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GENERALIZATIONS OF SPECIAL BIHARI TYPE INTEGRAL INEQUALITIES

LÁSZLÓ HORVÁTH

(*communicated by A. M. Fink*)

Abstract. In this paper we consider Bihari type integral inequalities in measure spaces. We give explicit bounds for the solutions under very weak conditions. The studied inequality essentially contains all inequalities of similar forms that was considered previously, the results and the proofs give a unified approach of the problem. The results are applied to establish the existence of a solution to the integral equation corresponding to the integral inequality.

1. Introduction

Throughout this paper, $(X, \mathscr{A}, \mu)$ denotes a measure space. We consider integral inequalities

$$y(x) \leqslant f(x) + g(x) \int\limits_{S(x)} y^{\alpha} d\mu, \quad x \in X \tag{1.1}$$

with $0 < \alpha < 1$. The formal assumptions on the functions y, f, $g : X \to \mathbb{R}$ and $S : X \to \mathscr{A}$ are listed in the main theorems. Special cases of these inequalities have been studied by many authors. See, for example [1], [3], [7]-[10] and the references therein. Note that inequalities of this type are connected to the well known Bihari type inequalities (see [2]). The results are very useful in the theory of differential and integral equations.

In most of the concrete cases $X = [0, \infty[$, the functions y, f and g are continuous, $S(x) = [0, x]$ or $S(x) = [x, \infty[$, and Riemann integral is used. The essential idea of the proof of such results is to transforme the considered integral inequalities to differential inequalities, but this approach can not be applied for (1.1). The present work succesfully resolves this problem. It is emphasised that we are able to give explicit bounds for the solutions of (1.1) under very weak hypotheses on the function S. We assume only that it satisfies the condition (see [4])

$$\text{if} \quad y \in S(x), \quad \text{then} \quad S(y) \subset S(x), x \in X. \tag{C2}$$

This hypotheses is satisfied, in particular, if $X = [0, \infty[$, $\mathscr{A}$ is the σ-algebra of Lebesgue measurable sets, and $S(x) = [0, \beta x]$ or $S(x) = [\gamma x, \infty[$ ($0 \leqslant \beta \leqslant 1$ and

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$1 \leqslant \gamma$) or $S(x) = [0, a[$ ($0 < a$ or $a = \infty$ fixed), but the sets $S(x)$ are not even intervals. Analoges of these examples can also be defined in higher dimensions. For further examples see [6].

If the additional condition

$$H := \{(x_1, x_2) \in X^2 \mid x_2 \in S(x_1)\} \quad \text{is} \quad \mu^2\text{-measurable} \tag{C3}$$

is also supposed, then it is possible to show deeper properties of the bounds for the solutions of (1.1).

Finally we illustrate the scope of the results by applying them to establish the existence of a solution to the integral equations corresponding to (1.1).

2. Preliminaries

$\mathscr{A}$ always represents a σ-algebra in X, and the μ-integrable functions over X are considered to be almost measurable on X. The product of the measure space $(X, \mathscr{A}, \mu)$ with itself is understood as in [5], and it is denoted by $(X^2, \mathscr{A}^2, \mu^2)$. If f is a function and A is a subset of the domain of f, we denote by $f|A$ the restriction of f to A. We let $\mathbb{N} := \{0, 1, 2, \ldots\}$.

In this section a few preliminary results are presented which we need later. Some of them are rather simple, but others are also of independent interest.

LEMMA 1.
(a) *If* $0 < \alpha < 1$, *then* $(x+y)^\alpha \leqslant x^\alpha + y^\alpha$ *for every* $x,\ y \geqslant 0$.
(b) *If* $\alpha > 1$, *then* $(x+y)^\alpha \leqslant 2^{\alpha-1}(x^\alpha + y^\alpha)$ *for every* $x,\ y \geqslant 0$.

Proof. The proofs are elementary and we omit them. □

LEMMA 2. *Consider the algebric equation*

$$x = a + bx^\alpha, \quad x \geqslant 0, \tag{2.1}$$

where $0 < \alpha < 1$ *and* $a,\ b \geqslant 0$.

