Mathematical
Inequalities
& Applications
Volume 8, Number 3 (2005), 459–475

LIAPUNOV–TYPE INEQUALITIES AND NEUMANN BOUNDARY VALUE PROBLEMS AT RESONANCE

A. CAÑADA, J. A. MONTERO AND S. VILLEGAS

(*communicated by J. Mawhin*)

Abstract. This paper is devoted to the study of resonant nonlinear boundary value problems with Neumann boundary condition. First we consider the linear situation doing a careful analysis on the existence of nontrivial solutions. This analysis involves Liapunov-type inequalities with the L_p- norm of the coefficient function for $1 \leqslant p \leqslant \infty$. We carry out a complete treatment of the problem for any constant $p \geqslant 1$. Then, this is combined with Schauder fixed point theorem to obtain new results about the existence and uniqueness of solutions for resonant nonlinear problems.

1. Introduction

Let us consider the Neumann problem

$$u''(x) + f(x, u(x)) = 0, \quad x \in (0, L), \quad u'(0) = u'(L) = 0 \tag{1.1}$$

where $f : [0, L] \times \mathbb{R} \to \mathbb{R}$, $(x, u) \to f(x, u)$, satisfies the condition

(H) f, f_u are continuous on $[0, L] \times \mathbb{R}$ and $0 \leqslant f_u(x, u)$ on $[0, L] \times \mathbb{R}$.

The existence of a solution of (1.1) implies

$$\int_0^L f(x, z)\, dx = 0 \tag{1.2}$$

for some $z \in \mathbb{R}$. However, conditions (H) and (1.2) are not sufficient for the existence of solutions of (1.1). Indeed, consider the problem

$$u''(x) + \pi^2 u(x) + \cos(\pi x) = 0, \quad x \in (0, 1), \quad u'(0) = u'(1) = 0. \tag{1.3}$$

The function $f(x, u) = \pi^2 u + \cos(\pi x)$ satisfies (H) and (1.2), but the Fredholm alternative theorem shows that there is no solution of (1.3).

If (H) and (1.2) are assumed, and for instance, $L = 1$ for simplicity, different supplementary assumptions have been given which imply the existence of a solution of (1.1). For example

---

Mathematics subject classification (2000): 34B15, 34B05.

Key words and phrases: Neumann boundary value problems, resonance, Liapunov inequalities, existence and uniqueness.

The authors have been supported by the Ministry of Science and Technology of Spain (BFM2002-02649).

$(h1)$ $f_u(x,u) \leqslant \beta(x)$ on $[0,1]\times\mathbb{R}$ with $\beta \in L^\infty(0,1)$, $\beta(x) \leqslant \pi^2$ on $[0,1]$ and $\beta(x) < \pi^2$ on a subset of $(0,1)$ of positive measure.

Conditions of this type are referred to as non-uniform non-resonance conditions with respect to the first positive eigenvalue of the associated linear homogeneous problem. By using variational methods, it is proved in [8] that (H), (1.2) and $(h1)$ imply the existence of solutions of (1.1).

On the other hand, in [6] it is supposed

$(h2)$ $f_u(x,u) \leqslant \beta(x)$ on $[0,1]\times\mathbb{R}$ with $\beta \in L^1(0,1)$ and $\int_0^1 \beta(x)\,dx \leqslant 4$.

The authors use Optimal Control theory methods to prove that (H), (1.2) and $(h2)$ imply the existence and uniqueness of solutions of (1.1). Restriction $(h2)$ is related to Liapunov-type inequalities for linear second order equations (see, for instance Corollary 5.1 in [5] for the case of Dirichlet boundary conditions and [1] for a survey paper on Lyapunov inequalities).

Let us observe that supplementary conditions $(h1)$ and $(h2)$ are given respectively in terms of $\|\beta\|_\infty$ and $\|\beta\|_1$, the usual norms in the spaces $L^\infty(0,1)$ and $L^1(0,1)$. Also, it is clear that under the hypotheses (H) and (1.2), $(h1)$ and $(h2)$ are not related.

