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HARDY–TYPE INEQUALITIES VIA CONVEXITY

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Abstract. A recently discovered Hardy-Pólya type inequality described by a convex function is considered and further developed both in weighted and unweighted cases. Also some corresponding multidimensional and reversed inequalities are pointed out. In particular, some new multidimensional Hardy and Pólya-Knopp type inequalities and some new integral inequalities with general integral operators (without additional restrictions on the kernel) are derived.

1. Introduction

The classical Hardy inequality reads

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \quad p>1. \tag{1.1}$$

It was proved in 1925 by G. H. Hardy in [7] but it has also an interesting prehistory, e.g. it was formulated almost in the paper [6]. By replacing f with $f^{\frac{1}{p}}$ in (1.1) and letting $p \to \infty$ we obtain the limiting inequality

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right) dx \leqslant e\int_0^\infty f(x)dx. \tag{1.2}$$

This inequality is many times referred to as Knopp's inequality, with the reference to the paper [11]. However, inequality (1.2) was known before and Hardy himself (see [7, p. 156]) claimed that it was G. Pólya who pointed it out to him (probably by using just the limit argument above). Note that the discrete version of (1.1) is surely due to T. Carleman [2]. We also remark that the constants $\left(\frac{p}{p-1}\right)^p$ and e in (1.1) and (1.2) are sharp. The inequalities (1.1) and (1.2) have been investigated and generalized in several directions, e.g. one chapter of the book [14] is devoted to this subject. Moreover, there are two books ([12] and [17]) completely devoted to this subject. We also refer to the references in these books and the classical book [8]. For some complementary

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historical remarks connected to this development we also refer to [13]. Concerning several proofs, generalizations and the history of Carleman's inequality we refer to [9] and the references given there. Recently it was pointed out by S. Kaijser et al. in [10] that both (1.1) and (1.2) are just special cases of the much more general (Hardy-Knopp type) inequality

$$\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x f(t)dt\right)\frac{dx}{x} \leqslant \int_0^\infty \Phi(f(x))\frac{dx}{x}, \tag{1.3}$$

where Φ is a convex function on $(0,\infty)$. And obviously, (1.3) just follows by using a standard application of Jensen's inequality and the Fubini theorem. This idea was further developed and applied in the thesis [18] and [5]. Concerning some new developments in the theory of convex functions in this connection we also refer to the book [15].

In this paper we generalize the crucial inequality (1.3) in various ways and point out some corresponding applications. In Section 2 we prove some multidimensional versions of (1.3) and also some corresponding reversed inequalities for concave functions. Moreover, we point out that these results imply new multidimensional versions of (1.1) and (1.2) and the corresponding reversed inequalities. In Section 3 we prove some weighted versions of the results in Section 2 and derive the corresponding generalizations of Hardy and Pólya-Knopp type inequalities. Finally, in Section 4 we prove some new results of the type (1.3), where the Hardy operator H defined by $Hf(x) = \frac{1}{x}\int_0^x f(t)dt$ is replaced by the more general integral operator H_K defined by

$$H_Kf(x) = \frac{1}{K(x)}\int_0^x k(x,y)f(y)dy,$$

where $K(x) = \int_0^x k(x,y)dy$ and $k(x,y) \geqslant 0$ without any further (e.g. Oinarov type) restrictions on the kernel. Also results for the dual operator of H_K are proved and it is pointed out that most of the results in [5] follow by just using our results with $k(x,y) = 1$. Hence our results unify, generalize and complement also several other recent results e.g. some results in [3], [4], [19] and [20].

Conventions. Throughout this paper all functions are assumed to be positive and measurable and expressions on the form $0\cdot\infty$, $\frac{\infty}{\infty}$ and $\frac{0}{0}$ are taken to be equal to zero. Moreover, by a weight $u = u(x)$ we mean a non-negative measurable function on the actual interval or more general set.

2. A multidimensional Hardy-type inequality

In this Section we prove and discuss the following Hardy-type inequality:

THEOREM 2.1. *Let* $0 < b_i \leqslant \infty$, $i = 1,2,\dots,n$ $(n \in \mathbb{Z}_+)$, $-\infty \leqslant a < c \leqslant \infty$ *and let* Φ *be a positive function on* $[a,c]$.

(a) *If* Φ *is convex, then*

$$\int_0^{b_1}\cdots\int_0^{b_n}\Phi\left(\frac{1}{x_1\dots x_n}\int_0^{x_1}\cdots\int_0^{x_n}f(t_1,\dots,t_n)dt_1\dots dt_n\right)\frac{dx_1\dots dx_n}{x_1\dots x_n} \tag{2.1}$$
$$\leqslant \int_0^{b_1}\cdots\int_0^{b_n}\Phi\left(f(x_1,\dots,x_n)\right)\left(1-\frac{x_1}{b_1}\right)\dots\left(1-\frac{x_n}{b_n}\right)\frac{dx_1\dots dx_n}{x_1\dots x_n},$$

for every function f *on* $(\mathbf{0},\mathbf{b})$ *such that* $a<f(\mathbf{x})<c$.

