Mathematical
Inequalities
& Applications
Volume 8, Number 3 (2005), 451–458

CERTAIN INEQUALITIES AND THEIR APPLICATIONS TO MULTIVALENTLY ANALYTIC FUNCTIONS

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*This paper is dedicated
to the memory of
Professor Dr. Doğan Çoker*

(*communicated by S. Owa*)

Abstract. In the present investigation, by making use of fractional calculus operator, two theorem involving certain inequalities of multivalent functions and their derivatives which are analytic in the open unit disk are stated. In addition, some interesting and/or mentioned results which will be important for Analytic and Geometric Function Theory (see, [1], [4], and also [2]) are also pointed.

1. Introduction and Definitions

Let $\mathscr{T}(p)$ denote the class of functions $f(z)$ of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbf{N} := \{1,\ 2,\ 3,\ \cdots\}), \tag{1.1}$$

which are *analytic* and *multivalent* in the open unit disk $\mathbf{U} = \{z \in \mathbf{C} : |z| < 1\}$.

A function $f(z) \in \mathscr{T}(p)$ is said to be in the class $\mathscr{S}(p;\alpha)$ of *multivalently starlike* (or, *starlike* when $p = 1$) *of order* α in $\mathbf{U}$ if it satisfies the inequality:

$$\mathscr{R}e\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbf{U}; 0 \leqslant \alpha < p; p \in \mathbf{N}). \tag{1.2}$$

On the other hand, a function $f(z) \in \mathscr{T}(p)$ is said to be in the class $\mathscr{C}(p;\alpha)$ of *multivalently convex* (or, *convex* when $p = 1$) *of order* α in $\mathbf{U}$ if it also satisfies the inequality:

$$\mathscr{R}e\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbf{U}; 0 \leqslant \alpha < p; p \in \mathbf{N}). \tag{1.3}$$

Mathematics subject classification (2000): 30C45, 26A33, 33C20.

Key words and phrases: Multivalently analytic function, multivalently starlike function, multivalently convex function, multivalently close-to-convex function, unit open disk, fractional calculus, principal values, inequalities, Jack's Lemma.

Paper MIA-08-41

Furthermore, a function $f(z) \in \mathscr{T}(p)$ is said to be in the class $\mathscr{K}(p;\alpha)$ of *multivalently close-to-convex* (or, *close-to-convex* when $p=1$) *of order* α in $\mathbf{U}$ if it satisfies:

$$\mathscr{R}e\left(\frac{f'(z)}{z^{p-1}}\right) > \alpha \quad (z \in \mathbf{U}; 0 \leqslant \alpha < p; p \in \mathbf{N}). \tag{1.4}$$

(See, for details, Duren [4], Goodman [1], and see also, Srivastava and Owa [2].)

We now denote by $\mathscr{V}_\delta^\mu(p)$ and $\mathscr{W}_\delta^\mu(p)$ the subclasses of functions $f(z)$ in $\mathscr{T}(p)$ which satisfy:

$$\mathscr{R}e\left\{\frac{\Gamma(p-\mu+1)[zD_z^{1+\mu}f(z)-(p-\mu)D_z^\mu f(z)]}{\Gamma(p-\mu+1)D_z^\mu f(z)-\Gamma(p+1)z^{p-\mu}}\right\}\begin{cases} <\frac{1}{\delta} & when\ \delta>0 \\ >\frac{1}{\delta} & when\ \delta<0, \end{cases} \tag{1.5}$$

and

$$\mathscr{R}e\left\{\frac{z\{(D_z^\mu f(z))(D_z^{1+\mu}f(z))+z[(D_z^\mu f(z))(D_z^{2+\mu}f(z))-(D_z^{1+\mu}f(z))^2]\}}{z(D_z^\mu f(z))(D_z^{1+\mu}f(z))-(p-\mu)(D_z^\mu f(z))^2}\right\}$$

$$\begin{cases} <\frac{1}{\delta} & when\ \delta>0 \\ >\frac{1}{\delta} & when\ \delta<0, \end{cases} \tag{1.6}$$

$$(z \in \mathbf{U}; \delta \neq 0; p \in \mathbf{N}; 0 \leqslant \mu < 1),$$

respectively. In (1.5), (1.6), and throughout this paper, D_z^μ denotes an operator of fractional calculus, which is defined as follows (cf., [2], [5], [6], and, see (also), for example, [7-9]):