In this paper we provide supplementary conditions in terms of $\|\beta\|_p$, $1 < p < \infty$. In fact, this was the original motivation of our work, but the proofs are based in a previous analysis of the linear case which involves Liapunov-type inequalities with the L_p- norm of the coefficient function for $1 \leqslant p \leqslant \infty$. Really, this is the main contribution of this paper where we carry out a complete treatment of the linear problem for any $p \geqslant 1$. As a consequence, a natural relation between $(h1)$ and $(h2)$ arises if one studies the limits of $\|\beta\|_p$ for $p \to 1^+$ and $p \to \infty$.

One of the main results of our paper is given by Lemma 2.6 below where we prove that the best constant of our problem, β_p defined in (2.5), can be computed by using a certain minimization problem. Motivated by a completely different problem (an isoperimetric inequality known as Wulff theorem, of interest in crystallography), the authors studied in [3] a similar variational problem for the case of periodic or Dirichlet boundary conditions (see also [7] for the case $p = 2, 3, 3/2$ and Dirichlet boundary conditions and [2] for more general minimization problems). We study in this paper the case of Neumann boundary conditions and our treatment of the Euler equation associated to the mentioned minimization problem is different from that of [3]. Finally, it is clear from the proofs that one can deal with other boundary conditions and more general second order equations. Also, some results for PDE problems may be obtained. This will be published elsewhere.

2. Liapunov-type inequalities for the linear problem

This section will be concerned with the existence of nontrivial solutions of a homogeneous linear problem of the form

$$u''(x) + a(x)u(x) = 0, \quad x \in (0,L), \quad u'(0) = u'(L) = 0 \tag{2.1}$$

where $a \in \Lambda$ and Λ is defined by

$$\Lambda = \left\{ a \in L^\infty(0,L) \setminus \{0\} : \int_0^L a(x)dx \geqslant 0 \text{ and (2.1) has nontrivial solutions} \right\}. \tag{2.2}$$

Here $u \in H^1(0,L)$, the usual Sobolev space. For each p with $1 \leqslant p \leqslant \infty$ we can define the functional $I_p : \Lambda \to \mathbb{R}$ given by the expression

$$I_p(a) = \|a\|_p = \left(\int_0^L |a(x)|^p \, dx \right)^{1/p}, \forall\, a \in \Lambda,\ 1 \leqslant p < \infty \tag{2.3}$$

$$I_\infty(a) = \text{sup ess } a,\ \forall a \in \Lambda.$$

Obviously, the positive eigenvalues of the eigenvalue problem

$$u''(x) + \lambda u(x) = 0, \quad x \in (0,L), \quad u'(0) = u'(L) = 0, \tag{2.4}$$

belong to Λ. Therefore Λ is not empty and

$$\beta_p \equiv \inf_{a \in \Lambda} I_p(a), \quad 1 \leqslant p \leqslant \infty \tag{2.5}$$

is well defined. The main result of this section is the following.

THEOREM 2.1. *The following statements hold:*

1. *β_p is attained if and only if $1 < p \leqslant \infty$. In this case, β_p is attained in a unique element $a_p \in \Lambda$ which is not constant if $1 < p < \infty$.*

2. *The quantity β_p is given by*

$$\beta_p = \begin{cases} \dfrac{4}{L}, \text{ if } p = 1, \\ \dfrac{4(p-1)^{1+\frac{1}{p}}}{L^{2-\frac{1}{p}} p(2p-1)^{1/p}} \left(\displaystyle\int_0^{\pi/2} (\sin x)^{-1/p} \, dx \right)^2, \text{ if } 1 < p < \infty, \\ \dfrac{\pi^2}{L^2}, \text{ if } p = \infty \end{cases} \tag{2.6}$$

3. $a_\infty(x) \equiv \dfrac{\pi^2}{L^2}$. *If $1 < p < \infty$, the function a_p is given by $a_p = |u_p|^{\frac{2}{p-1}}$, where in the interval $(0, \frac{L}{2})$, u_p is the unique positive solution of the problem*

$$-u''(x) = u(x)^{\frac{p+1}{p-1}},\ u'(0) = 0,\ u(L/2) = 0$$

and in the interval $(L/2, L)$ u_p is defined by the formula $u_p(x) = -u_p(L-x)$, $\forall\, x \in (L/2, L)$ (see the proof of Lemma 2.7).

