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ON DUAL BRUNN–MINKOWSKI INEQUALITIES

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Dedicated to the memory of prof. Mladen Alić

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Abstract. The main purpose of this paper is first to improve two classical dual Brunn-Minkowski inequalities, then we generalize another dual Brunn-Minkowski inequality from generic volume to Quermassintegral.

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathscr{C}^n$ denote the set of non-empty convex figures (compact, convex subsets) and $\mathscr{K}^n$ denote the subset of $\mathscr{C}^n$ consisting of all convex bodies (compact convex subsets with non-empty interiors) in $\mathbb{R}^n$. We reserve the letter u for unit vectors, and the letter B is reserved for the unit ball centered at the origin. The surface of B is S^{n-1}.

We use $V(K)$ for the n-dimensional volume of convex body K. Let $h(K,\cdot):S^{n-1}\to\mathbb{R}$, denote the support function of $K\in\mathscr{K}^n$; i.e.,

$$h(K,u)=\mathrm{Max}\{u\cdot x : x\in K\}, u\in S^{n-1}, \tag{1}$$

where $u\cdot x$ denotes the usual inner product of u and x in $\mathbb{R}^n$.

The radial function $\rho(K,\cdot):S^{n-1}\to\mathbb{R}$ of a compact subset K of $\mathbb{R}^n$ is defined by the relation

$$\rho(K,u)=\mathrm{Max}\{\lambda\geqslant 0 : \lambda u\in K\}.$$

If $\rho(K,\cdot)$ is positive and continuous, K is called a star body. Let φ^n denote the set of star bodies in $\mathbb{R}^n$.

The polar coordinate formula for volume in $\mathbb{R}^n$ is

$$V(K)=\frac{1}{n}\int_{S^{n-1}}\rho(K,u)^n dS(u) \tag{2}$$

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1. Definitions

If $K_i \in \mathscr{K}^n$ $(i = 1,2,\ldots,r)$ and λ_i $(i = 1,2,\ldots,r)$ are nonnegative real numbers, then of fundamental importance is the fact that the volume of $\sum_{i=1}^r \lambda_i K_i$ is a homogeneous polynomial in λ_i given by [7]

$$V(\sum_{i=1}^{r} \lambda_i K_i) = \sum_{i_1,\ldots,i_n} \lambda_{i_1} \ldots \lambda_{i_n} V_{i_1\ldots i_n}, \tag{3}$$

where the sum is taken over all n-tuples $(i_1,\ldots,i_n)$ of positive integers not exceeding r. The coefficient $V_{i_1\ldots i_n}$ depends only on the bodies $K_{i_1},\ldots,K_{i_n}$ and is uniquely determined by (3), it is called the mixed volume of $K_i,\ldots,K_{i_n}$, and is written as $V(K_{i_1},\ldots,K_{in})$. Let $K_1 = \ldots = K_{n-i} = K$ and $K_{n-i+1} = \ldots = K_n = L$, then the mixed volume $V(K_1\ldots K_n)$ is written as $V_i(K,L)$.

The radial Minkowski linear combination, $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$, is defined by Lutwak [9]

$$\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r = \{\lambda_1 x_1 \tilde{+} \cdots \tilde{+} \lambda_r x_r : x_i \in K_i\}, \tag{4}$$

for $K_1,\ldots,K_r \in \varphi^n$ and $\lambda_1,\ldots,\lambda_r \in \mathbb{R}$.

It has the following important property:

$$\rho(\lambda K \tilde{+} \mu L, \cdot) = \lambda \rho(K,\cdot) + \mu \rho(L,\cdot). \tag{5}$$

for $K, L \in \varphi^n$ and $\lambda, \mu \geqslant 0$.

For $K_1,\ldots,K_r \in \varphi^n$ and $\lambda_1,\ldots,\lambda_r \geqslant 0$, the volume of the radial Minkowski linear combination $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$ is a homogeneous nth polynomial in the λ_i,

$$V(\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r) = \sum \tilde{V}_{i_1,\ldots,i_n} \lambda_{i_1} \cdots \lambda_{i_n}, \tag{6}$$

where the sum is taken over all n-tuples $(i_1,\ldots,i_n)$ whose entries are positive integers not exceeding r. If we require the coefficients of the polynomial in (6) to be symmetric in their argument, then they are uniquely determined. The coefficient $\tilde{V}_{i_1,\ldots,i_n}$ is nonnegative and depends only on the bodies $K_{i_1},\ldots,K_{i_n}$. Here we denote $\tilde{V}_{i_1,\ldots,i_n}$ to $\tilde{V}(K_{i_1},\ldots,K_{i_n})$ and is called the dual mixed volume of $K_{i_1},\ldots,K_{i_n}$. The dual mixed volumes $\tilde{V}_i(K,B)$ is written as $\tilde{W}_i(K)$, if $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = B$.