DEFINITION 1. Let a function $f(z)$ be analytic in a simply-connected region of the z-plane containing the origin. The fractional integral of order μ is defined by

$$D_z^{-\mu}f(z) = \frac{1}{\Gamma(\mu)}\int_0^z \frac{f(\xi)}{(z-\xi)^{1-\mu}}d\xi \quad (\mu>0), \tag{1.7}$$

and fractional derivative of order μ is defined by

$$D_z^\mu f(z) = \frac{1}{\Gamma(1-\mu)}\frac{d}{dz}\int_0^z \frac{f(\xi)}{(z-\xi)^\mu}d\xi \quad (0 \leqslant \mu < 1), \tag{1.8}$$

where the multiplicity of $(z-\xi)^{\mu-1}$ involved in (1.7) and that of $(z-\xi)^{-\mu}$ in (1.8) are removed by requiring $log(z-\xi)$ to be real when $z-\xi>0$.

DEFINITION 2. Under the hypotheses of Definition 1, the fractional derivative of order $m+\mu$ is defined by

$$D_z^{m+\mu}f(z) = \frac{d^m}{dz^m}D_z^\mu f(z), \quad (m \in \mathbf{N}_0 := \mathbf{N}\cup\{0\}; 0 \leqslant \mu < 1). \tag{1.9}$$

We note that $\mathscr{T} \equiv \mathscr{T}(1)$,

$$\begin{aligned}
&\mathscr{E}_1(\mu;\delta) \equiv \mathscr{V}_\mu^\delta(1), \mathscr{E}_2(\mu;\delta) \equiv \mathscr{W}_\mu^\delta(1),\\
&\mathscr{A}_1(p;\mu) \equiv \mathscr{V}_1^\mu(p), \mathscr{B}_1(p;\mu) \equiv \mathscr{W}_1^\mu(p),\\
&\mathscr{A}_2(p;\mu) \equiv \mathscr{V}_{-1}^\mu(p), \mathscr{B}_2(p;\mu) \equiv \mathscr{W}_{-1}^\mu(p),\\
&\mathscr{A}_3(p;\delta) \equiv \mathscr{V}_\delta^0(p), \mathscr{B}_3(p;\delta) \equiv \mathscr{W}_\delta^0(p),\\
&\mathscr{A}_4(p;\delta) \equiv \mathscr{V}_\delta^1(p) \quad \textit{and} \quad \mathscr{B}_4(p;\delta) \equiv \mathscr{W}_\delta^1(p).
\end{aligned}$$

In proving the main results, we shall need the following Lemma known as Jack's Lemma in the literature.

LEMMA. (cf., [3]). *Let $w(z)$ be non-constant and analytic in $\mathbf{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point z_0, then*

$$z_0 w'(z_0) = c w(z_0), \tag{1.10}$$

where c is real number and $c \geqslant 1$.

2. The Main Results

An application of the above Lemma leads to

THEOREM 1. *Let $z \in \mathbf{U}, p \in \mathbf{N}, 0 \leqslant \mu < 1, 0 \leqslant \alpha < \Gamma(p+1)/\Gamma(p-\mu+1)$, and $f(z) \in \mathscr{T}(p)$. If the function $f(z)$ belongs to the class $\mathscr{V}_\mu^\delta(p)$, then*

$$\mathscr{R}e\left(\frac{D_z^\mu f(z)}{z^{p-\mu}}\right) > \alpha. \tag{2.1}$$

Proof. First of them, Definition 1 readily privides us the following fractional derivative formula for a pover function :

$$D_z^\mu\{z^\kappa\} = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\mu+1)} z^{\kappa-\mu} \quad (\kappa > -1; 0 \leqslant \mu < 1). \tag{2.2}$$

