslant t\leqslant t_1$ and if

$$\sum_{j=1}^n V(w_{ij},t)P_j\geqslant 0\qquad(i=1,\dots,m),$$

then $\frac{H_t'(t)}{H(t)}>0$ for any t. From Lemma 5 we get $0\leqslant P_j\leqslant 1$ $(j=1,\dots,n)$ and $\sum_{j=1}^n P_j=1$, and as the conditions of Lemma 6 are satisfied and $w_{ij}=\frac{x_{ij}}{g(\mathbf{x}_i)}$, we get

$$\sum_{j=1}^n P_jV(w_{ij},t)\geqslant V\left(\frac{\prod_{j=1}^n x_{ij}^{P_j}}{g(\mathbf{x}_i)},t\right)\geqslant 0\qquad(i=1,\dots,m).$$

This completes the proof of Theorem 2.

EXAMPLE 2. If, for $\mathbf{x} = (x_1, \dots, x_n) \in \mathbf{R}^n$ with $x_i > 0$ $(i = 1, \dots, n)$,

$$G(\mathbf{x}) = g(\mathbf{x}) = \prod_{j=1}^{n} x_j^{\alpha_j} \qquad \left(\mathbf{x} = (x_1, \dots, x_n) \in \mathbf{R}^n\right)$$

with $0 \leqslant \alpha_j \leqslant 1$ and $\sum_{j=1}^{n} \alpha_j = 1$, and if $r(x,t) = x^t$ $(0 \leqslant t \leqslant 1)$ and $w_{ij} = \frac{x_{ij}}{y_i}$, then all conditions of Theorem 2 are satisfied, and for $\mathbf{x}_i = (x_{i1}, \dots, x_{in}) \in \mathbf{R}^n$ $(i = 1, \dots, m)$ with $x_{ij} > 0$ $(j = 1, \dots, n)$ the function H given by

$$H(t) = \prod_{j=1}^{n} \left(\sum_{i=1}^{m} g(\mathbf{x}_i) w_{ij}^t \right)^{\alpha_j}$$

is monotone increasing on $[t_0, t_1]$. This was proved in [4] for the function h given in (1).

EXAMPLE 3. An immediate example of Theorem 1 for

$$f(x) = x^{\beta}, \;\; F(x) = (1 + x^q)^{\frac{1}{q}}, \;\; r(x,t) = x^t \qquad (0 < x;\, 0 \leqslant t, \beta, q \leqslant 1)$$

and $\mathbf{x}_i = (x_{i1}, x_{i2})$ $(i = 1, \dots, m)$ with $x_{ij} > 0$ $(j = 1, 2)$ is the Minkowski-Hölder-type inequality

$$\begin{aligned} \sum_{i=1}^{m} A(\mathbf{x}_i, t) &= \sum_{i=1}^{m} (x_{i1}^{1-\beta} x_{i2}^{\beta})^{1-t} (x_{i1}^{qt} + x_{i2}^{qt})^{\frac{1}{q}} \\ &\leqslant \left(\sum_{i=1}^{m} x_{i1} \right)^{(1-\beta)(1-t)} \left(\sum_{i=1}^{m} x_{i2} \right)^{\beta(1-t)} \left(\left(\sum_{i=1}^{m} x_{i1} \right)^{qt} + \left(\sum_{i=1}^{m} x_{i2} \right)^{qt} \right)^{\frac{1}{q}} \\ &= A\left(\sum_{i=1}^{m} (\mathbf{x}_i, t) \right), \end{aligned}$$

and as a result of Theorem 2 we get that the function H given by

$$H(t) = \left(\left(\sum_{i=1}^{m} (x_{i1}^{1-\beta} x_{i2}^{\beta})^{1-t} x_{i1}^t \right)^q + \left(\sum_{i=1}^{m} (x_{i1}^{1-\beta} x_{i2}^{\beta})^{1-t} x_{i2}^t \right)^q \right)^{\frac{1}{q}}$$

is increasing on $\{t \geqslant 0\}$ for $q \geqslant 0$ in the following cases:

1. $\beta = \frac{1}{2}$ and $\frac{x_{i2}}{x_{i1}} \geqslant 1$ or $\frac{x_{i2}}{x_{i1}} \leqslant 1$ $(i = 1, \dots, m)$.
2. $\beta \leqslant \frac{1}{2}$ and $\frac{x_{i2}}{x_{i1}} \geqslant 1$ $(i = 1, \dots, m)$.
3. $\beta \geqslant \frac{1}{2}$ and $\frac{x_{i2}}{x_{i1}} \leqslant 1$ $(i = 1, \dots, m)$.