(b) *If* Φ *is concave, then*

$$\int_0^{b_1}\cdots\int_0^{b_n}\Phi\left(\frac{1}{x_1\dots x_n}\int_0^{x_1}\cdots\int_0^{x_n}f(t_1,\dots,t_n)dt_1\dots dt_n\right)\frac{dx_1\dots dx_n}{x_1\dots x_n} \tag{2.2}$$
$$\geqslant \int_0^{b_1}\cdots\int_0^{b_n}\Phi\left(f(x_1,\dots,x_n)\right)\left(1-\frac{x_1}{b_1}\right)\dots\left(1-\frac{x_n}{b_n}\right)\frac{dx_1\dots dx_n}{x_1\dots x_n},$$

for every function f *on* $(\mathbf{0},\mathbf{b})$ *such that* $a<f(\mathbf{x})<c$.

Here and in the sequel the notations $\mathbf{b},\mathbf{x}$, etc. as usual means $\mathbf{b}=(b_1,b_2,\dots,b_n)$, $\mathbf{x}=(x_1,x_2,\dots,x_n),\dots$ and $\mathbf{b}<\mathbf{x}$ means that $b_i<x_i,\ i=1,2,\dots,n$.

Proof. Let Φ be convex. Then, according to Jensen's inequality and the Fubini theorem, we have

$$\int_0^{b_1}\cdots\int_0^{b_n}\Phi\left(\frac{1}{x_1\dots x_n}\int_0^{x_1}\cdots\int_0^{x_n}f(t_1,\dots,t_n)dt_1\dots dt_n\right)\frac{dx_1\dots dx_n}{x_1\dots x_n}$$
$$\leqslant \int_0^{b_1}\cdots\int_0^{b_n}\left(\int_0^{x_1}\cdots\int_0^{x_n}\Phi\left(f(t_1,\dots,t_n)\right)dt_1\dots dt_n\right)x_1^{-2}\dots x_n^{-2}dx_1\dots dx_n$$
$$=\int_0^{b_1}\cdots\int_0^{b_n}\Phi\left(f(t_1,\dots,t_n)\right)\left(\int_{t_1}^{b_1}\cdots\int_{t_n}^{b_n}x_1^{-2}\dots x_n^{-2}dx_1\dots dx_n\right)dt_1\dots dt_n$$
$$=\int_0^{b_1}\cdots\int_0^{b_n}\Phi\left(f(t_1,\dots,t_n)\right)\left(1-\frac{t_1}{b_1}\right)\dots\left(1-\frac{t_n}{b_n}\right)\frac{dt_1\dots dt_n}{t_1\dots t_n}.$$

By making the same calculation with Φ concave we see that only the inequality sign will be reversed and the proof is complete. □

By choosing $\Phi(t)=t^p$ in Theorem 2.1. we obtain some natural multidimensional forms of the classical Hardy and reversed Hardy inequalities.

COROLLARY 2.2. *Let* $0<d_i\leqslant\infty,\ i=1,2,\dots,n\ \ (n\in\mathbb{Z}_+)$.

(*a*) *If* $p>1$ *or* $p<0$, *then*

$$\int_0^{d_1}\cdots\int_0^{d_n}\left(\frac{1}{y_1\dots y_n}\int_0^{y_1}\cdots\int_0^{y_n} g(s_1,\dots,s_n)ds_1\dots ds_n\right)^p dy_1\dots dy_n \tag{2.3}$$
$$\leqslant\left(\frac{p}{p-1}\right)^{pn}\int_0^{d_1}\cdots\int_0^{d_n} g^p(y_1,\dots,y_n)\left(1-\left(\frac{y_1}{d_1}\right)^{\frac{p-1}{p}}\right)$$
$$\cdot\ldots\cdot\left(1-\left(\frac{y_n}{d_n}\right)^{\frac{p-1}{p}}\right)dy_1\dots .dy_n,$$

for each positive function g *on* $(\mathbf{0},\mathbf{d})$.

(*b*) *If* $0<p<1$, *then*

$$\int_{d_1}^{\infty}\cdots\int_{d_n}^{\infty}\left(\frac{1}{y_1\dots y_n}\int_{y_1}^{\infty}\cdots\int_{y_n}^{\infty} g(s_1,\dots,s_n)ds_1\dots ds_n\right) dy_1\dots dy_n \tag{2.4}$$
$$\geqslant\left(\frac{p}{1-p}\right)^{pn}\int_{d_1}^{\infty}\cdots\int_{d_n}^{\infty} g^p(y_1,\dots,y_n)\left(1-\left(\frac{d_1}{y_1}\right)^{\frac{1-p}{p}}\right)$$
$$\cdot\ldots\cdot\left(1-\left(\frac{d_n}{y_n}\right)^{\frac{1-p}{p}}\right)dy_1\dots dy_n,$$

for each positive function g *on* $(\mathbf{0},\mathbf{d})$.

REMARK 2.1. For the case $n=1$, $p>1$, the improvement (2.3) of Hardy's inequality was proved in [5]. It has been known for a long time that Hardy's inequality (for $n=1$) in fact holds also for $p<0$. See e.g. [1] and the book [12] and the references given there. This multidimensional generalization of these facts seems to be new.

