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ASYMPTOTIC STABILITY AND INTEGRAL INEQUALITIES FOR SOLUTIONS OF LINEAR SYSTEMS ON RADON–NIKODÝM SPACES

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Dedicated to the memory of prof. Mladen Alić

(*communicated by N. Elezović*)

Abstract. We consider the mild solution $u_f(\cdot,0)$ of a well-posed nonhomogeneous Cauchy problem

$$\begin{cases} \dot{u}(t) &= A(t)u(t)+f(t), \quad t \geqslant 0 \\ u(0) &= 0 \end{cases}$$

on a Radon-Nikodým space X, where $A(\cdot)$ is a linear operator-valued function. Under certain additional conditions we will prove that if the homogeneous system

$$\dot{u}(t) = A(t)u(t), \quad t \geqslant 0$$

is exponentially stable, then for each function f belonging to the Sobolev space $W^0_{p1}(\mathbb{R}_+,X)$, $1 \leqslant p < \infty$, the solution $u_f(\cdot,0)$ lies in the same space. The converse statement is more subtle, but it certainly works in the autonomous case. Integral inequalities of Landau type for the evolution semigroup associated with the system $(A(t))$ on the space $W^0_{p1}(\mathbb{R}_+,X)$ are also derived.

1. Introduction

In what follows X will denote a Banach space and $\mathscr{L}(X)$ will be the Banach space of all bounded linear operators acting on X. The norms on X and $\mathscr{L}(X)$ will be denoted by $\|\cdot\|$.

Let $\mathbf{T} = \{T(t)\}_{t\geqslant 0}$ be a strongly continuous semigroup of bounded linear operators acting on X, with infinitesimal generator $(G,D(G))$ and uniformly growth bound

$$\omega_0(\mathbf{T}) := \inf_{t>0} \frac{\ln(\|T(t)\|)}{t}.$$

We will denote by $\rho(G)$, $R(z,G)$ and respectively $\sigma(G)$ the resolvent set, the resolvent operator and the spectrum of G. It is well-known (see e.g. [11]) that the mild solution of the well-posed abstract Cauchy problem

$$\begin{cases} \dot{u}(t) = G(u(t)), \quad t \geqslant 0 \\ u(0) = x \in X \end{cases}$$

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is a trajectory of the semigroup $\mathbf{T}$.

We consider now the non-autonomous Cauchy problem

$$\begin{cases} \dot{u}(t) = A(t)u(t) + f(t), & t \geqslant s \geqslant 0, \\ u(s) = x \end{cases}$$

where $A(t)$ are linear operators on X, $x \in X$ and f is a X-valued, locally Bochner integrable function on $\mathbb{R}_+$. For suitable differential operators $A(t)$, these equations describe, for instance, diffusion processes with time dependent diffusion coefficients or boundary conditions, as well as the evolution of quantum mechanical systems with time varying potential. Further, important partial differential equations on non-cylindrical domains can be formulated as non-autonomous Cauchy problems, see [4]. The mild solutions of such well-posed Cauchy problems lead to a *strongly continuous evolution family* $\mathscr{U} := \{U(t,s)\}_{t \geqslant s \geqslant 0}$ of bounded linear operators on X, that is,

(i) $U(t,s) = U(t,r)U(r,s)$ for all $t \geqslant r \geqslant s \geqslant 0$;

(ii) $U(t,t)x = x$ for all $t \geqslant 0$ and all $x \in X$;

(iii) for every $x \in X$, the map $(t,s) \mapsto U(t,s)x$ is continuous on the set

$$\{(t,s) : t \geqslant s \geqslant 0\}.$$

For more details about the well-posedness of abstract differential equations, see [17] and the references therein.

The evolution family $\mathscr{U}$ is called *exponentially bounded* if there exist $\omega \in \mathbb{R}$ and $K \geqslant 1$ such that

$$\|U(t,s)\| \leqslant K e^{\omega(t-s)} \text{ for every } t \geqslant s \geqslant 0. \tag{EB}$$

If

$$U(t,s) = U(t-s,0) \text{ for all } t \geqslant s \geqslant 0, \tag{SC}$$

then the one-parameter family $\mathbf{T} := \{U(t,0) : t \geqslant 0\}$ is a strongly continuous semigroups on X and the relation (EB) is automatically verified.