In these three cases we get especially that

$$\sum_{i=1}^{m} A(\mathbf{x}_i, 0) = H(0) \leqslant H(t) \leqslant H(1) = A\left(\sum_{i=1}^{m} (\mathbf{x}_i, 1) \right)$$

for $0 \leqslant t \leqslant 1$.

Proof. Denote $y_i = x_{i1}(\frac{x_{i2}}{x_{i1}})^\beta$ and $w_{ij} = \frac{x_{ij}}{y_i}$ for $i = 1,\dots,m$ and $j = 1,2$. Then, according to Lemma 3 and Lemma 4,

$$\begin{aligned}\frac{H'(t)}{H(t)} &= (1-P)\frac{\sum_{1\leqslant i\leqslant k\leqslant m} y_i y_k (w_{i1}^t - w_{k1}^t)(\ln w_{i1} - \ln w_{k1})}{(\sum_{i=1}^m y_i w_{i1}^t)(\sum_{i=1}^m y_k)} \\ &\quad + P\frac{\sum_{1\leqslant i\leqslant k\leqslant m} y_i y_k (w_{i2}^t - w_{k2}^t)(\ln w_{i2} - \ln w_{k2})}{(\sum_{i=1}^m y_i w_{i2}^t)(\sum_{i=1}^m y_k)} \\ &\quad + \frac{1}{\sum_{i=1}^m y_i}\sum_{i=1}^m \big((1-P)\ln w_{i1} + P\ln w_{i2}\big) \end{aligned} \tag{8}$$

where

$$P = \frac{(\sum_{i=1}^m y_i w_{i2}^t)^q}{(\sum_{i=1}^m y_i w_{i1}^t)^q + (\sum_{i=1}^m y_i w_{i2}^t)^q}.$$

As $\frac{x_{i2}}{x_{i1}} = \frac{w_{i2}}{w_{i1}}$, we get that

$$\frac{1}{2} \leqslant P \leqslant 1 \quad \text{when} \quad \frac{x_{i2}}{x_{i1}} \geqslant 1$$

$$0 < P \leqslant \frac{1}{2} \quad \text{when} \quad \frac{x_{i2}}{x_{i1}} \leqslant 1 \qquad (i = 1,\dots,m;\ t \geqslant 0, q > 0).$$

Hence, if $\beta = \frac{1}{2}, t \geqslant 0, q > 0$ we get

$$\frac{1}{\sum_{i=1}^m y_i}\sum_{i=1}^m y_i \ln\left(\frac{x_{i2}}{x_{i1}}\right)^{P-\beta} \geqslant 0. \tag{9}$$

It is easy to verify that (9) holds too when $\beta \leqslant \frac{1}{2}, q > 0, t > 0$ for $\frac{x_{i2}}{x_{i1}} \geqslant 1$ $(i = 1,\dots,m)$ and also when $\beta \geqslant \frac{1}{2}, q > 0, t \geqslant 0$ for $0 < \frac{x_{i2}}{x_{i1}} \leqslant 1$ $(i = 1,\dots,m)$. As

$$\frac{1}{(\sum_{i=1}^m y_i)}\sum_{i=1}^m \big((1-P)\ln w_{i1} + P\ln w_{i2}\big) = \frac{1}{(\sum_{i=1}^m y_i)}\sum_{i=1}^m y_i \ln\left(\frac{x_{i2}}{x_{i1}}\right)^{P-\beta}$$

and as $t \geqslant 0$ we get from (8) and (9) that $H'(t) \geqslant 0$ in all the three cases which are stated in this example.

EXAMPLE 4. For $t \geqslant 1$ and $x \geqslant 0$, the functions $r(x,t) = x^t t^x$ and $V(x,t) = \frac{r_t'(x,t)}{r(x,t)} = \ln x + \frac{x}{t}$ are similarly ordered in x for every fixed t, $V(1,t) \geqslant 0$ and $V(e^x,t)$ is convex in x. Therefore, under the conditions of Theorem 2, its result holds here for $t \geqslant 1$. In particular, if $n = 2$, $f(x) = x^{\frac{1}{2}}$ and $F(x) = (1+x^q)^{\frac{1}{q}}$, then

$$\begin{aligned} H(1) &= \left(\left(\sum_{i=1}^m x_{i1}\right)^q + \left(\sum_{i=1}^m x_{i2}\right)^q\right)^{\frac{1}{q}} \\ &\leqslant \left(\left(\sum_{i=1}^m x_{i1}^{\frac{1}{2}+\frac{t}{2}} x_{i2}^{\frac{1}{2}-\frac{t}{2}} t^{(\frac{x_{i1}}{x_{i2}})^{\frac{1}{2}}}\right)^q + \left(\sum_{i=1}^m x_{i1}^{\frac{1}{2}-\frac{t}{2}} x_{i2}^{\frac{1}{2}+\frac{t}{2}} t^{(\frac{x_{i2}}{x_{i1}})^{\frac{1}{2}}}\right)^q\right)^{\frac{1}{q}} \\ &= H(t) \leqslant H(t_1) \end{aligned}$$

for $1 \leqslant t \leqslant t_1$.