Proof. Apply (2.1) with $\Phi(u)=u^p$, $p>1$ or $p<0$ and we obtain that

$$\int_0^{b_1}\cdots\int_0^{b_n}\left(\frac{1}{x_1\dots x_n}\int_0^{x_1}\cdots\int_0^{x_n} f(t_1,\dots,t_n)dt_1\dots dt_n\right)^p\frac{dx_1\dots dx_n}{x_1\dots x_n} \tag{2.5}$$
$$\leqslant\int_0^{b_1}\cdots\int_0^{b_n} f^p(x_1,\dots,x_n)\left(1-\frac{x_1}{b_1}\right)\cdots\left(1-\frac{x_n}{b_n}\right)\frac{dx_1\dots dx_n}{x_1\dots x_n}.$$

Let $p>1$. Now, first put $t_i=s_i^{\frac{p-1}{p}}$ in (2.5) and then let $x_i=y_i^{\frac{p-1}{p}}$, $b_i^{\frac{p}{p-1}}=d_i$ for $i=1,2,\dots,n$. Then the proof of (2.3) follows by defining the relation between the functions f and g by

$$f\left(x_1^{\frac{p-1}{p}}, x_2^{\frac{p-1}{p}}, \dots, x_n^{\frac{p-1}{p}}\right) x_1^{-\frac{1}{p}} x_2^{-\frac{1}{p}} \dots x_n^{-\frac{1}{p}} = g(x_1, x_2, \dots, x_n).$$

The proof of the case $p < 0$ is completely similar. Moreover, for the case $0 < p < 1$ the function Φ is concave and the proof of (2.4) follows in an analogous way by applying (2.2).

The proof is complete. $\square$

By choosing $\Phi(t) = \exp(t)$ in Theorem 2.1. and replacing f by $\ln g^p$ we obtain the following multidimensional form of the so called Pólya-Knopp's inequality:

COROLLARY 2.3. *Let* $0 < b_i \leqslant \infty$, $i = 1, 2, \dots, n$. *If* $p > 0$, *then*

$$\left(\int_0^{b_1} \dots \int_0^{b_n} \left[\exp\left(\frac{1}{x_1 \dots x_n} \int_0^{x_1} \dots \int_0^{x_n} \ln g(t_1, \dots, t_n) dt_1 \dots dt_n\right)\right]^p dx_1 \dots dx_n\right)^{\frac{1}{p}}$$
$$\leqslant e^{\frac{n}{p}} \left(\int_0^{b_1} \dots \int_0^{b_n} g^p(x_1, \dots x_n) \prod_{i=1}^n \left(1 - \frac{x_i}{b_i}\right) dx_1 \dots dx_n\right)^{\frac{1}{p}},$$

for each positive function g *on* $(0, b)$.

Proof. It is obviously sufficient to prove the inequality for the case $p = 1$. By applying (2.1) with $\Phi(u) = \exp u$ and replacing f by $\ln f$ we obtain that

$$\left(\int_0^{b_1} \dots \int_0^{b_n} \exp\left(\frac{1}{x_1 \dots x_n} \int_0^{x_1} \dots \int_0^{x_n} \ln f(t_1, \dots, t_n) dt_1 \dots dt_n\right) \frac{dx_1 \dots dx_n}{x_1 \dots x_n}\right) \quad (2.6)$$
$$\leqslant \left(\int_0^{b_1} \dots \int_0^{b_n} f(x_1, \dots x_n) \prod_{i=1}^n \left(1 - \frac{x_i}{b_i}\right) \frac{dx_1 \dots dx_n}{x_1 \dots x_n}\right).$$

The proof follows by using this inequality with $g(x_1, \dots, x_n) = \frac{f(x_1, \dots x_n)}{x_1 \dots x_n}$. $\square$

REMARK 2.2. For the case $n = 1$ Corollary 2.3. was proved in [3] by using another technique (via a mixed mean inequality).

REMARK 2.3. By instead applying (2.2) with $\Phi(u) = \ln u$ and replacing f by $\exp f$ we obtain the following reversed version of (2.6) :

$$\left(\int_0^{b_1} \dots \int_0^{b_n} \ln\left(\frac{1}{x_1 \dots x_n} \int_0^{x_1} \dots \int_0^{x_n} \exp f(t_1, \dots, t_n) dt_1 \dots dt_n\right) \frac{dx_1 \dots dx_n}{x_1 \dots x_n}\right)$$
$$\geqslant \left(\int_0^{b_1} \dots \int_0^{b_n} f(x_1, \dots x_n) \prod_{i=1}^n \left(1 - \frac{x_i}{b_i}\right) \frac{dx_1 \dots dx_n}{x_1 \dots x_n}\right),$$

which we think is new also for $n = 1$.

3. On the weighted case

Our main result in this section reads:

THEOREM 3.1. *Let* $0 < p \leqslant q < \infty$ *and* $0 < b_i \leqslant \infty$, $i = 1, \ldots, n$. *Let* $\Phi(\mathbf{x})$ *be a positive and convex function on* (a,c), $-\infty \leqslant a < c \leqslant \infty$, *and let* $u(\mathbf{x})$ *and* $w(\mathbf{x})$ *be weight functions on* $(\mathbf{0},\mathbf{b})$ *such that* $a < f(\mathbf{x}) < c$. *Then*

$$\left(\int_0^{b_1}\cdots\int_0^{b_n}\left(\Phi\left(\frac{1}{x_1\ldots x_n}\int_0^{x_1}\cdots\int_0^{x_n} f(t_1,\ldots,t_n)dt_1\ldots dt_n\right)\right)^{\frac{q}{p}} w(x_1,\ldots,x_n)\frac{dx_1\ldots dx_n}{x_1\ldots x_n}\right)^{\frac{1}{q}}$$
$$\leqslant C\left(\int_0^{b_1}\cdots\int_0^{b_n}\Phi\left(f(x_1,\ldots,x_n)\right)u(x_1,\ldots,x_n)\frac{dx_1\ldots dx_n}{x_1\ldots x_n}\right)^{\frac{1}{p}} \tag{3.1}$$

holds for some finite positive constant C *if*

$$A := \sup_{0<t_i\leqslant b_i}\left(\frac{t_1\ldots t_n}{u(\mathbf{t})}\right)^{\frac{1}{p}}\left(\int_{t_1}^{b_1}\cdots\int_{t_n}^{b_n} w(\mathbf{x})x_1^{-\left(\frac{q}{p}+1\right)}\ldots x_n^{-\left(\frac{q}{p}+1\right)}dx_1\ldots dx_n\right)^{\frac{1}{q}} < \infty.$$