The following result is well-known, see [5] and [9]: If the Cauchy problem

$$\begin{cases} \dot{u}(t) = A(t)u(t) + f(t), & t \geqslant 0 \\ u(0) = 0 \end{cases} \tag{(A(t),f,0)}$$

has a bounded solution for every continuous and bounded function $f : \mathbb{R}_+ \to X$, then the homogeneous system

$$\dot{u}(t) = A(t)u(t), \quad t \geqslant 0 \tag{A(t)}$$

is exponentially stable, that is, there exist the constants $N > 0$ and $\nu > 0$ such that $\|U(t,s)\| \leqslant Ne^{-\nu(t-s)}$ for every $t \geqslant s \geqslant 0$. Here $\mathscr{U} = \{U(t,s) : t \geqslant s \geqslant 0\} \subset \mathscr{L}(X)$ is the evolution family on X associated with the system $(A(t))$.

In order to define the evolution semigroup associated with a strongly continuous and exponentially bounded evolution family on X, let us consider the following Banach functions spaces:

$$E_p := \begin{cases} L^p(\mathbb{R}_+, X), & \text{if} \quad 1 \leqslant p < \infty \\ C_{00}(\mathbb{R}_+, X), & \text{if} \quad p = \infty. \end{cases}$$

Here $L^p(\mathbb{R}_+,X)$ denotes the Banach space of all Bochner measurable functions $f:\mathbb{R}_+\to X,$ such that

$$\|f\|_p := \left(\int_0^\infty \|f(s)\|^p ds\right)^{1/p} < \infty.$$

$C_{00}(\mathbb{R}_+,X)$ is the Banach space (endowed with the sup norm) of all continuous functions $f:\mathbb{R}_+\to X$ such that

$$f(0)=\lim_{t\to\infty} f(t)=0.$$

The evolution semigroup

$$\mathscr{T}_p := \{\mathscr{T}_p(t) : t \geqslant 0\} \subset \mathscr{L}(E_p)$$

associated with the family $\mathscr{U}$ on the space $E_p,$ is defined by

$$(\mathscr{T}_p(t)f)(s) := \begin{cases} U(s,s-t)f(s-t), & \text{if} \quad s \geqslant t \\ 0, & \text{if} \quad 0 \leqslant s < t. \end{cases}$$

Let G_p be its infinitesimal generator of $\mathscr{T}_p$. The following result on evolution families and their associated evolution semigroups was established in [1]:

THEOREM 1. *Let $\mathscr{U}=\{U(t,s)\}_{t\geqslant s\geqslant 0}$ be a strongly continuous and exponentially bounded evolution family on $X,$ $1\leqslant p\leqslant\infty,$ and $\mathscr{T}_p$ the evolution semigroup associated with U on the space E_p. Then following four statements are equivalent:*

(i) $\mathscr{U}$ is exponentially stable i.e., (EB) holds for some negative ω;

(ii) $\omega_0(\mathscr{T}_p)$ is negative;

(iii) $\sigma(G_p)$ is a subset of $\mathbb{C}_- := \{z\in\mathbb{C} : \operatorname{Re} z<0\}$;

(iv) Given $\mu\in\mathbb{R}$ and $f\in E_p,$ the mild solution of the Cauchy problem

$$\begin{cases} dotu(t) = G_p u(t) + e^{i\mu t} f, & t\geqslant 0 \\ u(0)=0 \end{cases}$$

is bounded, that is,

$$\sup_{t\geqslant 0}\left\|\int_0^t e^{-i\mu s}\mathscr{T}_p(s)f\, ds\right\|_{E_p} < \infty.$$