4. *The mapping $[1,\infty) \to \mathbb{R}$, $p \to \beta_p$, is continuous and $\lim_{p\to\infty} \beta_p = \beta_\infty$. Moreover, the mapping $[1,\infty) \to \mathbb{R}$, $p \to L^{-1/p}\beta_p$ is strictly increasing.*

Proof. It is based on some lemmas. In the first one we study the case $p = \infty$. □

LEMMA 2.2. *β_∞ is attained in a unique element $a_\infty \in \Lambda$. Moreover $a_\infty(x) \equiv \dfrac{\pi^2}{L^2}$.*

Proof. If $a \in \Lambda$ and $u \in H^1(0,L)$ is a nontrivial solution of

$$-u''(x) = a(x)u(x), \quad x \in (0,L), \quad u'(0) = u'(L) = 0, \tag{2.7}$$

then

$$\int_0^L u'v' = \int_0^L auv, \ \forall\, v \in H^1(0,L).$$

In particular, we have

$$\int_0^L u'^2 = \int_0^L au^2, \qquad \int_0^L au = 0. \tag{2.8}$$

Therefore, for each $k \in \mathbb{R}$, we have

$$\int_0^L (u+k)'^2 = \int_0^L u'^2 = \int_0^L au^2 \leqslant \int_0^L au^2 + k^2 \int_0^L a$$
$$= \int_0^L au^2 + \int_0^L k^2 a + 2k \int_0^L au = \int_0^L a(u+k)^2.$$

This implies

$$\int_0^L (u+k)'^2 \leqslant \|a\|_\infty \int_0^L (u+k)^2.$$

Also, since u is a nonconstant solution of (2.7), $u+k$ is a nontrivial function. Consequently

$$\|a\|_\infty \geqslant \frac{\displaystyle\int_0^L (u+k)'^2}{\displaystyle\int_0^L (u+k)^2}.$$

Now, choose $k_0 \in \mathbb{R}$ satisfying $\int_0^L (u+k_0) = 0$. Then,

$$\|a\|_\infty \geqslant \frac{\displaystyle\int_0^L (u+k_0)'^2}{\displaystyle\int_0^L (u+k_0)^2} \geqslant \inf_{v \in X_\infty \setminus \{0\}} \frac{\displaystyle\int_0^L (v)'^2}{\displaystyle\int_0^L (v)^2} = \frac{\pi^2}{L^2}, \ \forall\, a \in \Lambda \tag{2.9}$$

where $X_\infty = \left\{ v \in H^1(0,L) : \int_0^L v = 0 \right\}$.

Hence $\beta_\infty \geqslant \frac{\pi^2}{L^2}$. Since the constant function $\frac{\pi^2}{L^2}$ is an element of Λ, we deduce $\beta_\infty = \frac{\pi^2}{L^2}$. Furthermore, if $a \in \Lambda$ is such that $\|a\|_\infty = \frac{\pi^2}{L^2}$, it follows from (2.9) that

$$\frac{\pi^2}{L^2} = \frac{\displaystyle\int_0^L (u+k_0)'^2}{\displaystyle\int_0^L (u+k_0)^2}.$$

The variational characterization of the constant $\frac{\pi^2}{L^2}$ (this constant is the second eigenvalue of the eigenvalue problem (2.4)) implies that $u(x)+k_0 = c\cos(\pi x/L)$, for some nonzero constant c. Then $a(x) \equiv \frac{\pi^2}{L^2}$. This completes the proof of the lemma. □

Now we deal with the case $p=1$. Previously, we need the following result.

LEMMA 2.3. *Let* $X_1 = \{u \in H^1(0,L) : \max_{x\in[0,L]} u(x) + \min_{x\in[0,L]} u(x) = 0\}$. *Then*

$$\inf_{u\in X_1\setminus\{0\}} \frac{\int_0^L u'^2}{\|u\|_\infty^2} = \frac{4}{L}. \tag{2.10}$$

Moreover, this infimum is attained in a function $u \in X_1 \setminus \{0\}$ *if and only there exists a nonzero constant* k *such that* $u(x) = k(x-\frac{L}{2}), \forall x \in [0,L]$.