REMARK 1. The inequality $\sum_{j=1}^{n} \frac{x_{ij}^{P_j}}{y_i} \geqslant 1 \quad (i = 1, \ldots, m)$ in Theorem 2 means that, in the case $n = 2$, F has to satisfy the differential inequality $x^{x^{\frac{F'(x)}{F(x)}}} \geqslant f(x)$. For instance, the inequality holds if $F(x) = x^{\beta}$ $(\frac{1}{2} \leqslant \beta \leqslant 1)$ and $f(x) = \arctan x$ $(x \geqslant 0)$. As these functions are concave too, Theorem 1 holds for any $r(x,t) \geqslant 0$ which is concave on $\mathbf{R}^+$, for every $t_0 \leqslant t \leqslant t_1$.

Theorem 2 holds if $r = r(x,t)$ satisfies the relevant conditions stated there. If, for instance, $r(x,t) = x^t$ $(x > 0, 0 \leqslant t \leqslant 1)$, then for the functions F and f above both theorems hold for $n = 2$. The result of Theorem 2, if we choose $F(x) = (1 + x^q)^{\frac{1}{q}}$ with $0 < q < 1$ and $f(x) = \arctan x$, holds too for $\frac{x_{i2}}{x_{i1}} \geqslant 1$ $(i = 1, \ldots, m)$.

REMARK 2. The differential equation $x^{x^{\frac{F'(x)}{F(x)}}} = f(x)$ is solvable for $F(x) = f(x) = x^{\beta}$ only, and in order for Theorems 1 and 2 to hold in the case $n = 2$, F has to be concave and hence $0 \leqslant \beta \leqslant 1$.

The following theorem can be proved in a similar way as Theorem 2 and therefore the proof is omitted.

THEOREM 3. *Let* $Y : [a,b] \to \mathbf{R}$ *and* $f_j : [a,b] \to \mathbf{R}^+$ $(j = 1, \ldots, n)$ *be continuous functions, and let* $F_j : \mathbf{R}^+ \to \mathbf{R}^+$ $(j = 1, \ldots, n)$ *be concave differentiable functions. Further, let* $r = r(x,t)$ *with* $r(x,t) > 0$ *for* $(x,t) \in \mathbf{R}^+ \times [t_0, t_1]$ *be differentiable on* $\mathbf{R}^+ \times (t_0, t_1)$, *set* $V = \frac{r_t'}{r}$ *and suppose* $V(x,t)$ *to be non-negative on* $\{x \geqslant 1\}$, $V(e^x, t)$ *to be convex on* $\mathbf{R}^+$, *and* $r(x,t)$ *and* $V(x,t)$ *to be similarly ordered in* x, *for all* $t_0 \leqslant t \leqslant t_1$. *At least, let* $\prod_{i=1}^{n} f_i^{P_i}(x) \geqslant Y(x)$ $(x > 0)$ *where* P_i *are defined in (7). Then the function* H *defined by*

$$H(t) = G(z_1(t), \ldots, z_n(t)) \qquad (t_0 \leqslant t \leqslant t_1)$$

with

$$z_i(t) = \int_a^b Y(x) r\Big(\frac{f_i(x)}{Y(x)}, t\Big) dx \qquad (i = 1, \ldots, n)$$

is monotone increasing on $[t_0, t_1]$.

3. Continuous sharpening of the inverse Hölder's inequality

In this chapter we consider the reverse Hölder's inequality in both forms: discrete and integral. For this let $p > 0$ and $q < 0$ be real numbers such that $p + q = 1$. The *discrete* reverse Hölder's inequality states that if $x_{i1}, x_{i2} > 0$ $(i = 1, \ldots, m)$, then

$$\sum_{i=1}^{m} x_{i1}^{p} x_{i2}^{q} > \Big(\sum_{i=1}^{m} x_{i1}\Big)^{p} \Big(\sum_{i=1}^{m} x_{i2}\Big)^{q}.$$

The *integral* reverse Hölder's inequality states that if $f, g \geqslant 0$ are integrable real-valued functions on $[a,b]$, then

$$\int_a^b f(x)^p g(x)^q dx \geqslant \Big(\int_a^b f(x)\, dx\Big)^{p} \Big(\int_a^b g(x)\, dx\Big)^{q}.$$

The following theorem shows that the above inequalities can be sharpened too using a function similar to that in [4].