Clearly, $(i)\Rightarrow(ii)\Rightarrow(iii)$. The implication $(iii)\Rightarrow(ii)$ is a consequence of the spectral mapping theorem. See [10], Theorem 2.2, Corollary 2.4, for the case $p=\infty$, and [8], Theorem 2.2 and Corollary 2.3, for $1\leqslant p<\infty$. The implication $(ii)\Rightarrow(i)$ follows by [15] or [3] Theorem 2.2. Finally, the equivalence between (i) and (iv) follows from the following result, which was proved in [1] and [14]:

THEOREM 2. *A bounded strongly continuous semigroup $T=\{T(t)\}_{t\geqslant 0}$ on a Banach space X satisfies the condition*

$$\sup_{t\geqslant 0}\left\|\int_0^t e^{-i\mu\xi}T(\xi)x d\xi\right\| = M(\mu,x) < \infty \textit{ for every } \mu\in\mathbb{R} \textit{ and every } x\in X$$

if and only if the spectrum of its infinitesimal generator belongs to $\mathbb{C}_-$.

2. Results

It is known, see for example [3], that $u_f(\cdot,0)$ belongs to E_p for every $f \in E_p$ if and only if the evolution family $\mathscr{U}$ is exponentially stable. Related result is given in the Theorem 3 below. For each $t \geqslant 0$ we consider the function g_t defined by:

$$g_t(r) := \left(\int_0^t \mathscr{T}_p(s)f\, ds\right)(r).$$

LEMMA 1. *We suppose that the evolution family $\mathscr{U}$ is uniformly stable, that is,*

$$\sup_{t\geqslant s\geqslant 0} \|U(t,s)\| = M < \infty. \tag{2.1}$$

Let $1 \leqslant p < \infty$ and $f \in L^p(\mathbb{R}_+,X)$. If the mild solution $u_f(\cdot,0)$ of the Cauchy problem $(A(t),f,0)$ belongs to $L^p(\mathbb{R}_+,X)$, then for each $t \geqslant 0$ the function g_t belongs to $L^p(\mathbb{R}_+,X)$, and moreover,

$$\sup_{t\geqslant 0}\|g_t\|_p \leqslant (2+M^p)^{\frac{1}{p}}\|u_f(\cdot,0)\|_p < \infty.$$

Proof. An easy computation, based on [16], Lemma 2.2, yields

$$g_t(r) = \begin{cases} \displaystyle\int_0^t U(r,r-s)f(r-s)ds, & \text{if} \quad r \geqslant t \\ \displaystyle\int_0^r U(r,r-s)f(r-s)ds, & \text{if} \quad 0 \leqslant r < t. \end{cases}$$

In order to prove that g_t is a L^p-function we remark that $\|g_t\|_p^p = I_1 + I_2$ where

$$I_1 := \int_t^\infty \|\int_0^t U(r,r-s)f(r-s)ds\|^p dr$$

and

$$I_2 := \int_0^t \|\int_0^r U(r,r-s)f(r-s)ds\|^p dr.$$

Clearly,

$$I_2 \leqslant \left(\|\int_0^{\cdot} U(\cdot,\cdot-s)f(\cdot-s)ds\|_p\right)^p = \|u_f(\cdot,0)\|_p^p$$

while for I_1 we have

$$I_1 \leqslant \|\int_0^{\cdot} U(\cdot,\cdot-s)f(\cdot-s)ds\|_p^p + \int_t^\infty \|\int_t^r U(r,r-s)f(r-s)ds\|^p dr$$

and

$$\begin{aligned} \|\int_t^r U(r,r-s)f(r-s)ds\|^p &= \|\int_0^{r-t} U(r,\xi)f(\xi)d\xi\|^2 \\ &\leqslant \|U(r,r-t)\int_0^{r-t} U(r-t,\xi)f(\xi)d\xi\|^p \\ &\leqslant M^p\|\int_0^{r-t} U(r-t,\xi)f(\xi)d\xi\|^p. \end{aligned}$$

Then

$$\int_t^{\infty} \| \int_t^r U(r, r-s) f(r-s) ds \|^p dr$$
$$\leqslant M^p \int_0^{\infty} \| \int_0^{\rho} U(\rho, \xi) f(\xi) d\xi \|^p d\rho = M^p \|u_f(\cdot, 0)\|_p^p$$

Finally we obtain:

$$\sup_{t \geqslant 0} \|g_t\|_p \leqslant (2 + M^p)^{1/p} \|u_f(\cdot, 0)\|_p < \infty.$$

The proof of Lemma 1 is complete. □

We make the following assumption about the family $\mathscr{U}$:
Assumption 1

$$\frac{1}{h} \|U(t+h, s+h)x - U(t,s)x\| \to 0 \text{ as } h \to 0+,$$

for each $t \geqslant s \geqslant 0$ and each $x \in X$.