Proof. If $u \in X_1\setminus\{0\}$, and $x_1, x_2 \in [0,L]$ are such that $u(x_1) = \max_{[0,L]} u$, $u(x_2) = \min_{[0,L]} u$, then $\|u\|_\infty = \max_{[0,L]} u = -\min_{[0,L]} u$. Clearly, it is not restrictive to assume that $x_1 < x_2$. Let us denote $I = [x_1, x_2]$. Then, it follows from the Cauchy-Schwartz inequality

$$\int_0^L u'^2 \geqslant \int_I u'^2 \geqslant \frac{\left(\int_I |u'|\right)^2}{x_2-x_1} \geqslant \frac{\left(\int_I u'\right)^2}{x_2-x_1} = \frac{(u(x_2)-u(x_1))^2}{x_2-x_1} = \frac{4\|u\|_\infty^2}{x_2-x_1} \geqslant \frac{4}{L}\|u\|_\infty^2. \tag{2.11}$$

Therefore,

$$\inf_{u\in X_1\setminus\{0\}} \frac{\int_0^L u'^2}{\|u\|_\infty^2} \geqslant \frac{4}{L}.$$

On the other hand, if $v(x) = x - \frac{L}{2}$, $\forall\, x \in [0,L]$, then $v \in X_1 \setminus \{0\}$ and $\dfrac{\int_0^L v'^2}{\|v\|_\infty^2} = \dfrac{4}{L}$.

This proves (2.10). Finally, if $u \in X_1 \setminus \{0\}$ is such that $\dfrac{\int_0^L u'^2}{\|u\|_\infty^2} = \dfrac{4}{L}$, then all the inequalities of (2.11) transform into equalities. In particular, $x_2 = L$, $x_1 = 0$ and $\left(\int_0^L u'\right)^2 = L\int_0^L u'^2$. Again, the Cauchy-Schwartz inequality (equality in this case) implies that the function u' is constant in $[0,L]$. Taking into account that $u \in X_1 \setminus \{0\}$, we have the existence of a nontrivial constant k such that $u(x) = k(x-\frac{L}{2}), \forall x \in [0,L]$. □

With the help of the previous lemma, we obtain the next result concerning the case $p=1$.

LEMMA 2.4. *We have* $\beta_1 = \frac{4}{L}$ *and* $\|a\|_1 > \frac{4}{L}$, $\forall\, a \in \Lambda$.

Proof. As in Lemma 2.2, if $a \in \Lambda$ and $u \in H^1(0,L)$ is a nontrivial solution of (2.7), then we obtain, for each $k \in \mathbb{R}$,

$$\int_0^L (u+k)'^2 \leqslant \int_0^L a(u+k)^2.$$

This implies

$$\int_0^L (u+k)'^2 \leqslant \|a\|_1 \|(u+k)\|_\infty^2$$

and consequently

$$\|a\|_1 \geqslant \frac{\int_0^L (u+k)'^2}{\|(u+k)\|_\infty^2}.$$

Now, if we choose $k_0 \in \mathbb{R}$ satisfying $u+k_0 \in X_1$, we deduce

$$\|a\|_1 \geqslant \frac{\int_0^L (u+k_0)'^2}{\|u+k_0\|_\infty^2} \geqslant \frac{4}{L}, \quad \forall\, a \in \Lambda. \tag{2.12}$$

Therefore, $\beta_1 \geqslant \frac{4}{L}$. Also, we can define a minimizing sequence in the following way. Let $\{u_n\} \subset C^2[0,L]$ be a sequence such that $u_n(x) = (x-\frac{L}{2})$, $\forall\, x \in (\frac{1}{n}, L-\frac{1}{n})$; $u_n'(0) = u_n'(L) = 0$; $u_n''(x) > 0$, $\forall\, x \in [0,\frac{1}{n})$; $u_n''(x) < 0$, $\forall\, x \in (L-\frac{1}{n}, L]$. Then, if we define the sequence of continuous functions $a_n : [0,L] \to \mathbb{R}$, as $a_n(x) = 0$, $\forall\, x \in [\frac{1}{n}, L-\frac{1}{n}]$; $a_n(x) = \frac{-u_n''(x)}{u_n(x)}$, $\forall\, x \in [0,\frac{1}{n}] \cup [L-\frac{1}{n}, L]$, we have that $a_n \in L^\infty(0,L)$, $a_n \geqslant 0$, a.e. in $(0,L)$, a_n is nontrivial and moreover $u_n''(x) + a_n(x)u_n(x) = 0$, in $(0,L)$, $u_n'(0) = u_n'(L) = 0$. Therefore, $a_n \in \Lambda$, $\forall\, n \in \mathbb{N}$. Also,