Then, under the hypothesis of Theorem 1, define $w(z)$ by

$$\left(\frac{D_z^\mu f(z)}{z^{p-\mu}} - \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}\right)^\delta = \left(\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} - \alpha\right)^\delta w(z), \tag{2.3}$$

where the value of

$$\left(\frac{D_z^\mu f(z)}{z^{p-\mu}} - \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}\right)^\delta \quad (\delta \neq 0; p \in \mathbf{N}; 0 \leqslant \mu < 1), \tag{2.4}$$

is taken to be as its principal value. Then, clearly $w(z)$ is an analytic function in $\mathbf{U}$ and $w(0) = 0$. Upon differentiating both sides of (2.3) with repect to the variable z, we have that

$$\left(\frac{D_z^\mu f(z)}{z^{p-\mu}} - \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}\right)^{\delta-1}$$

$$\times\left(\frac{D_z^{1+\mu}f(z)}{z^{p-\mu-1}}-(p-\mu)\frac{D_z^{\mu}f(z)}{z^{p-\mu}}\right)=\frac{zw'(z)}{\delta}. \tag{2.5}$$

By using of (2.3) in (2.5), we then find that

$$\frac{\Gamma(p-\mu+1)[zD_z^{1+\mu}f(z)-(p-\mu)D_z^{\mu}f(z)]}{\Gamma(p-\mu+1)D_z^{\mu}f(z)-\Gamma(p+1)z^{p-\mu}}=\frac{1}{\delta}\frac{zw'(z)}{w(z)}. \tag{2.6}$$

If now we suppose that

$$\max_{|z|\leqslant|z_0|}|w(z)|=|w(z_0)|=1 \quad (z\in\mathbf{U};w(z_0)\neq 0). \tag{2.7}$$

and apply Jack's Lemma, we find that

$$z_0w'(z_0)=cw(z_0) \quad (c\geqslant 1;z_0\in\mathbf{U}), \tag{2.8}$$

and also setting $z=z_0$ in (2.6), we easily obtain that

$$\mathscr{R}e\left\{\left.\frac{\Gamma(p-\mu+1)[zD_z^{1+\mu}f(z)-(p-\mu)D_z^{\mu}f(z)]}{\Gamma(p-\mu+1)D_z^{\mu}f(z)-\Gamma(p+1)z^{p-\mu}}\right|_{z=z_0}\right\}$$

$$=\mathscr{R}e\left(\frac{1}{\delta}\frac{z_0w'(z_0)}{w(z_0)}\right)=\frac{c}{\delta}\begin{cases}\geqslant\frac{1}{\delta} & \text{when } \delta>0\\ \leqslant\frac{1}{\delta} & \text{when } \delta<0,\end{cases} \tag{2.9}$$

which obviously contradicts our hypothesis that $f(z)\in\mathscr{V}_\mu^\delta(p)$. Therefore, we must have $|w(z)|<1 \ (z\in\mathbf{U})$, and it immediately follows from (2.3) that

$$\left|\frac{D_z^{\mu}f(z)}{z^{p-\mu}}-\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}\right|^{\delta}<\left(\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}-\alpha\right)^{\delta}, \tag{2.10}$$

which implies to the assertion in (2.1). This evidently completes the proof of the Theorem 1.

We next derive

THEOREM 2. *Let $z\in\mathbf{U}$, $p\in\mathbf{N}$, $0\leqslant\mu<1$, $0\leqslant\alpha<p$, and $f(z)\in\mathscr{T}(p)$. If the function $f(z)$ belongs to the class $\mathscr{W}_\mu^\delta(p)$, then*

$$\mathscr{R}e\left(\frac{zD_z^{1+\mu}f(z)}{D_z^{\mu}f(z)}\right)>\alpha \quad (0\leqslant\mu+\alpha<p). \tag{2.11}$$

Proof. Under the hypothesis of Theorem 2, we again define a function $w(z)$ by

$$\left(\frac{zD_z^{1+\mu}f(z)}{D_z^{\mu}f(z)}-(p-\mu)\right)^{\delta}=(p-\mu-\alpha)^{\delta}w(z), \tag{2.12}$$

where the value of

$$\left(\frac{zD_z^{1+\mu}f(z)}{D_z^{\mu}f(z)}-(p-\mu)\right)^{\delta} \quad (\delta\neq 0;\ p\in\mathbf{N};\ 0\leqslant\mu<1) \tag{2.13}$$

is considered to be as its principal value. From the logarithmic derivation of both sides of (2.12), we easily recieve:

$$\frac{z\{(D_z^{\mu}f(z))(D_z^{1+\mu}f(z))+z[(D_z^{\mu}f(z))(D_z^{2+\mu}f(z))-(D_z^{1+\mu}f(z))^2]\}}{z(D_z^{\mu}f(z))(D_z^{1+\mu}f(z))-(p-\mu)(D_z^{\mu}f(z))^2}$$

$$=\frac{1}{\delta}\frac{zw'(z)}{w(z)} \quad (z\in\mathbf{U};\delta\neq 0;p\in\mathbf{N};0\leqslant\mu<1). \tag{2.14}$$