Moreover, if C *is the least constant for* (3.1) *to hold it yields that*

$$C \leqslant A.$$

Proof. We apply first Jensen's inequality and then Minkowski's inequality and find that

$$\left(\int_0^{b_1}\cdots\int_0^{b_n}\left(\Phi\left(\frac{1}{x_1\ldots x_n}\int_0^{x_1}\cdots\int_0^{x_n} f(t_1,\ldots,t_n)dt_1\ldots dt_n\right)\right)^{\frac{q}{p}} w(x_1,\ldots,x_n)\frac{dx_1\ldots dx_n}{x_1\ldots x_n}\right)^{\frac{1}{q}}$$
$$\leqslant \left(\int_0^{b_1}\cdots\int_0^{b_n}\left(\frac{1}{x_1\ldots x_n}\int_0^{x_1}\cdots\int_0^{x_n}\Phi\left(f(t_1,\ldots,t_n)\right)dt_1\ldots dt_n\right)^{\frac{q}{p}} w(x_1,\ldots,x_n)\frac{dx_1\ldots dx_n}{x_1\ldots x_n}\right)^{\frac{1}{q}}$$
$$\leqslant \left(\int_0^{b_1}\cdots\int_0^{b_n}\Phi\left(f(t_1,\ldots,t_n)\right)t_1\ldots t_n\frac{u(\mathbf{t})}{u(\mathbf{t})}\left(\int_{t_1}^{b_1}\cdots\int_{t_n}^{b_n} w(\mathbf{x})x_1^{-\left(\frac{q}{p}+1\right)}\right.\right.$$
$$\left.\left.\cdot\ldots\cdot x_n^{-\left(\frac{q}{p}+1\right)}dx_1\ldots dx_n\right)^{\frac{p}{q}}\frac{dt_1\ldots dt_n}{t_1\ldots t_n}\right)^{\frac{1}{p}}$$
$$\leqslant A\left(\int_0^{b_1}\cdots\int_0^{b_n}\Phi\left(f(t_1,\ldots,t_n)\right)u(\mathbf{t})\frac{dt_1\ldots dt_n}{t_1\ldots t_n}\right)^{\frac{1}{p}}.$$

Thus, (3.1) holds with $C = A$ so the proof is complete. □

REMARK 3.1. If $p = q$, then Theorem 3.1. implies that for Φ convex

$$\int_0^{b_1} \dots \int_0^{b_n} \Phi\left(\frac{1}{x_1 \dots x_n} \int_0^{x_1} \dots \int_0^{x_n} f(t_1, \dots, t_n) dt_1 \dots dt_n\right) w(x_1, \dots, x_n) \frac{dx_1 \dots dx_n}{x_1 \dots x_n} \tag{3.2}$$
$$\leqslant A_1 \int_0^{b_1} \dots \int_0^{b_n} \Phi\left(f(t_1, \dots, t_n)\right) u(t_1, \dots t_n) \frac{dt_1 \dots dt_n}{t_1 \dots t_n},$$

where

$$A_1 := \sup_{0<t_i \leqslant b_i} \frac{t_1, \dots t_n}{u(t_1, \dots t_n)} \int_{t_1}^{b_1} \dots \int_{t_n}^{b_n} w(x_1, \dots, x_n) \frac{dx_1 \dots dx_n}{x_1^2 \dots x_n^2}. \tag{3.3}$$

Moreover, by modifying the proof of Theorem 3.1. we find that if Φ is concave, then (3.2) holds in the opposite direction with (3.3) replaced by

$$A_2 := \inf_{0<t_i \leqslant b_i} \frac{t_1, \dots t_n}{u(t_1, \dots t_n)} \int_{t_1}^{b_1} \dots \int_{t_n}^{b_n} w(x_1, \dots, x_n) \frac{dx_1 \dots dx_n}{x_1^2 \dots x_n^2}. \tag{3.4}$$

EXAMPLE 3.1. In the case $p = q$, $w \equiv 1$ and $u(x_1, \dots, x_n) = \left(1 - \frac{x_1}{b_1}\right) \dots \left(1 - \frac{x_n}{b_n}\right)$ a simple calculation shows that $A_1 = A_2 = 1$ (A_1 and A_2 are defined by (3.3) and (3.4)) and we get (2.1) and (2.2)).

4. Hardy-type inequalities with a general kernel

In this section we consider the following general Hardy-type arithmetic mean operator:

$$A_K f(x) := \frac{1}{K(x)} \int_0^x k(x, y) f(y) dy, \qquad x > 0, \tag{4.1}$$

where $f(x)$ is real-valued and measurable, $k(x, y)$ is measurable and nonnegative, and

$$K(x) := \int_0^x k(x, y) dy.$$

We also consider the natural limit (geometric mean) operator

$$G_K f(x) := \exp\left(\frac{1}{K(x)} \int_0^x k(x, y) \ln f(y) dy\right).$$

There are some results concerning the mapping properties of these operators (see e.g. [16] and the book [12] and the references given there) but we need always to assume some extra assumptions on the kernel $k(x,y)$ (i.e. that it satisfies the so called Oinarov condition). Here we will present some results which can be achieved without any restriction on k except that it is nonnegative. We start by stating the following elementary but surprisingly powerful generalization of many forms of the classical Hardy and Pólya-Knopp inequalities (see e.g. the examples and remarks below).