$$\begin{aligned}
\int_0^L a_n &= \int_0^{\frac{1}{n}} \frac{-u_n''(x)}{u_n(x)} + \int_{L-\frac{1}{n}}^L \frac{-u_n''(x)}{u_n(x)} \\
&\leqslant \int_0^{\frac{1}{n}} \frac{u_n''(x)}{\min_{[0,\frac{1}{n}]}(-u_n)} + \int_{L-\frac{1}{n}}^L \frac{-u_n''(x)}{\min_{[L-\frac{1}{n},L]}(u_n)} \\
&= \frac{u_n'(\frac{1}{n})}{\frac{L}{2}-\frac{1}{n}} + \frac{u_n'(L-\frac{1}{n})}{\frac{L}{2}-\frac{1}{n}} = \frac{1}{\frac{L}{2}-\frac{1}{n}} + \frac{1}{\frac{L}{2}-\frac{1}{n}}.
\end{aligned}$$

Taking limits as $n \to \infty$, we deduce $\beta_1 = \frac{4}{L}$.

Finally, let $a \in \Lambda$ be such that $\|a\|_1 = \frac{4}{L}$. By choosing u a nontrivial solution of (2.1) and $k_0 \in \mathbb{R}$ such that $u+k_0 \in X_1$, we obtain

$$\int_0^L (u+k_0)'^2 \leqslant \frac{4}{L}\|(u+k_0)\|_\infty^2.$$

From Lemma 2.3 we obtain that $u+k_0 = k(x-\frac{L}{2})$, $\forall\, x \in [0,L]$ and for some constant k. Then from (2.1) we deduce $a \equiv 0$, which is a contradiction. $\square$

REMARK 1. Previous lemma was proved by Huaizhong and Yong (see Theorem 3 in [6]) by using methods from Optimal Control theory. More precisely, the authors used the Pontryagin's maximum principle. The proof that we have presented here motivates some of the main ideas that we will use in the case $1 < p < \infty$.

Next we concentrate on the case $1 < p < \infty$. This is the most difficult one and we will need some auxiliary lemmas.

LEMMA 2.5. *Assume* $1 < p < \infty$ *and let* $X_p = \left\{ u \in H^1(0,L) : \int_0^L |u|^{\frac{2}{p-1}} u = 0 \right\}$ *If* $J_p : X_p \setminus \{0\} \to \mathbb{R}$ *is defined by*

$$J_p(u) = \frac{\displaystyle\int_0^L u'^2}{\left(\displaystyle\int_0^L |u|^{\frac{2p}{p-1}} \right)^{\frac{p-1}{p}}} \tag{2.13}$$

and $m_p \equiv \inf_{X_p \setminus \{0\}} J_p$, m_p *is attained. Moreover, if* $u_p \in X_p \setminus \{0\}$ *is a minimizer, then* u_p *satisfies the problem*

$$u_p''(x) + A_p(u_p)|u_p(x)|^{\frac{2}{p-1}} u_p(x) = 0, \; x \in (0,L), \;\; u_p'(0) = u_p'(L) = 0, \tag{2.14}$$

where

$$A_p(u_p) = m_p \left(\int_0^L |u_p|^{\frac{2p}{p-1}} \right)^{\frac{-1}{p}}. \tag{2.15}$$