If the evolution family $\mathscr{U}$ is a strongly continuous semigroup, that is, if the condition (SC) holds (or, equivalently, if the map $t \mapsto A(t)$ is constant), then the *Assumption 1* is fulfilled.

If $X = \mathbb{C}$ and $U(t,s) := \frac{1+s}{1+t}$ for every $t \geqslant s \geqslant 0$, then the family $\mathscr{U} = \{U(t,s) : t \geqslant s \geqslant 0\}$ is a strongly continuous and uniformly stable evolution family on $\mathbb{C}$ which fails *Assumption 1*.

We remind that a X-valued function f on $\mathbb{R}_+$ is called absolutely continuous if for each positive real number a the restriction of f to the interval $[0, a]$ is absolutely continuous.

In the following we will suppose that X is a Radon-Nikodým space, that is, every X-valued, absolutely continuous function is differentiable almost everywhere on compact subintervals of $\mathbb{R}_+$. Examples of such spaces are the reflexive Banach spaces (particularly, the Hilbert spaces). If X is a Radon-Nikodým space and f is a X-valued absolutely continuous function on $\mathbb{R}_+$, then its derivative f' is locally Bochner integrable. See [6].

The Sobolev space $W_{p1}^0(\mathbb{R}_+, X)$ consists of all X-valued, absolutely continuous functions f on $\mathbb{R}_+$ such that $f(0) = 0$ and

$$\|f\|_{p1}^0 := \|f\|_p + \|f'\|_p < \infty.$$

LEMMA 2. *Let X be a Radon-Nikodým space and let f be an X-valued, absolutely continuous function on $\mathbb{R}_+$ such that f and its derivative f' are L^p-functions. If $\mathscr{U}$ is a uniformly stable evolution family on X, such that the Assumption 1 is fulfilled, then*

$$(\mathscr{T}_p(t)f)' = \mathscr{T}_p(t)f' \quad \textit{almost everywhere on } \mathbb{R}_+. \tag{2.2}$$

for every $t \geqslant 0$.

Proof. Using the relation (2.1), it follows that

$$U(s+h,s+h-t)\frac{f(s+h-t)-f(s-t)}{h}-U(s+h,s+h-t)f'(s-t)\to 0,$$

as $h\to 0$ and thus

$$U(s+h,s+h-t)\frac{f(s+h-t)-f(s-t)}{h}\to U(s,s-t)f'(s-t) \text{ as } h\to 0.$$

Due to *Assumption 1* we conclude that

$$\frac{U(s+h,s+h-t)f(s-t)-U(s,s-t)f(s-t)}{h}\to 0 \text{ as } h\to 0.$$

Then (2.2) holds and the proof is complete. □

We remark that the argument in the proof of Lemma 2, works in the case of strongly continuous semigroups, even if it is not uniformly stable.

PROPOSITION 1. *Let $\mathscr{U}=\{U(t,s)\}_{t\geqslant s\geqslant 0}$ be a strongly continuous, uniformly stable evolution family on X such that Assumption 1 is fulfilled. For each $t\geqslant 0$, some $1\leqslant p<\infty$ and each function $f\in L^p(\mathbb{R}_+,X)$ we consider the map $t\mapsto \mathscr{T}_0(t)f:=\mathscr{T}_p(t)f$. Then the linear operator $f\mapsto \mathscr{T}_0(t)f$ acts on $W^0_{p1}(\mathbb{R}_+,X)$ and the family $\mathscr{T}_0:=\{\mathscr{T}_0(t)\}_{t\geqslant 0}$ is a strongly continuous semigroup which is called evolution semigroup associated to $\mathscr{U}$ on the space $W^0_{p1}(\mathbb{R}_+,X)$.*