THEOREM 4.1. *Let u be a weight function on $(0,b)$, $0 < b \leqslant \infty$, and let $k(x,y) \geqslant 0$ on $(0,b)\times(0,b)$. Assume that $\frac{k(x,y)u(x)}{xk(x)}$ is locally integrable on $(0,b)$ for each fixed $y \in (0,b)$ and define v by*

$$v(y) := y\int_y^b \frac{k(x,y)}{K(x)}u(x)\frac{dx}{x} < \infty, \quad y \in (0,b).$$

If Φ is a positive and convex function on (a,c), $-\infty \leqslant a < c \leqslant \infty$, then

$$\int_0^b \Phi\left(A_K f(x)\right) u(x)\frac{dx}{x} \leqslant \int_0^b \Phi\left(f(x)\right) v(x)\frac{dx}{x},$$

for all f with $a < f(x) < c$, $0 \leqslant x \leqslant b$, where A_K is defined by (4.1).

Proof. By using Jensen's inequality and the Fubini theorem we find that

$$\begin{aligned}\int_0^b \Phi\left(A_K f(x)\right) u(x)\frac{dx}{x} &= \int_0^b \Phi\left(\frac{1}{K(x)}\int_0^x k(x,y)f(y)dy\right) u(x)\frac{dx}{x}\\ &\leqslant \int_0^b \left(\frac{1}{K(x)}\int_0^x k(x,y)\Phi\left(f(y)\right)dy\right) u(x)\frac{dx}{x}\\ &= \int_0^b \Phi\left(f(y)\right)\left(\int_y^b \frac{1}{K(x)}k(x,y)u(x)\frac{dx}{x}\right)dy\\ &= \int_0^b \Phi\left(f(x)\right) v(x)\frac{dx}{x}.\end{aligned}$$

The proof is complete. □

For the special case $k(x,y) = 1$ we have:

COROLLARY 4.2. *Let u be a weight function on $(0,b)$, $0 < b \leqslant \infty$, and let v be defined by $v(y) = y\int_y^b \frac{u(x)}{x^2}dx$. If Φ is a positive and convex function on (a,c),*

$-\infty \leqslant a < c \leqslant \infty$, *then*

$$\int_0^b \Phi\left(\frac{1}{x}\int_0^x f(y)dy\right) u(x)\frac{dx}{x} \leqslant \int_0^b \Phi(f(x))\, v(x)\frac{dx}{x},$$

for all f *such that* $a < f(x) < c,\ 0 \leqslant x \leqslant b.$

REMARK 4.1. The result in Corollary 4.2. was also recently presented and discussed in [5, Theorem 1]. For the unweighted case see also [10].

EXAMPLE 4.1. By applying Theorem 4.1. with $\Phi(x) = \exp x$ and replacing f by $\ln f^p$ we obtain the following generalization of the Pólya-Knopp type inequality (see [9] and [12] and the references given there):

$$\int_0^b \left[\exp\left(\frac{1}{K(x)}\int_0^x k(x,y)\ln f(y)dy\right)\right]^p u(x)\frac{dx}{x} \leqslant \int_0^b f^p(x)v(x)\frac{dx}{x}, \quad p>0, \quad (4.2)$$

where $k(x,y),\ K(x),\ u(x)$ and $v(x)$ are defined as in Theorem 4.1.

REMARK 4.2. By applying (4.2) with $p = 1,\ u(x) \equiv 1$ (and hence $v(x) = 1-\frac{x}{b}$) we get

$$\int_0^b \exp\left(\frac{1}{K(x)}\int_0^x k(x,y)\ln f(y)dy\right)\frac{dx}{x} \leqslant \int_0^b f(x)\left(1-\frac{x}{b}\right)\frac{dx}{x},$$

which is a generalization of (14) from [5] and hence we can get e.g. the results from their Corollary 2 (i) with $\alpha = 1$, in particular that

$$\int_0^b \exp\left(\frac{1}{x}\int_0^x \ln f(y)dy\right) x^{\varepsilon}dx \leqslant e^{1+\varepsilon}\int_0^b \left(1-\frac{x}{b}\right) f(x)x^{\varepsilon}dx.$$

EXAMPLE 4.2. Let $k(x,y) = \frac{\gamma}{x^\gamma}(x-y)^{\gamma-1}$ if $0 \leqslant x \leqslant y$, and $k(x,y) = 0$ if $x > y$. Then $K(x) \equiv 1$ and

$$A_k f(x) = \frac{\gamma}{x^\gamma}\int_0^x (x-y)^{\gamma-1} f(y)dy = R_\gamma f(x), \quad \gamma > 0,$$

where R_γ is the Riemann-Liouville operator. If $u(x) \equiv 1$, then $v(x) = (1-\frac{x}{b})^\gamma$, and the inequality from Theorem 4.1. reads

$$\int_0^b \Phi\left(R_\gamma f\right)(x)\frac{dx}{x} \leqslant \int_0^b \Phi(f(x))\left(1-\frac{x}{b}\right)^\gamma \frac{dx}{x}.$$