For $K_i \in \varphi^n$, then [9]

$$\tilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K,u)^{n-i} dS(u). \tag{7}$$

Lutwak defines the radial Blaschke linear combination [9]. If $K, L \in \varphi^n$ and $\lambda, \mu \geqslant 0$, then $\lambda \cdot K \check{+} \mu \cdot L$, is the star body whose radial function is given by:

$$\rho(\lambda \cdot K \check{+} \mu \cdot L, \cdot)^{n-1} = \lambda \rho(K,\cdot)^{n-1} + \mu \rho(L,\cdot)^{n-1}. \tag{8}$$

We shall call the addition and scalar multiplication radial Blaschke addition and scalar multiplication.

A new addition, harmonic Blaschke addition, be defined by Lutwak [8]. Suppose $K, L \in \varphi^n$, and $\lambda, \mu \geqslant 0$ (not both zero). To define the harmonic Blaschke linear combination, $\lambda K \hat{+} \mu L$, first define $\xi > 0$ by

$$\xi^{1/(n+1)} = \frac{1}{n} \int_{S^{n-1}} [\lambda V(K)^{-1} \rho(K,u)^{n+1} + \mu V(L)^{-1} \rho(L,u)^{n+1}]^{n/(n+1)} dS(u). \tag{9}$$

The body $\lambda K \hat{+} \mu L \in \varphi^n$ is defined as the body whose radial function is given by [8]

$$\xi^{-1} \rho(\lambda K \hat{+} \mu L, \cdot)^{n+1} = \lambda V(K)^{-1} \rho(K, \cdot)^{n+1} + \mu V(L)^{-1} \rho(L, \cdot)^{n+1}. \tag{10}$$

From this definition and (2), it follows immediately that $\xi = V(\lambda K \hat{+} \mu L)$, and hence

$$V(\lambda K \hat{+} \mu L)^{-1} \rho(\lambda K \hat{+} \mu L, \cdot)^{n+1} = \lambda V(K)^{-1} \rho(K, \cdot)^{n+1} + \mu V(L)^{-1} \rho(L, \cdot)^{n+1}. \tag{11}$$

2. Main results

In recent years some authors including Ball [1-3], Bourgain [4], Gardner [5-6], Schneider [7], Lutwak [8-11] and Leng *et al* [12] have given good-sized attention to the Brunn-Minkowski theory and Brunn-Minkowski-Firey theory and their various generalizations. In particular, Lutwak had established the following three important Brunn-Minkowski type inequalities:

THEOREM A (see [11]) *If $K, L \in \mathscr{K}^n$, then*

$$V(K \tilde{+} L)^{1/n} \leqslant V(K)^{1/n} + V(L)^{1/n}, \tag{12}$$

with equality if and only if K and L are homothetic.

THEOREM B (see [9]) *If $K, L \in \varphi^n$, then*

$$V(K \check{+} L)^{(n-1)/n} \leqslant V(K)^{(n-1)/n} + V(L)^{(n-1)/n}, \tag{13}$$

with equality if and only if K and L are dilates.

THEOREM C (see [8]) *If $K, L \in \varphi^n$, and $\lambda, \mu > 0$, then*

$$V(\lambda K \hat{+} \mu L)^{1/n} \geqslant \lambda V(K)^{1/n} + \mu V(L)^{1/n}, \tag{14}$$

with equality if and only if K and L are dilates.

First, in this paper, we prove inequalities (12) and (13), then we generalize inequality (14) to Quermassintegral.

THEOREM 1. *If $K, L \in \mathscr{K}^n$, then*

$$\begin{aligned} V(K \tilde{+} L)^{1/n} &\leqslant V(\alpha K \tilde{+} (1-\alpha) L)^{1/n} + V((1-\alpha) K \tilde{+} \alpha L)^{1/n} \\ &\leqslant V(K)^{1/n} + V(L)^{1/n}, \text{ for } \alpha \in [0,1], \end{aligned} \tag{15}$$

In each case, the sign of equality holds if and only if K and L are homothetic.