Proof. Based on the relation (2.1) it is easy to see that for every $t\geqslant 0$, the map $s\mapsto(\mathscr{T}_0(t)f)(s)$ is an absolutely continuous function. Using the fact that $(\mathscr{T}_0(t)f)'=\mathscr{T}_p(t)f'$ almost everywhere on $\mathbb{R}_+$ and the fact that $\mathscr{T}_p(t)$ acts on $L^p(\mathbb{R}_+,X)$ it follows that $\mathscr{T}_0(t)$ acts on $W^0_{p1}(\mathbb{R}_+,X)$. Moreover, the maps

$$t\mapsto \mathscr{T}_p(t)f:\mathbb{R}_+\to L^p(\mathbb{R}_+,X),\quad t\mapsto \mathscr{T}_p(t)f':\mathbb{R}_+\to L^p(\mathbb{R}_+,X),$$

are continuous, see [2], [3] for details, and therefore

$$\|\mathscr{T}_0(t)f-f\|^0_{p1}=\|\mathscr{T}_p(t)f-f\|_p+\|\mathscr{T}_p(t)f'-f'\|_p\to 0$$

when $t\to 0$, that is, the the semigroup $\mathscr{T}_0$ on $W^0_{p1}(\mathbb{R}_+,X)$, is strongly continuous. □

THEOREM 3. *Let $\mathscr{U}$ be an evolution family on X generated by the family of linear operators $\{A(t)\}_{t\geqslant 0}$. We suppose that $\mathscr{U}$ is strongly continuous, uniformly stable and such that the Assumption 1 is fulfilled. Let $\mathscr{T}_0$ the evolution semigroup associated to $\mathscr{U}$ on the space $W^0_{p1}(\mathbb{R}_+,X)$ for some $1\leqslant p<\infty$, and G_0 its infinitesimal generator. If for each f in the space $W^0_{p1}(\mathbb{R}_+,X)$ the mild solution of $(A(t),f,0)$ lies in the same space then for each real number μ and each $f\in W^0_{p1}(\mathbb{R}_+,X)$ one has*

$$\sup_{t\geqslant 0}\left\|\int_0^t e^{-i\mu s}\mathscr{T}_0 f(s)ds\right\|_{W^0_{p1}(\mathbb{R}_+,X)}=K_{\mu,p}(f)<\infty \tag{2.3}$$

or equivalently-conform to the above Theorem 2-the spectrum of G_0 is a subset of $\mathbb{C}_-$.

Proof. First we prove that (2.3) holds for $\mu = 0$. Let $f \in W^0_{p1}(\mathbb{R}_+, X)$ and $t \geqslant 0$. For almost everywhere $\tau \in [0,t]$, one has:

$$\frac{d}{d\tau}\left[\left(\int_0^t \mathscr{T}_p(s)f\,ds\right)(\tau)\right] = \frac{d}{d\tau}\left[\int_0^\tau U(\tau,\tau-s)f(\tau-s)ds\right]$$
$$= U(\tau,0)f(0) + \int_0^\tau \frac{d}{d\tau}[U(\tau,\tau-s)f(\tau-s)]ds$$
$$= \left[\int_0^t \mathscr{T}_p(s)f'ds\right](\tau)$$

where the result from the above Lemma 2 was used. Similar result holds for almost everywhere $\tau > t$. Then using the above Lemma 1, we get:

$$\sup_{t\geqslant 0}\|\int_0^t \mathscr{T}_0(s)f\,ds\|^0_{p1} \leqslant \sup_{t\geqslant 0}\|\int_0^t \mathscr{T}_p(s)f\,ds\| + \sup_{t\geqslant 0}\|\int_0^t \mathscr{T}_p(s)f'ds\|$$
$$\leqslant (2+M^p)^{1/p}[\|u_f(\cdot,0)\|_p + \|u_{f'}(\cdot,0)\|_p] < \infty.$$