In particular, for the case $\Phi(x) = x^p,\ p \geqslant 1$ or $p < 0$ and $k > 1$ we get, after some variable substitutions and changes of notations,

$$\int_0^b \frac{1}{x^{\gamma(k-1)+1}} \left(\int_0^x \left(x^{\frac{k-1}{p}} - t^{\frac{k-1}{p}} \right)^{\gamma-1} f(t)dt \right)^p dx \leqslant \left(\frac{p}{\gamma(k-1)} \right)^p \int_0^b f^p(x) \left[1 - \left(\frac{x}{b} \right)^{\frac{k-1}{p}} \right]^\gamma x^{p-k} dx.$$

Note that if $\gamma = 1$ we get Corollary 1 (i) from [5] and an equivalent form is

$$\int_0^b \left(\frac{1}{x} \int_0^x f(t)dt \right)^p x^\varepsilon dx \leqslant \left(\frac{p}{p-1-\varepsilon} \right)^p \int_0^b f^p(x) \left[1 - \left(\frac{x}{b} \right)^{\frac{p-1-\varepsilon}{p}} \right] x^\varepsilon dx, \tag{4.3}$$

where $p > 1$ or $p < 0$ and $\varepsilon < p-1$.

REMARK 4.3. For the case $b = \infty$ (4.3) reads

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)dt \right)^p x^\varepsilon dx \leqslant \left(\frac{p}{p-1-\varepsilon} \right)^p \int_0^\infty f^p(x) x^\varepsilon dx, \varepsilon < p-1, \tag{4.4}$$

which for the case $p > 1$ is the most elementary form of weighted Hardy's inequality (see [8]). By making some substitutions we find that (4.4) can be equivalently rewritten on the basic form ((1.3) with $\Phi(u) = u^p$)

$$\int_0^\infty \left(\frac{1}{x} \int_0^x g(t)dt \right)^p \frac{dx}{x} \leqslant \int_0^\infty g^p(x) \frac{dx}{x}, \tag{4.5}$$

where $f(t) = g(t^{\frac{p-1-\varepsilon}{p}}) t^{-\frac{1+\varepsilon}{p}}$. Note that (4.5) also holds for $p = 1$. This shows that (4.4) in fact is not more general than the unweighted Hardy inequality (1.1) because both equivalently be rewritten on the same form (4.5).

REMARK 4.4. The fact that Hardy's inequality holds also for the case $p < 0$, $b = \infty$ was known earlier (see e.g. [1] and the references given there). Our approach give both cases $p < 0$ and $p > 1$ directly as a consequence of Theorem 4.1. Some other new information concerning the case $p < 0$ can also be found in the PhD thesis [16] of V. D. Prokhorov.

Now let $\widetilde{K}(x) := \int_x^\infty k(x,y)dy < \infty$ and define

$$A_{\widetilde{K}}(f(x)) := \frac{1}{\widetilde{K}(x)} \int_x^\infty f(y)k(x,y)dy.$$

Then from Jensen's inequality we have

$$\Phi\left(A_{\tilde{K}}(f(x))\right) \leqslant \frac{1}{\widetilde{K}(x)} \int_x^\infty \Phi\left(f(y)\right) k(x,y)dy,\ x>0.$$

By making similar calculations as in the proof of Theorem 4.1. we also have the following:

THEOREM 4.3. *For* $0 \leqslant b < \infty$, *let* u *be a weight function such that* $\frac{k(x,y)u(x)}{x\tilde{K}(x)}$ *is locally integrable on* (b,∞) *for every fixed* $y \in (b,\infty)$. *Let the function* v *be defined by*

$$v(y) = y\int_b^y \frac{k(x,y)u(x)}{x\tilde{K}(x)}dx < \infty, \quad y \in (b,\infty).$$

If Φ *is a positive and convex function on* (a,c), $-\infty \leqslant a < c \leqslant \infty$, *then*

$$\int_b^\infty \Phi(A_{\tilde{K}}f(x))u(x)\frac{dx}{x} \leqslant \int_b^\infty \Phi(f(x))v(x)\frac{dx}{x},$$

for all f *with* $a < f(x) < c$, $0 \leqslant x \leqslant b$.

REMARK 4.5. In fact Theorem 4.3. may be seen as a generalization of [5, Theorem 2]. This fact can be seen by choosing $k(x,y) = \frac{1}{y^2}$. Then $\widetilde{K}(x) = \int_x^\infty \frac{1}{y^2}dy = \frac{1}{x}$,

$$v(y) = y\int_b^y \frac{u(x)}{x\frac{1}{x}y^2}dx = \frac{1}{y}\int_b^y u(x)dx$$

and

$$A_{\tilde{K}}f(x) = x\int_x^\infty f(y)\frac{dy}{y^2},$$

and the statement follows.