(a) *For* $a = b = 0$, $x_0 = 0$ *is the only solution of* (2.1). *For* $a + b > 0$ *there is exactly one positive root of* (2.1) *which is denoted by* x_0.

(b) *If* $0 \leqslant x \leqslant x_0$, *then* $x \leqslant a + bx^\alpha$. *Similarly, if* $x > x_0$, *then* $x > a + bx^\alpha$.

(c) *The following inequality holds*

$$a + b^{\frac{1}{1-\alpha}} \leqslant x_0 \leqslant \frac{a}{1-\alpha} + b^{\frac{1}{1-\alpha}}. \tag{2.2}$$

Proof. For $b > 0$ let $f : [0, \infty[\to \mathbb{R}$ be given by $f(x) = a + bx^\alpha - x$. Then a simple covering argument shows that f is strictly increasing on $[0, (\alpha b)^{\frac{1}{1-\alpha}}]$, strictly decreasing on $[(\alpha b)^{\frac{1}{1-\alpha}}, \infty[$, and strictly concave.

(a) If $b = 0$, then $x = a$ is the only solution of (2.1). Suppose $b > 0$. Since $f(0) = a \geqslant 0$, $\lim\limits_{\infty} f = -\infty$, and f is continuous, it follows from the monotonicity properties of f that there exists exactly one $x_0 > 0$ satisfying $f(x_0) = 0$.

(b) One cheks easily.

(c) If $a \geqslant 0$ and $b = 0$, then $x_0 = a$, so that (2.2) holds. Suppose $b > 0$. Since f takes on its largest value at $(\alpha b)^{\frac{1}{1-\alpha}}$ and f is strictly concave, the inequality $b^{\frac{1}{1-\alpha}} > (\alpha b)^{\frac{1}{1-\alpha}}$ implies that the x-coordinate of the point at which the tangent to the graph of f at $(b^{\frac{1}{1-\alpha}}, f(b^{\frac{1}{1-\alpha}}))$ intersects the x-axis is an upper bound for x_0. This gives the second inequality in (2.2).

To prove the first part of (2.2), an easy computation shows that $a + b^{\frac{1}{1-\alpha}}$ is a solution of the inequality $x \leqslant a + bx^{\alpha}$, so that we can apply (b). □

REMARK 3. An argument entirely similar to that of Lemma 2. (c) (using the tangent to the graph of f at $(a + b^{\frac{1}{1-\alpha}}, f(a + b^{\frac{1}{1-\alpha}}))$) shows that

$$x_0 \leqslant \frac{(1-\alpha)b(a + b^{\frac{1}{1-\alpha}})^{\alpha} + a}{1 - \alpha b(a + b^{\frac{1}{1-\alpha}})^{\alpha-1}}.$$

It follows from $b^{\frac{1}{1-\alpha}} \leqslant a + b^{\frac{1}{1-\alpha}} \leqslant x_0$ that the previous upper bound for x_0 is sharper then the upper bound in (2.2), but we don't use this stronger result in the paper.

We need the following function space.

DEFINITION 4. Suppose $S : X \to \mathscr{A}$, and $0 < \alpha < 1$. By $\mathscr{L}^{\alpha}$ we denote the vector space

$$\{z : X \to \mathbb{R} \mid z \text{ is } \mu\text{-almost measurable on } S(x),$$
$$\text{and } |z|^{\alpha} \text{ is } \mu\text{-integrable over } S(x) \text{ for every } x \in X\}.$$

The purpose of the next lemma is to describe the almost measurability of certain functions.

LEMMA 5.

(a) Let $pr_i : X^2 \to X$, $pr_i(x_1, x_2) = x_i$ $(i = 1, 2)$. If $A \in \mathscr{A}$ and the function $r : A \to \mathbb{R}$ is μ-almost measurable on A, then the function $r \circ pr_1$ ($r \circ pr_2$) is μ^2-almost measurable on $A \times X$ ($X \times A$).

(b) Let $S : X \to \mathscr{A}$ satisfy (C3), and let $A \in \mathscr{A}$ such that $S(x) \subset A$ for every $x \in A$. Suppose $p : A \to \mathbb{R}$ is μ-integrable over A, $q : A \to \mathbb{R}$ is μ-almost measurable on A, and there exists a measurable subset C of A such that $\mu(C)$ is σ-finite and $q(x) = 0$ for all $x \in A \setminus C$. Then the function

$$x \to q(x) \int_{S(x)} p d\mu, \quad x \in A$$

is μ-almost measurable on A.