It is easy to see that the defined function $w(z)$ satisfies the conditions of the Lemma. If we make use of the same technique as in the proof of the Theorem 1 in the equality (2.14), then we easily arrive at the desired proof of the Theorem 2. Therefore, its details need not be presented.

We clearly get that the main results (Theorems 1 and 2) have many useful consequences concerning analytic and/or multivalently analytic functions. To state them, it will be sufficient to choose suitable values of the papameters $\delta,\ \mu,\ \alpha$ and/or p. Some of them are the following:

If we first take $p=1$ in Theorems 1 and 2 together with definition (1.5) and (1.6), respectively, then $\mathscr{E}_1(\mu;\delta)\equiv\mathscr{V}_{\mu}^{\delta}(1), \mathscr{E}_2(\mu;\delta)\equiv\mathscr{W}_{\mu}^{\delta}(1),$ and we then have the following results:

COROLLARY 1. *Let* $z\in\mathbf{U},\ 0\leqslant\mu<1,\ 0\leqslant\alpha<1/\Gamma(2-\mu),$ *and* $f(z)\in\mathscr{T}$. *If the function* $f(z)$ *belongs to the class* $\mathscr{E}_1(\mu;\delta)$, *namely it satisfies:*

$$\mathscr{R}e\left\{\frac{\Gamma(2-\mu)[zD_z^{1+\mu}f(z)-(1-\mu)D_z^{\mu}f(z)]}{\Gamma(2-\mu)D_z^{\mu}f(z)-z^{1-\mu}}\right\}\begin{cases}<\frac{1}{\delta} & \text{when } \delta>0\\ >\frac{1}{\delta} & \text{when } \delta<0,\end{cases}$$

then

$$\mathscr{R}e\left(\frac{D_z^{\mu}f(z)}{z^{1-\mu}}\right)>\alpha.$$

COROLLARY 2. *Let* $z\in\mathbf{U},\ 0\leqslant\mu<1,\ 0\leqslant\alpha<1,$ *and* $f(z)\in\mathscr{T}$. *If the function* $f(z)$ *belongs to the class* $\mathscr{E}_2(\mu;\delta)$, *namely it satisfies the inequality:*

$$\mathscr{R}e\left\{\frac{z\{(D_z^{\mu}f(z))(D_z^{1+\mu}f(z))+z[(D_z^{\mu}f(z))(D_z^{2+\mu}f(z))-(D_z^{1+\mu}f(z))^2]\}}{z(D_z^{\mu}f(z))(D_z^{1+\mu}f(z))-(1-\mu)(D_z^{\mu}f(z))^2}\right\}$$

$$\begin{cases}<\frac{1}{\delta} & \text{when } \delta>0\\ >\frac{1}{\delta} & \text{when } \delta<0,\end{cases}$$

then

$$\mathscr{R}e\left(\frac{zD_z^{1+\mu}f(z)}{D_z^{\mu}f(z)}\right)>\alpha.$$

If we set $\delta=1$ in Theorem 1 together with definition (1.5), then $\mathscr{A}_1(p;\mu)\equiv\mathscr{V}_1^{\mu}(p)$ and we also obtain:

COROLLARY 3. *Let* $z \in \mathbf{U},\ p \in \mathbf{N},\ 0 \leqslant \mu < 1,\ 0 \leqslant \alpha < \Gamma(p+1)/\Gamma(p-\mu+1),$ *and* $f(z) \in \mathscr{T}(p)$. *If the function* $f(z)$ *belongs to the class* $\mathscr{A}_1(p;\mu)$, *namely it satisfies:*

$$\mathscr{R}e\left\{\frac{\Gamma(p-\mu+1)[zD_z^{1+\mu}f(z)-(p-\mu)D_z^{\mu}f(z)]}{\Gamma(p-\mu+1)D_z^{\mu}f(z)-\Gamma(p+1)z^{p-\mu}}\right\}<1,$$

then

$$\mathscr{R}e\left(\frac{D_z^{\mu}f(z)}{z^{p-\mu}}\right)>\alpha.$$

If we let $\delta = 1$ in Theorem 2 together with definition (1.6), then $\mathscr{B}_1(p;\mu) \equiv \mathscr{W}_1^{\mu}(p)$ and we recieve:

COROLLARY 4. *Let* $z \in \mathbf{U},\ p \in \mathbf{N},\ 0 \leqslant \mu < 1,\ 0 \leqslant \alpha < p,$ *and* $f(z) \in \mathscr{T}(p)$. *If the function* $f(z)$ *belongs to the class* $\mathscr{B}_1(p;\mu)$, *namely it satisfies:*

$$\mathscr{R}e\left\{\frac{z\{(D_z^{\mu}f(z))(D_z^{1+\mu}f(z))+z[(D_z^{\mu}f(z))(D_z^{2+\mu}f(z))-(D_z^{1+\mu}f(z))^2]\}}{z(D_z^{\mu}f(z))(D_z^{1+\mu}f(z))-(p-\mu)(D_z^{\mu}f(z))^2}\right\}<1,$$

then

$$\mathscr{R}e\left(\frac{zD_z^{1+\mu}f(z)}{D_z^{\mu}f(z)}\right)>\alpha.$$

By taking $\delta = -1$ in Theorem 1 together with definition (1.5), then $\mathscr{A}_2(p;\mu) \equiv \mathscr{V}_{-1}^{\mu}(p)$ and we also get:

COROLLARY 5. *Let* $z \in \mathbf{U},\ p \in \mathbf{N},\ \delta \neq 0,\ 0 \leqslant \alpha < \Gamma(p+1)/\Gamma(p-\mu+1),$ *and* $f(z) \in \mathscr{T}(p)$. *If the function* $f(z)$ *belongs to the class* $\mathscr{A}_2(p;\delta)$, *namely it satisfies:*

$$\mathscr{R}e\left\{\frac{\Gamma(p-\mu+1)[zD_z^{1+\mu}f(z)-(p-\mu)D_z^{\mu}f(z)]}{\Gamma(p-\mu+1)D_z^{\mu}f(z)-\Gamma(p+1)z^{p-\mu}}\right\}>-1,$$

then

$$\mathscr{R}e\left(\frac{f(z)}{z^p}\right)>\alpha.$$

By setting $\delta = -1$ in Theorem 2 together with definition (1.6), then $\mathscr{B}_2(p;\mu) \equiv \mathscr{W}_{-1}^{\mu}(p)$ and we have:

COROLLARY 6. *Let* $z \in \mathbf{U},\ p \in \mathbf{N},\ \delta \neq 0;\ 0 \leqslant \alpha < p,$ *and* $f(z) \in \mathscr{T}(p)$. *If the function* $f(z)$ *belongs to the class* $\mathscr{B}_2(p;\delta)$, *namely it satisfies:*

$$\mathscr{R}e\left\{\frac{z\{(D_z^{\mu}f(z))(D_z^{1+\mu}f(z))+z[(D_z^{\mu}f(z))(D_z^{2+\mu}f(z))-(D_z^{1+\mu}f(z))^2]\}}{z(D_z^{\mu}f(z))(D_z^{1+\mu}f(z))-(p-\mu)(D_z^{\mu}f(z))^2}\right\}>-1,$$

then

$$\mathscr{R}e\left(\frac{zf'(z)}{f(z)}\right)>\alpha, \quad i.e., \quad f(z)\in\mathscr{S}(p;\alpha).$$

By letting $\mu = 0$ in Theorem 1 together with definition (1.5), then $\mathscr{A}_3(p;\delta) \equiv \mathscr{V}_{\delta}^{0}(p)$ and we arrive at:

COROLLARY 7. *Let* $z \in \mathbf{U},\ p \in \mathbf{N},\ \delta \neq 0,\ 0 \leqslant \alpha < 1,$ *and* $f(z) \in \mathscr{T}(p)$. *If the function* $f(z)$ *belongs to the class* $\mathscr{A}_3(p;\delta)$, *namely it satisfies:*

$$\mathscr{R}e\left(\frac{zf'(z) - pf(z)}{f(z) - z^p}\right)\begin{cases} < \frac{1}{\delta} & \text{when } \delta > 0 \\ > \frac{1}{\delta} & \text{when } \delta < 0 \end{cases},$$

then

$$\mathscr{R}e\left(\frac{f'(z)}{z^{p-1}}\right) > \alpha, \quad i.e., \quad f(z) \in \mathscr{K}(p;\alpha).$$

By putting $\mu = 0$ in Theorem 2 together with definition (1.6), then $\mathscr{B}_3(p;\delta) \equiv \mathscr{W}_\delta^0(p)$ and we have:

COROLLARY 8. *Let* $z \in \mathbf{U},\ p \in \mathbf{N},\ \delta \neq 0;\ 0 \leqslant \alpha < p,$ *and* $f(z) \in \mathscr{T}(p)$. *If the function* $f(z)$ *belongs to the class* $\mathscr{B}_3(p;\delta)$, *namely it satisfies:*

$$\mathscr{R}e\left\{\frac{z\{(f(z))(f'(z)) + z[(f(z))(f''(z)) - (f'(z))^2]\}}{z[f(z)f'(z) - p(f(z))^2]}\right\}\begin{cases} < \frac{1}{\delta} & \text{when } \delta > 0 \\ > \frac{1}{\delta} & \text{when } \delta < 0 \end{cases},$$

then

$$\mathscr{R}e\left(\frac{zf''(z)}{f'(z)}\right) > \alpha, \quad i.e., \quad f(z) \in \mathscr{C}(p;\alpha).$$

If we take $\mu \to 1-$ in Theorem 1 together with definition (1.5), then $\mathscr{A}_4(p;\delta) \equiv \mathscr{V}_\delta^1(p)$ and we also get:

COROLLARY 9. *Let* $z \in \mathbf{U},\ p \in \mathbf{N},\ \delta \neq 0,\ 0 \leqslant \alpha < p,$ *and* $f(z) \in \mathscr{T}(p)$. *If the function* $f(z)$ *belongs to the class* $\mathscr{A}_4(p;\delta)$, *namely it satisfies:*

$$\mathscr{R}e\left(\frac{zf''(z) - (p-1)f(z)}{f'(z) - z^{p-1}}\right)\begin{cases} < \frac{1}{\delta} & \text{when } \delta > 0 \\ > \frac{1}{\delta} & \text{when } \delta < 0 \end{cases},$$

then

$$\mathscr{R}e\left(\frac{f'(z)}{z^{p-1}}\right) > \alpha, \quad i.e., \quad f(z) \in \mathscr{K}(p;\alpha).$$

If we take $\mu \to 1-$ in Theorem 2 together with definition (1.6), then $\mathscr{B}_4(p;\delta) \equiv \mathscr{W}_\delta^1(p)$ and we obtain:

COROLLARY 10. *Let* $z \in \mathbf{U},\ p \in \mathbf{N},\ \delta \neq 0;\ 0 \leqslant \alpha < p,$ *and* $f(z) \in \mathscr{T}(p)$. *If the function* $f(z)$ *belongs to the class* $\mathscr{B}_4(p;\delta)$, *namely it satisfies:*

$$\mathscr{R}e\left\{\frac{z\{f'(z)f''(z) + z[f'(z)f'''(z) - (f''(z))^2]\}}{zf'(z)f''(z) - (p-1)(f'(z))^2]}\right\}\begin{cases} < \frac{1}{\delta} & \text{when } \delta > 0 \\ > \frac{1}{\delta} & \text{when } \delta < 0 \end{cases},$$

then

$$\mathscr{R}e\left(\frac{zf''(z)}{f'(z)}\right) > \alpha, \quad i.e., \quad f(z) \in \mathscr{C}(p;\alpha).$$

Acknowledgment. This present investigation was supported by TÜBİTAK (The Scientific and Technical Research Council of TURKEY), and Başkent University (Ankara, TURKEY). The author would also like to acknowledge Professor Dr. Mehmet HABERAL, Rector of Başkent University, who generously supports scientific researches in all aspects.

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(Received March 19, 2003)

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