THEOREM 4. *Let* $p > 0$ *and* $q < 0$ *be real numbers such that* $p + q = 1$.

(a) *If* $f, g : [a, b] \to \mathbf{R}^+$ *are continuous functions and* $Y = f^p g^q$, *then the function*

$$h_I(t) = \left(\int_a^b Y(x) \Big(\frac{g(x)}{Y(x)} \Big)^t dx \right)^p \left(\int_a^b Y(x) \Big(\frac{f(x)}{Y(x)} \Big)^t dx \right)^q$$

is decreasing on $[0, 1]$. *In particular, for* $t \in [0, 1]$ *we have*

$$\int_a^b f(x)^p g(x)^q dx \geqslant h_I(t) \geqslant \left(\int_a^b f(x)\, dx \right)^p \left(\int_a^b g(x)\, dx \right)^q$$

which is a sharpening of the integral reverse Hölder's inequality.

(b) *If* $x_{i1}, x_{i2} > 0$ *are real numbers and* $y_i = x_{i1}^p x_{i2}^q$ $(i = 1, \ldots, m)$, *then the function*

$$h_D(t) = \left(\sum_{i=1}^m y_i \Big(\frac{x_{i1}}{y_i} \Big)^t \right)^p \left(\sum_{i=1}^m y_i \Big(\frac{x_{i2}}{y_i} \Big)^t \right)^q$$

is decreasing on $[0, 1]$.

Proof. (a) As a consequence of Chebyshev inequality [3: p. 198]

$$\int_a^b p(x)\, dx \int_a^b p(x) g_1(x) g_2(x)\, dx \geqslant \int_a^b p(x) g_1(x)\, dx \int_a^b p(x) g_2(x)\, dx$$

for similarly ordered functions g_1 and g_2 and weight p, choosing $g_1(x) = \ln \frac{g(x)}{f(x)}$ and $g_2(x) = \big(\frac{g(x)}{f(x)}\big)^t$ $(t \in [0, 1])$ and as weight function $p(x) = Y(x) \big(\frac{g(x)}{f(x)}\big)^{-tq}$, we get

$$\int_a^b Y(x) \Big(\frac{g(x)}{f(x)} \Big)^{tp} \ln \frac{g(x)}{f(x)}\, dx \int_a^b Y(x) \Big(\frac{g(x)}{f(x)} \Big)^{-tq} dx \\ - \int_a^b Y(x) \Big(\frac{g(x)}{f(x)} \Big)^{-tq} \ln \frac{g(x)}{f(x)}\, dx \int_a^b Y(x) \Big(\frac{g(x)}{f(x)} \Big)^{tp} dx \geqslant 0$$

and, because $p > 0$ and $q < 0$,

$$\begin{aligned} \frac{h_I'(t)}{h_I(t)} &= pq \frac{\int_a^b Y(x) (\frac{f(x)}{g(x)})^{tq} \ln \frac{f(x)}{g(x)} dx}{\int_a^b Y(x) (\frac{f(x)}{g(x)})^{tq} dx} + pq \frac{\int_a^b Y(x) (\frac{g(x)}{f(x)})^{tp} \ln \frac{g(x)}{f(x)} dx}{\int_a^b Y(x) (\frac{g(x)}{f(x)})^{tp} dx} \\ &= pq \frac{\int_a^b Y(x) (\frac{g(x)}{f(x)})^{tp} \ln \frac{g(x)}{f(x)} dx \int_a^b Y(x) (\frac{g(x)}{f(x)})^{-tq} dx}{\int_a^b Y(x) (\frac{g(x)}{f(x)})^{-tq} dx \int_a^b Y(x) (\frac{g(x)}{f(x)})^{tp} dx} \\ &\quad - pq \frac{\int_a^b Y(x) (\frac{g(x)}{f(x)})^{-tq} \ln \frac{g(x)}{f(x)} dx \int_a^b Y(x) (\frac{g(x)}{f(x)})^{tp} dx}{\int_a^b Y(x) (\frac{g(x)}{f(x)})^{-tq} dx \int_a^b Y(x) (\frac{g(x)}{f(x)})^{tp} dx} \\ &\leqslant 0. \end{aligned}$$

Hence $h_I'(t) \leqslant 0$ for $t \in [0,1]$, i.e. h_I is a decreasing function on $[0,1]$. The proof of statement (b) is similar and is therefore omitted

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