Proof. (a) Let B be a measurable subset of A such that $\mu(A \setminus B) = 0$ and r is measurable on B. Since pr_1 is $\mathscr{A}^2 - \mathscr{A}$ measurable, $r \circ pr_1$ is measurable on $B \times X$, and therefore it is enough to show that

$$\mu^2((A \times X) \setminus (B \times X)) = 0.$$

Clearly

$$\mu^2((A \times X) \setminus (B \times X)) = \mu^2((A \setminus B) \times X) = \mu(A \setminus B) \cdot \mu(X) = 0.$$

(b) It is obviously enough to confine ourselves to those p and q that are nonnegative. Since p is μ-integrable over A, there is a measurable subset D of A such that $\mu(D)$ is σ-finite and $p(x) = 0$ for all $x \in A \setminus D$. Let

$$h : A^2 \to \mathbb{R}, \quad h(x,u) = q(x)p(u)\chi_H(x,u),$$

where χ_H denotes the characteristic function of H. By (a), the function

$$(x,u) \to q(x)p(u), \quad (x,u) \in A^2$$

is μ^2-almost measurable on A^2, and hence, by (C3), the function h is μ^2-almost measurable on A^2. It therefore follows from the nonnegativity of h that the integral

$$\int_{A^2} h d\mu^2$$

exists. Since $\mu^2(C \times D)$ is σ-finite and $h(x,u) = 0$ for all $(x,u) \in A^2 \setminus (C \times D)$, we can apply the Fubini's theorem (see [5]) which implies, by the second condition on S, that the function

$$x \to \int_A h(x,u)d\mu(u) = q(x) \int_{S(x)} pd\mu, \quad x \in A$$

is μ-almost measurable on A. $\square$

3. Main results

Our first result gives explicit bounds for the solutions of (1.1).

THEOREM 6. *Suppose* $0 < \alpha < 1$, *and* f, $g \in \mathscr{L}^\alpha$ *are nonnegative.*
(a) *If* $y \in \mathscr{L}^\alpha$ *is nonnegative such that*

$$y(x) \leqslant f(x) + g(x) \int_{S(x)} y^\alpha d\mu, \quad x \in X, \tag{3.1}$$

then

$$y(x) \leqslant f(x) + \frac{g(x)}{1-\alpha} \int_{S(x)} f^\alpha d\mu + g(x) \left(\int_{S(x)} g^\alpha d\mu \right)^{\frac{1}{1-\alpha}}, \quad x \in X. \tag{3.2}$$

(b) *If* $y \in \mathscr{L}^\alpha$ *is nonnegative such that*

$$y(x) \geqslant f(x) + g(x) \int_{S(x)} y^\alpha d\mu, \quad x \in X, \tag{3.3}$$

then

$$y(x) \geqslant f(x) + g(x) \int_{S(x)} f^\alpha d\mu, \quad x \in X. \tag{3.4}$$

(c) *If the property (C3) holds, then the functions defined by the right hand sides of* (3.2) *and* (3.4) *belong to* $\mathscr{L}^\alpha$.

Proof. (a) When $S(x)=\emptyset$ for some $x\in X$, then for all such x $y(x)\leqslant f(x)$ by (3.2), and this is equivalent to (3.1). Now suppose $x\in X$ with $S(x)\neq\emptyset$. Then by (3.1), Lemma 1. (a) and the property $(C2)$,

$$y^{\alpha}(u)\leqslant\left(f(u)+g(u)\int\limits_{S(u)}y^{\alpha}d\mu\right)^{\alpha}$$

$$\leqslant f^{\alpha}(u)+g^{\alpha}(u)\left(\int\limits_{S(u)}y^{\alpha}d\mu\right)^{\alpha}\leqslant f^{\alpha}(u)+g^{\alpha}(u)\left(\int\limits_{S(x)}y^{\alpha}d\mu\right)^{\alpha},\quad u\in S(x),$$

so that

$$\int\limits_{S(x)}y^{\alpha}d\mu\leqslant\int\limits_{S(x)}f^{\alpha}d\mu+\int\limits_{S(x)}g^{\alpha}d\mu\left(\int\limits_{S(x)}y^{\alpha}d\mu\right)^{\alpha}.$$