Our final result reads:

THEOREM 4.4. *Let* $1 < p \leqslant q < \infty$, $0 < b \leqslant \infty$, $s \in (1,p)$, *let* Φ *be a convex and strictly monotone function on* (a,c), $-\infty \leqslant a < c \leqslant \infty$, *and let* A_K *be a general Hardy type operator defined by* (4.1). *Then the inequality*

$$\left(\int_0^b \left[\Phi\left(A_K f(x)\right)\right]^q u(x)\frac{dx}{x}\right)^{\frac{1}{q}} \leqslant C\left(\int_0^b \Phi^p(f(x))v(x)\frac{dx}{x}\right)^{\frac{1}{p}} \tag{4.6}$$

holds for all functions $f(x)$, $a < f(x) < c$, $x \in [0,b]$, *if*

$$A(s) := \sup_{0<t<b}\left(\int_t^b \left(\frac{k(x,t)}{K(x)}\right)^q u(x)V(x)^{\frac{q(p-s)}{p}}\frac{dx}{x}\right)^{\frac{1}{q}} V(t)^{\frac{s-1}{p}} < \infty, \tag{4.7}$$

holds, where $V(t)=\int_0^t \frac{v^{1-p'}(x)}{x^{1-p'}}dx$. *Moreover, if* C *is the best possible constant in* (4.6), *then*

$$C \leqslant \inf_{1<s<p}\left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}} A(s). \tag{4.8}$$

Proof. First we apply Jensen's inequality to the left hand side of (4.6), and get

$$\begin{aligned}\left(\int_0^b\left[\Phi\left(A_K f(x)\right)\right]^q u(x)\frac{dx}{x}\right)^{\frac{1}{q}} &= \left(\int_0^b\left[\Phi\left(\frac{1}{K(x)}\int_0^x k(x,t)f(t)dt\right)\right]^q u(x)\frac{dx}{x}\right)^{\frac{1}{q}}\\ &\leqslant \left(\int_0^b\left[\frac{1}{K(x)}\int_0^x k(x,t)\Phi\left(f(t)\right)dt\right]^q u(x)\frac{dx}{x}\right)^{\frac{1}{q}}.\end{aligned}$$

Thus we can give an upper bound for the inequality (4.6) if we can prove the following estimate:

$$\left(\int_0^b\left[\frac{1}{K(x)}\int_0^x k(x,t)\Phi\left(f(t)\right)dt\right]^q u(x)\frac{dx}{x}\right)^{\frac{1}{q}} \leqslant C\left(\int_0^b \Phi^p(f(x))v(x)\frac{dx}{x}\right)^{\frac{1}{p}}. \tag{4.9}$$

Now let $\Phi^p(f(x))\frac{v(x)}{x}=\Phi(g(x))$ in (4.9). Then (4.9) is equivalent to

$$\left(\int_0^b\left[\frac{1}{K(x)}\int_0^x k(x,t)\Phi^{\frac{1}{p}}\left(g(t)\right)\left(\frac{t}{v(t)}\right)^{\frac{1}{p}}dt\right]^q u(x)\frac{dx}{x}\right)^{\frac{1}{q}} \leqslant C\left(\int_0^b \Phi(g(x))dx\right)^{\frac{1}{p}}. \tag{4.10}$$

We apply Hölder's inequality, Minkowski's inequality, (4.7) and find that

$$\begin{aligned}&\left(\int_0^b\left[\frac{1}{K(x)}\int_0^x k(x,t)\Phi^{\frac{1}{p}}\left(g(t)\right)\left(\frac{t}{v(t)}\right)^{\frac{1}{p}}dt\right]^q u(x)\frac{dx}{x}\right)^{\frac{1}{q}}\\ &= \left(\int_0^b\left[\frac{1}{K(x)}\int_0^x k(x,t)\Phi^{\frac{1}{p}}\left(g(t)\right)V(t)^{\frac{s-1}{p}}V(t)^{\frac{-(s-1)}{p}}v(t)^{-\frac{1}{p}}t^{\frac{1}{p}}dt\right]^q u(x)\frac{dx}{x}\right)^{\frac{1}{q}}\\ &\leqslant \left(\int_0^b\left[\int_0^x k^p(x,t)\Phi\left(g(t)\right)V(t)^{s-1}dt\right]^{\frac{q}{p}}\left[\int_0^x V(t)^{\frac{-p'(s-1)}{p}}v(t)^{1-p'}t^{p'-1}dt\right]^{\frac{q}{p'}}\frac{u(x)}{K(x)^q}\frac{dx}{x}\right)^{\frac{1}{q}}\\ &= \left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}}\left(\int_0^b\left[\int_0^x k^p(x,t)\Phi\left(g(t)\right)V(t)^{s-1}dt\right]^{\frac{q}{p}}V^{\frac{q(p-s)}{p}}(x)\frac{u(x)}{K(x)^q}\frac{dx}{x}\right)^{\frac{1}{q}}\end{aligned}$$

$$\leqslant \left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}} \left(\int_0^b \Phi\left(g(t)\right) V(t)^{s-1} \left(\int_t^b V^{\frac{q(p-s)}{p}}(x) u(x) \left(\frac{k(x,t)}{K(x)}\right)^q \frac{dx}{x}\right)^{\frac{p}{q}} dt\right)^{\frac{1}{p}}$$

$$\leqslant \left(\frac{p-1}{p-s}\right)^{\frac{1}{p'}} A(s) \left(\int_0^b \Phi\left(g(t)\right) dt\right)^{\frac{1}{p}}.$$

Hence (4.10) and, thus, (4.9) holds with a constant C satisfying the inequality (4.8) and the proof is complete. $\square$