From the Principle of Uniform Boundedness it follows that there exists a positive constant K_p such that:

$$\sup_{t\geqslant 0}\left\|\int_0^t \mathscr{T}_0(s)f\,ds\right\|_{W^0_{p1}(\mathbb{R}_+,X)} \leqslant K_p\|f\|_{W^0_{p1}(\mathbb{R}_+,X)}.$$

Let μ be a given real number. In order to obtain (2.3) put $\mathscr{U}_\mu := \{e^{-i\mu(t-s)}U(t,s) : t \geqslant s \geqslant 0\}$ instead of $\mathscr{U}$. Nothing that the evolution semigroup associated to $\mathscr{U}_\mu$ is given by $\mathscr{T}_{0,\mu}(t) = e^{-i\mu t}\mathscr{T}_0(t)$. □

COROLLARY 1. *Under the conditions in Theorem 3 we have:*

$$\lim_{t\to\infty}\mathscr{T}_0(t)f = 0 \text{ for every } f \in W^0_{p1}(\mathbb{R}_+, X).$$

If, in addition, $\mathscr{T}_0$ *is analytic then* $\mathscr{T}_0$ *is exponentially stable. In this case if the family* $\mathbf{T} := \{U(t,0) : t \geqslant 0\}$ *is a strongly continuous semigroup then it is exponentially stable too.*

Proof. The first part follows by Theorem 3 above and ([13], Theorem 3). The second part can be obtained as follows. Let us consider the function $s \mapsto \phi(s) := se^{-s}$ on $\mathbb{R}_+$. It is clear that ϕ belongs to $W^0_{p1}(\mathbb{R}_+,\mathbb{R})$ and for each $x \in X$ the map

$$s \mapsto \phi_x(s) := \phi(s)x : \mathbb{R}_+ \to X$$

belongs to $W^0_{p1}(\mathbb{R}_+, X)$. Let N and ν be two positive constants such that

$$\|\mathscr{T}_0(t)\|_{\mathscr{L}(W^0_{p1}(\mathbb{R}_+,X))} \leqslant Ne^{-\nu t} \text{ for every } t \geqslant 0.$$

Then an easy computation yields

$$\|T(t)x\| \|\phi\|_{p1}^{0} = \|\mathscr{T}_0(t)\phi_x\|_{p1}^{0} \leqslant Ne^{-\nu t}\|\phi\|_{p1}^{0}\|x\|,$$

that is, $\mathbf{T}$ is exponentially stable. □

A recently result of van Neerven (see [12], Theorem 10), says that a strongly continuous semigroup $\mathbf{T} = \{T(t)\}_{t\geqslant 0}$ on a Banach space X is uniformly exponentially stable if for each f in $C_c((0,\infty),X)$ -the space of continuous X-valued functions with compact support in $(0,\infty)$-and some $1 \leqslant p < \infty$, one has:

$$\int_0^\infty \left\| \int_0^t T(s)f(t-s)ds \right\|^p dt < \infty.$$

The following result can be stated:

COROLLARY 2. *Let $\mathbf{T} = \{T(t) : t \geqslant 0\}$ be a strongly continuous semigroup on a Radon-Nikodým space X and $1 \leqslant p < \infty$. If for each f in the space $W_{p1}^{0}(\mathbb{R}_+,X)$ the map $t \mapsto \int_0^t T(s)f(t-s)ds$ lies in the same space then the semigroup $\mathbf{T}$ is uniformly exponentially stable.*

Proof. Let $f \in C_c((0,\infty),X) \subset W_{p1}^{0}(\mathbb{R}_+,X)$. According with the above Theorem 3 one has:

$$\begin{aligned}
\infty > \sup_{t\geqslant 0} \left\| \int_0^t \mathscr{T}_0(s)f\,ds \right\|_{W_{p1}^{0}(\mathbb{R}_+,X)} &\geqslant \sup_{t\geqslant 0} \left\| \int_0^t \mathscr{T}_p(s)f\,ds \right\|_p \\
&= \sup_{t\geqslant 0} \left(\int_0^\infty \left\| \left(\int_0^t \mathscr{T}_p(s)f\,ds \right)(\tau) \right\|^p d\tau \right)^{\frac{1}{p}} \\
&\geqslant \sup_{t\geqslant 0} \left(\int_0^t \left\| \int_0^\tau T(s)f(\tau-s)ds \right\|^p d\tau \right)^{\frac{1}{p}} \\
&= \left(\int_0^\infty \left\| \int_0^\tau T(s)f(\tau-s)ds \right\|^p d\tau \right)^{\frac{1}{p}}.
\end{aligned}$$