It therefore follows from Lemma 2. (b) and the second inequality in (c) that

$$\int\limits_{S(x)}y^{\alpha}d\mu\leqslant\frac{1}{1-\alpha}\int\limits_{S(x)}f^{\alpha}d\mu+\left(\int\limits_{S(x)}g^{\alpha}d\mu\right)^{\frac{1}{1-\alpha}}.\tag{3.5}$$

The substitution of (3.5) into (3.1) implies (3.2).

(b) We can assume that $x\in X$ with $S(x)\neq\emptyset$. Then, by (3.3), $y(u)\geqslant f(u)$ for all $u\in S(x)$, and hence (3.4) follows from another application of (3.3).

(c) We consider only the function defined by the right hand side of (3.2). Since $\mathscr{L}^{\alpha}$ is a vector space, it is enough to prove that the last two members of this function belong to $\mathscr{L}^{\alpha}$.

Let $x\in X$ with $S(x)\neq\emptyset$. Since $g\in\mathscr{L}^{\alpha}$, the subset $\{u\in S(x)\mid g(u)>0\}$ is of σ-finite measure.

By applying Lemma 5. (b) with $A:=S(x)$, $p:=f|S(x)$ and $q:=g|S(x)$, we obtain that the function

$$u\to g(u)\int\limits_{S(u)}f^{\alpha}d\mu,\quad u\in S(x)\tag{3.6}$$

is μ-almost measurable on $S(x)$. Since

$$g^{\alpha}(u)\left(\int\limits_{S(u)}f^{\alpha}d\mu\right)^{\alpha}\leqslant g^{\alpha}(u)\left(\int\limits_{S(x)}f^{\alpha}d\mu\right)^{\alpha},\quad u\in S(x),$$

this shows that the function (3.6) is μ-integrable over $S(x)$ in the αth power.

The hypotheses on g imply that $g^{1-\alpha}$ is μ-almost measurable on $S(x)$, and therefore, by Lemma 5. (b) with $A := S(x)$, $p := g^{\alpha}|S(x)$ and $q := g^{1-\alpha}|S(x)$, the function

$$u \to g^{1-\alpha}(u) \int\limits_{S(u)} g^{\alpha} d\mu, \quad u \in S(x)$$

is μ-almost measurable on $S(x)$, so that the function

$$u \to g(u) \left(\int\limits_{S(u)} g^{\alpha} d\mu \right)^{\frac{1}{1-\alpha}}, \quad u \in S(x) \tag{3.7}$$

is also μ-almost measurable on $S(x)$. Since

$$g^{\alpha}(u) \left(\int\limits_{S(u)} g^{\alpha} d\mu \right)^{\frac{\alpha}{1-\alpha}} \leqslant g^{\alpha}(u) \left(\int\limits_{S(x)} g^{\alpha} d\mu \right)^{\frac{\alpha}{1-\alpha}}, \quad u \in S(x),$$

the function (3.7) is μ-integrable over $S(x)$ in the αth power.

The proof is now completed. □

Some mention should be made here of the previous theorem. The inequality (3.1) essentially contains all inequalities of similar forms that was considered previously, but even if $X = \mathbb{R}^n$ it contains a lot of inequalities which have never been studied. Moreover, it is a generalization to measure spaces. The result and the proof give a unified approach of the problem. By using the inequality in Remark 3., we can obtain another upper bound for the solutions of (3.1) which is sharper than the upper bound in (3.2). Theorem 6. (c) does not hold without the property $(C3)$ in general as is easily seen by considering concrete examples.

The preceding theorem makes it possible for us to study another integral inequality. We show two different methods to get explicit upper bounds for the solutions of the inequality.