REMARK 1. Taking $\alpha = 1$ to (15), inequality (15) changes to inequality (12) which was given by Lutwak [11].

THEOREM 2. *If* $K, L \in \varphi^n$, *then*

$$V(K\check{+}L)^{(n-1)/n} \leqslant V(\alpha K\check{+}(1-\alpha)L)^{(n-1)/n} + V((1-\alpha)K\check{+}\alpha L)^{(n-1)/n}$$
$$\leqslant V(K)^{(n-1)/n} + V(L)^{(n-1)/n}, \text{ for } \alpha \in [0,1]. \tag{16}$$

In each case, the sign of equality holds if and only if K *and* L *are dilates.*

REMARK 2. Taking $\alpha = 1$ to (16), inequality (16) changes to inequality (13) which was given by Lutwak [9].

THEOREM 3. *If* $K, L \in \varphi^n, \lambda > 0$ *and* $\mu > 0$, *then*

$$\frac{\tilde{W}_i(\lambda K\hat{+}\mu L)^{(n+1)/(n-i)}}{V(\lambda K\hat{+}\mu L)}$$
$$\geqslant \frac{\lambda \tilde{W}_i(K)^{(n+1)/(n-i)}}{V(K)} + \frac{\mu \tilde{W}_i(L)^{(n+1)/(n-i)}}{V(L)}, \quad \text{for } n > i > -1, \tag{17}$$

with equality if and only if K *is a dilation of* L.

REMARK 3. Taking $i = 0$ to (17), inequality (17) changes to inequality (14) which was given by Lutwak [8].

Proof of Theorem 1. From (2), (5) and Minkowski inequality for integral, we obtain that

$$\begin{aligned}
V(K\tilde{+}L)^{1/n} &= \left(\frac{1}{n}\int_{S^{n-1}} \rho(K\tilde{+}L,u)^n dS(u)\right)^{1/n} \\
&= \left(\frac{1}{n}\int_{S^{n-1}} (\rho(K,u)+\rho(L,u))^n dS(u)\right)^{1/n} \\
&= \left(\frac{1}{n}\int_{S^{n-1}} (\alpha\rho(K,u) + (1-\alpha)\rho(L,u)) \right. \\
&\qquad \left. +((1-\alpha)\rho(K,u)+\alpha\rho(L,u)))^n \, dS(u)\right)^{1/n} \\
&\leqslant \left(\frac{1}{n}\int_{S^{n-1}} (\alpha\rho(K,u)+(1-\alpha)\rho(L,u))^n dS(u)\right)^{1/n} \\
&\qquad + \left(\frac{1}{n}\int_{S^{n-1}} ((1-\alpha)\rho(K,u)+\alpha\rho(L,u))^n dS(u)\right)^{1/n}
\end{aligned}$$

with equality if and only if K is a dilation of L.

From (5), it follows that

$$\begin{aligned}
&V(K\tilde{+}L)^{1/n}\\
&\quad\leqslant \left(\frac{1}{n}\int_{S^{n-1}}(\rho(\alpha K\tilde{+}(1-\alpha)L,u))^n dS(u)\right)^{1/n}\\
&\quad\quad+\left(\frac{1}{n}\int_{S^{n-1}}(\rho((1-\alpha)K\tilde{+}\alpha L,u))^n dS(u)\right)^{1/n}\\
&\quad= V(\alpha K\tilde{+}(1-\alpha)L)^{1/n}+V((1-\alpha)K\tilde{+}\alpha L)^{1/n}.
\end{aligned}$$

On the other hand, from (12), we have

$$V(\alpha K\tilde{+}(1-\alpha)L)^{1/n}\leqslant \alpha V^{1/n}(K)+(1-\alpha)V^{1/n}(L)$$

with equality if and only if K is a dilation of L, and

$$V((1-\alpha)K\tilde{+}\alpha L)^{1/n}\leqslant (1-\alpha)V^{1/n}(K)+\alpha V^{1/n}(L),$$

with equality if and only if K is a dilation of L. Therefore,

$$V(\alpha K\tilde{+}(1-\alpha)L)^{1/n}+V((1-\alpha)K\tilde{+}\alpha L)^{1/n}\leqslant V(K)^{1/n}+V(L)^{1/n},$$

with equality if and only if K is a dilation of L.

The proof is complete.