Now apply the above result of van Neerven in order to complete the proof. □

The following result was inspired by [10]:

LEMMA 3. *Let $\mathscr{T}_0$ be the evolution semigroup on $W_{p1}^{0}(\mathbb{R}_+,X)$, see Lemma 2 above, and let $(G_0, D(G_0))$ be its infinitesimal generator. If the set $\{u,f\}$ belongs to $W_{p1}^{0}(\mathbb{R}_+,X)$ then the following statements are equivalent:*

(i) $u \in D(G_0)$ *and* $G_0 u = -f$;

(ii) $u(t) = \int_0^t U(t,s)f(s)ds$ *for all* $t \geqslant 0$.

PROPOSITION 2. *If the family $\mathscr{U}$ is exponentially stable and Assumption 1 is fulfilled, then for each f belonging to $W^0_{p1}(\mathbb{R}_+,X)$ the mild solution $u_f(\cdot,0)$ of the Cauchy problem $(A(t),f,0)$, lies in the same space.*

Proof. The evolution semigroup $\mathscr{T}_0$ associated with $\mathscr{U}$ on the space $W^0_{p1}(\mathbb{R}_+,X)$ is exponentially stable, so its generator G_0 is an invertible operator. Then apply Lemma 3 above. $\square$

The following result is known in literature as *Kallman-Rota inequality*:

THEOREM 4. *Let $\mathbf{T} := \{T(t)\}_{t\geqslant 0}$ be a strongly continuous semigroup on a Banach space Y, and $(A,D(A))$ its infinitesimal generator. If there exists a positive constant M such that $\|T(t)\| \leqslant M$ for every $t \geqslant 0$ then*

$$\|Ay\|^2 \leqslant 4M^2\|A^2y\| \times \|y\|, \quad \textit{for every } y \in D(A^2).$$

See [7] for a proof. When $\mathbf{T}$ is the evolution semigroup $\mathscr{T}_0$ (on the Sobolev space $W^0_{p1}(\mathbb{R}_+,X)$), the discussion above yields the following result:

THEOREM 5. *Assume that $\mathscr{U}$ is a strongly continuous uniformly stable evolution family on the Radon-Nikodým space X, which verifies the Assumption 1 above. Consider a function $f \in W^0_{p1}(\mathbb{R}_+,X)$ such that:*

(i) the map $\int_0^{\cdot} U(\cdot,s)f(s)ds$ lies in $W^0_{p1}(\mathbb{R}_+,X)$;

(ii) the map $\int_0^{\cdot}(\cdot-s)U(\cdot-s)f(s)ds$ lies in $W^0_{p1}(\mathbb{R}_+,X)$.

Then the following inequality holds:

$$\left(\|\int_0^{\cdot} U(\cdot,s)f(s)ds\|^0_{p1}\right)^2 \leqslant 4M^2\|f\|^0_{p1} \times \|\int_0^{\cdot}(\cdot-s)U(\cdot,s)f(s)ds\|^0_{p1}.$$

Here M is the constant in (2.1).

COROLLARY 3. *Let F be a X-valued function on $\mathbb{R}_+$ such that F,F' and F'' belongs to $W^0_{p1}(\mathbb{R}_+,X)$. Then the following inequality holds:*

$$(\|F''\|_p + \|F'\|_p)^2 \leqslant 4(\|F''\|_p + \|F'''\|_p)(\|F''\|_p + \|F\|_p).$$

Proof. Apply Theorem 5 above for $U(t,s) = I$ and

$$F(\cdot) := \int_0^{\cdot}(\cdot-s)f(s)ds. \qquad \square$$

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