THEOREM 7. *Suppose* $0 < \alpha < 1$, *and* f, $g \in \mathscr{L}^1$ *are nonnegative.*

(a) *If* $y \in \mathscr{L}^1$ *is nonnegative such that*

$$y(x) \leqslant f(x) + g(x) \left(\int\limits_{S(x)} y d\mu \right)^{\alpha}, \quad x \in X. \tag{3.8}$$

Then

$(a1)$

$$y(x) \leqslant 2^{1-\alpha} \left(f^{\frac{1}{\alpha}}(x) + \frac{2^{1-\alpha} g^{\frac{1}{\alpha}}(x)}{1-\alpha} \int\limits_{S(x)} f d\mu + 2 g^{\frac{1}{\alpha}}(x) \left(\int\limits_{S(x)} g d\mu \right)^{\frac{1}{1-\alpha}} \right)^{\alpha}, \quad x \in X \tag{3.9}$$

$(a2)$

$$y(x) \leqslant f(x) + g(x)\Bigg(\int_{S(x)} f\,d\mu + \frac{1}{1-\alpha}\int_{S(x)} \Big(g(u)(\int_{S(u)} f\,d\mu)\Big)^{\alpha} d\mu(u) + \Big(\int_{S(x)} g d\mu\Big)^{\frac{1}{1-\alpha}}\Bigg)^{\alpha}, \quad x \in X, \tag{3.10}$$

whenever the property (C3) holds.

(b) *If $y \in \mathscr{L}^1$ is nonnegative such that*

$$y(x) \geqslant f(x) + g(x)\left(\int_{S(x)} y d\mu\right)^{\alpha}, \quad x \in X, \tag{3.11}$$

then

$$y(x) \geqslant f(x) + g(x)\left(\int_{S(x)} f\,d\mu\right)^{\alpha}, \quad x \in X. \tag{3.12}$$

(c) *If the property (C3) holds, then the functions defined by the right hand sides of* (3.9), (3.10) *and* (3.12) *belong to* $\mathscr{L}^1$.

Proof. $(a1)$ The inequalities (3.8) and Lemma $1.$ (b) imply that

$$y^{\frac{1}{\alpha}}(x) \leqslant \left(f(x)+g(x)\left(\int_{S(x)} y d\mu\right)^{\alpha}\right)^{\frac{1}{\alpha}} \leqslant 2^{\frac{1}{\alpha}-1}\left(f^{\frac{1}{\alpha}}(x)+g^{\frac{1}{\alpha}}(x)\int_{S(x)} y d\mu\right), \quad x \in X.$$

It therefore follows from Theorem $6.$ (a) that

$$y^{\frac{1}{\alpha}}(x) \leqslant 2^{\frac{1}{\alpha}-1}f^{\frac{1}{\alpha}}(x)+\frac{2^{1-\alpha}g^{\frac{1}{\alpha}}(x)}{1-\alpha}\int_{S(x)} 2^{1-\alpha}f\,d\mu+2^{1-\alpha}g^{\frac{1}{\alpha}}(x)\left(\int_{S(x)} 2^{1-\alpha}gd\mu\right)^{\frac{1}{1-\alpha}}, \ x \in X,$$

and this is equivalent to (3.9).

$(a2)$ By the inequality (3.8),

$$y(x) - f(x) \leqslant g(x)\left(\int_{S(x)} y d\mu\right)^{\alpha}, \quad x \in X.$$

Since the last expression is nonnegative on X, we have

$$\max(0, y(x) - f(x)) \leqslant g(x)\left(\int_{S(x)} y d\mu\right)^{\alpha}, \quad x \in X,$$

and it can be deduced from this that

$$\max(0, y(x) - f(x))^{\frac{1}{\alpha}} \leqslant g^{\frac{1}{\alpha}}(x) \int_{S(x)} y d\mu = g^{\frac{1}{\alpha}}(x) \int_{S(x)} f d\mu + g^{\frac{1}{\alpha}}(x) \int_{S(x)} (y - f) d\mu$$
$$\leqslant g^{\frac{1}{\alpha}}(x) \int_{S(x)} f d\mu + g^{\frac{1}{\alpha}}(x) \int_{S(x)} \max(0, y(x) - f(x)) d\mu, \quad x \in X. \tag{3.13}$$