Proof of Theorem 2. From (2), (8) and in view of Minkowski inequality, we obtain that

$$\begin{aligned}
V(K\check{+}L)^{(n-1)/n} &= \left(\frac{1}{n}\int_{S^{n-1}}\rho(K\check{+}L,u)^n dS(u)\right)^{(n-1)/n}\\
&= \left(\frac{1}{n}\int_{S^{n-1}}(\rho(K,u)^{n-1}+\rho(L,u)^{n-1})^{n/(n-1)}dS(u)\right)^{(n-1)/n}\\
&= \left(\frac{1}{n}\int_{S^{n-1}}\{[\alpha\rho(K,u)^{n-1}+(1-\alpha)\rho(L,u)^{n-1}]\right.\\
&\quad \left. +[(1-\alpha)\rho(K,u)+\alpha\rho(L,u)]\}^{(n/n-1)}dS(u)\right)^{(n-1)/n}\\
&\leqslant \left(\frac{1}{n}\int_{S^{n-1}}[\alpha\rho(K,u)^{n-1}+(1-\alpha)\rho(L,u)^{n-1}]^{(n/n-1)}dS(u)\right)^{(n-1)/n}\\
&\quad +\left(\frac{1}{n}\int_{S^{n-1}}[(1-\alpha)\rho(K,u)^{n-1}+\alpha\rho(L,u)^{n-1}]^{(n/n-1)}dS(u)\right)^{(n-1)/n}\\
&\quad \text{(with equality if and only if } K \text{ is a dilation of } L)\\
&= \left(\frac{1}{n}\int_{S^{n-1}}(\rho(\alpha K\check{+}(1-\alpha)L,u))^n dS(u)\right)^{(n-1)/n}\\
&\quad +\left(\frac{1}{n}\int_{S^{n-1}}(\rho((1-\alpha)K\check{+}\alpha L,u))^n dS(u)\right)^{(n-1)/n}
\end{aligned}$$

$$= V(\alpha K \check{+} (1-\alpha)L)^{(n-1)/n} + V((1-\alpha)K \check{+} \alpha L)^{(n-1)/n}.$$

On the other hand, by using inequality (13), we have

$$V(\alpha K \check{+} (1-\alpha)L)^{(n-1)/n} \leqslant \alpha V^{(n-1)/n}(K) + (1-\alpha)V^{(n-1)/n}(L),$$

with equality if and only if K is a dilation of L, and

$$V((1-\alpha)K \check{+} \alpha L)^{(n-1)/n} \leqslant (1-\alpha)V^{(n-1)/n}(K) + \alpha V^{(n-1)/n}(L),$$

with equality if and only if K is a dilation of L.

The proof is complete.

Proof of Theorem 3. From (7), (11) and in view of inverse Minkowski inequality for integral [13], we obtain that for $n > i > -1$

$$\begin{aligned}
&\tilde{W}_i(\lambda K \hat{+} \mu L)^{(n+1)/(n-i)} \\
&= \left(\frac{1}{n}\int_{S^{n-1}} \rho(\lambda K \hat{+} \mu L, u)^{n-i} dS(u)\right)^{(n+1)/(n-i)} \\
&= \left(\frac{1}{n}\int_{S^{n-1}} \Big(\big(\xi\lambda V(K)^{-1}\rho(K,u)^{n+1} \right. \\
&\qquad \left. + \xi\mu V(L)^{-1}\rho(L,u)^{n+1}\big)^{1/(n+1)}\Big)^{n-i} dS(u)\right)^{(n+1)/(n-i)} \\
&\geqslant \left(\frac{1}{n}\int_{S^{n-1}} \left(\xi\lambda V(K)^{-1}\rho(K,u)^{n+1}\right)^{(n-i)/(n+1)} dS(u)\right)^{(n+1)/(n-i)} \\
&\qquad + \left(\frac{1}{n}\int_{S^{n-1}} \left(\xi\mu V(L)^{-1}\rho(L,u)^{n+1}\right)^{(n-i)/(n+1)} dS(u)\right)^{(n+1)/(n-i)} \\
&= \xi\lambda V(K)^{-1} W_i(K)^{(n+1)/(n-i)} + \xi\mu V(L)^{-1} W_i(L)^{(n+1)/(n-i)}.
\end{aligned}$$

Notice that $\xi = V(\lambda K \hat{+} \mu L)$. This proof is complete.

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