EXAMPLE 4.3. Let $\Phi(x) = x$, $k(x,t) = 1$ and $p = q$ in (4.6). Then we get the following Hardy inequality

$$\left(\int_0^b \left(\frac{1}{x}\int_0^x f(t)dt\right)^p u(x)\frac{dx}{x}\right)^{\frac{1}{p}} \leqslant C\left(\int_0^b (f^p(x))v(x)\frac{dx}{x}\right)^{\frac{1}{p}}, \tag{4.11}$$

for which the well-known Muckenhoupt condition reads (see e.g. [12] or [17])

$$A := \sup_{0<x<b} \left(\int_x^b u(t)t^{-p}\frac{dt}{t}\right)^{\frac{1}{p}} \left(\int_0^x \frac{v^{1-p'}(t)}{t^{1-p'}}dt\right)^{\frac{1}{p'}} < \infty. \tag{4.12}$$

Moreover, for the best constant C in (4.11) it yields that

$$C \leqslant \left(p'\right)^{\frac{1}{p'}} p^{\frac{1}{p}} A. \tag{4.13}$$

If we now take in $p = 2$, $b = 1$, $u(x) = x^2$ and $v(x) = x(1-x)$ we get the following inequality:

$$\left(\int_0^1 \left(\int_0^x f(t)dt\right)^2 \frac{1}{x}dx\right)^{\frac{1}{2}} \leqslant C\left(\int_0^1 f^2(x)(1-x)dx\right)^{\frac{1}{2}}. \tag{4.14}$$

By using the weight characterization (4.12) and the estimate (4.13) we get that

$$C \leqslant \ln 4 \lesssim 1.386294361.$$

Obviously the condition (4.7) is fulfilled for the weights we have chosed and using the estimate (4.8) with $s = 1.27$ we get that

$$C \lesssim 1.131316436$$

which is a much better estimate of the best constant C in (4.14).

REMARK 4.6. Example 4.3 illustrates the important fact that in Theorem 4.4. we have a scale of conditions (one for each $s \in (1,p)$) and in each case we get an upper estimate of the best constant. Therefore, we get more possibilities to get better estimates of the best constant then by just using the (single) Muckenhoupt condition.

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REFERENCES

[1] P. R. BEESACK AND H. P. HEINIG, *Hardy's inequalities with indices less than 1*, Proc. Amer. Math. Soc. 83 (1981), no. 3, 532–536.

[2] T. CARLEMAN, *Sur les fonctions quasi-analytiques*, Comptes rendus du V ^{e} Congres des Mathematiciens Scandinaves, Helsingfors, (1922), 181–196.

[3] A. ČIŽMEŠIJA AND J. E. PEČARIĆ, *Some new generalisations of inequalities of Hardy and Levin-Cochran-Lee*, Bull. Austral. Math. Soc. 63 (1) (2001), 105–113.

[4] A. ČIŽMEŠIJA AND J. E. PEČARIĆ, *On Bicheng-Debnath's generalization of Hardy's integral inequality*, Int. J. Math. Math. Sci. 27 (4) (2001), 237–250.

[5] A. ČIŽMEŠIJA, J. E. PEČARIĆ AND L. E. PERSSON, *On strengthened Hardy and Pólya-Knopp's inequalities*, J. Approx. Theory 125 (2003), 74–84.

[6] G. H. HARDY, *Notes on a theorem of Hilbert*, Math. Z. 6 (1920), 314–317.

[7] G. H. HARDY, *Notes on some points in the integral calculus (60)*, Messenger of Math. 54 (1925), 150–156.

[8] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, 2 nd ed., Cambridge University Press, 1952 (1934).

[9] M. JOHANSSON, L. E. PERSSON AND A. WEDESTIG, *Carleman's inequality: history, proofs and some new generalizations*, J. Inequal. Pure Appl. Math., Vol 4, Issue 3, Article 53, (2003) (19 pages).

[10] S. KAIJSER, L. E. PERSSON AND A. ÖBERG, *On Carleman and Knopp's inequalities*, J. Approx. Theory 117 (2002), 140–151.

[11] K. KNOPP, *Über reihen mit positivern gliedern*, J. London Math. Soc. 3 (1928), 205–211.

[12] A. KUFNER AND L. E. PERSSON, *Weighted inequalities of Hardy type*, World Scientific Publishing Co, Singapore / New Jersey / London / Hong Kong, 2003.

[13] A. KUFNER AND L. E. PERSSON, *The Hardy inequality – about its history and current status*, Research report 6, Department of Mathematics, Luleå University of Technology, 2002.

[14] D. S. MITRINOVIĆ, J. E. PEČARIĆ AND A. M. FINK, *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, 1991.

[15] C. P. NICULESCU AND L. E. PERSSON, *Convex Function – Basic Theory and Applications*, Universitaria Press, 2003.

[16] D. V. PROKHOROV, *Weighted Inequalities involving Riemann-Liouville and Hardy-Type Operators*, Ph. D. thesis, Department of Mathematics, Luleå University of Technology, 2003.

[17] B. OPIC AND A. KUFNER, *Hardy-type inequalities*, Pitman Research Notes in Mathematics 219, Longman Scientific & Technical, Harlow, 1990.

[18] A. WEDESTIG, *Weighted Inequalities of Hardy-Type and their Limiting Ineqalities*, Ph. D. thesis, Department of Mathematics, Luleå University of Technology, 2003.

[19] B. YANG AND L. DEBNATH, *Generalizations of Hardy's integral inequalities*, Internat. J. Math. Sci. 22 (3) (1999), 535–542.

[20] B. YANG, Z. ZENG AND L. DEBNATH, *On new generalizations of Hardy's integral inequality*, J. Math. Anal. Appl. 217 (1998), 321–327.

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