Assuming for the present that the function

$$x \to g^{\frac{1}{\alpha}}(x) \int_{S(x)} f d\mu, \quad x \in X \tag{3.14}$$

belongs to $\mathscr{L}^{\alpha}$. Theorem 6. (a) can then be applied to (3.13), and it follows that

$$\max(0, y(x) - f(x))$$
$$\leqslant \left(g^{\frac{1}{\alpha}}(x) \int_{S(x)} f d\mu + \frac{g^{\frac{1}{\alpha}}(x)}{1-\alpha} \int_{S(x)} \left(g(u) \left(\int_{S(u)} f d\mu \right)^{\alpha} \right) d\mu(u) \right.$$
$$\left. + g^{\frac{1}{\alpha}}(x) \left(\int_{S(x)} g d\mu \right)^{\frac{1}{1-\alpha}} \right)^{\alpha}, \quad x \in X.$$

This inequality implies (3.10).

To justify that the function (3.14) lies in $\mathscr{L}^{\alpha}$, a similar argument used in the first part of the proof of Theorem 6. (c) can be applied.

(b) We argue as in the proof of Theorem 6. (b).

(c) The proof is, in its essentials, similar as that of Theorem 6. (c). $\square$

The following result provides an application of Theorem 6. We prove that the integral equation corresponding to the integral inequality (3.1) has at least one solution on X.

THEOREM 8. *Suppose $0 < \alpha < 1$, and f, $g \in \mathscr{L}^{\alpha}$ are nonnegative, and the property $(C3)$ holds. Then the integral equation*

$$y(x) = f(x) + g(x) \int_{S(x)} y^{\alpha} d\mu, \quad x \in X \tag{3.15}$$

has a solution on X, that is, there exists a nonnegative function $s \in \mathscr{L}^{\alpha}$ such that $y = s$ satisfies (3.15) for every $x \in X$.

Proof. Let $y_0 := f$, and let

$$y_{n+1}(x) := f(x) + g(x) \int_{S(x)} y_n^{\alpha} d\mu, \quad x \in X, \quad n \in \mathbb{N}. \tag{3.16}$$

We prove by induction on n that this recurrence relation defines a nonnegative sequence in $\mathscr{L}^{\alpha}$. By definition, y_0 is a nonnegative function from $\mathscr{L}^{\alpha}$. Suppose then that $n \in \mathbb{N}$ such that $y_n \in \mathscr{L}^{\alpha}$ and y_n is nonnegative. Then y_{n+1} is defined on X and nonnegative. Let $x \in X$ with $S(x) \neq \emptyset$. We can see exactly as in the first part of the proof of Theorem 6. (c) that y_{n+1}^{α} is μ-integrable over $S(x)$, so that $y_{n+1} \in \mathscr{L}^{\alpha}$. Another induction argument shows that the sequence (y_n) is increasing. Obviously $y_0 \leqslant y_1$. If $n \in \mathbb{N}$ for which $y_n \leqslant y_{n+1}$ holds, then, by the nonnegativity of g and the monotonicity of the integral, $y_{n+1} \leqslant y_{n+2}$. Since

$$y_n(x) \leqslant f(x) + g(x) \int\limits_{S(x)} y_n^{\alpha} d\mu, \quad x \in X, \quad n \in \mathbb{N},$$

Theorem 6. (a) and (c) imply that

$$y_n(x) \leqslant f(x) + \frac{g(x)}{1-\alpha} \int\limits_{S(x)} f^{\alpha} d\mu + g(x) \left(\int\limits_{S(x)} g^{\alpha} d\mu \right)^{\frac{1}{1-\alpha}}, \quad x \in X, \quad n \in \mathbb{N},$$

where the function defined by the right hand side of the previous inequality belongs to $\mathscr{L}^{\alpha}$. We have seen that (y_n) is increasing and bounded above by a function from $\mathscr{L}^{\alpha}$, and therefore it converges to a function $s \in \mathscr{L}^{\alpha}$, pointwise on X. It now follows from (3.16) and the monotone convergence theorem that s is a solution of the considered integral equation. The proof is complete. □

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Department of Mathematics and Computing
University of Veszprém
P.O. Box 158
8201 Veszprém
Hungary
e-mail: lhorvath@almos.vein.hu