Proof. It is clear that for any $u \in H^1(0,L)$, there exists some constant $k \in \mathbb{R}$ such that $u + k \in X_p$. Hence m_p is well defined. Now, let $\{u_n\} \subset X_p \setminus \{0\}$ be a minimizing sequence. Since the sequence $\{k_n u_n\}$, $k_n \neq 0$, is also a minimizing sequence, we can assume without loos of generality that $\int_0^L |u_n|^{\frac{2p}{p-1}} = 1$. Then $\left\{ \int_0^L |u_n'^2| \right\}$ is also bounded. Moreover, for each u_n there is $x_n \in (0,L)$ such that $u_n(x_n) = 0$. Therefore, $\{u_n\}$ is bounded in $H^1(0,L)$. So, we can suppose, up to a subsequence, that $u_n \rightharpoonup u_0$ in $H^1(0,L)$ and $u_n \to u_0$ in $C[0,L]$ (with the uniform norm). The strong convergence in $C[0,L]$ gives us $\int_0^L |u_0|^{\frac{2p}{p-1}} = 1$ and $u_0 \in X_p \setminus \{0\}$. The weak convergence in $H^1(0,L)$ implies $J_p(u_0) \leqslant \liminf J_p(u_n) = m_p$. Then u_0 is a minimizer.

Since $X_p = \{ u \in H^1(0,L) : \varphi(u) = 0\}$, $\varphi(u) = \int_0^L |u|^{\frac{2}{p-1}} u$, if $u_0 \in X_p \setminus \{0\}$ is any minimizer of J_p, Lagrange multiplier Theorem implies that there is $\lambda \in \mathbb{R}$ such that

$$H'(u_0) + \lambda \varphi'(u_0) = 0$$

where $H : H^1(0,L) \to \mathbb{R}$ is defined by

$$H(u) = \int_0^L u'^2 - m_p \left(\int_0^L |u|^{\frac{2p}{p-1}} \right)^{\frac{p-1}{p}}.$$

Also, since $u_0 \in X_p$ we have $H'(u_0)(1) = 0$. Moreover $H'(u_0)(v) = 0$, $\forall\, v \in H^1(0,L) :\; \varphi'(u_0)(v) = 0$. Finally, as any $v \in H^1(0,L)$ may be written in the form $v = \alpha + w$, $\alpha \in \mathbb{R}$, and w satisfying $\varphi'(u_0)(w) = 0$, we conclude $H'(u_0)(v) = 0$, $\forall\, v \in H^1(0,L)$, i.e., $H'(u_0) = 0$ which is (2.14). □

LEMMA 2.6. *If* $1 < p < \infty$, *we have* $\beta_p = m_p$.

Proof. As in Lemma 2.2, if $a \in \Lambda$ and $u \in H^1(0,L)$ is a nontrivial solution of (2.7), then for each $k \in \mathbb{R}$ we have

$$\int_0^L (u+k)'^2 \leqslant \int_0^L a(u+k)^2.$$

It follows from Hölder inequality

$$\int_0^L (u+k)'^2 \leqslant \|a\|_p \|(u+k)^2\|_{\frac{p}{p-1}}.$$

Also, since u is a nonconstant solution of (2.7), $u+k$ is a nontrivial function. Consequently,

$$\|a\|_p \geqslant \frac{\displaystyle\int_0^L (u+k)'^2}{\|(u+k)^2\|_{\frac{p}{p-1}}}.$$

Now, choose $k_0 \in \mathbb{R}$ satisfying $u+k_0 \in X_p$. Then, $\|a\|_p \geqslant m_p$, $\forall\, a \in \Lambda$ and consequently $\beta_p \geqslant m_p$. Reciprocally, if $u_p \in X_p \setminus \{0\}$ is any minimizer of J_p, then u_p satisfies (2.14). Therefore, $A_p(u_p)|u_p|^{\frac{2}{p-1}} \in \Lambda$. Also,

$$\|\, A_p(u_p)|u_p|^{\frac{2}{p-1}}\, \|_p^p = \int_0^L m_p^p |u_p|^{\frac{2p}{p-1}} = m_p^p.$$

Then $\beta_p \leqslant m_p$. $\square$

LEMMA 2.7. *If* $1 < p < \infty$, m_p *is given by*

$$m_p = \frac{4(p-1)^{1+\frac{1}{p}}}{L^{2-\frac{1}{p}} p(2p-1)^{1/p}} \left(\int_0^{\pi/2} (\sin x)^{-1/p}\, dx \right)^2. \tag{2.16}$$

Proof. By Lemma 2.5, if $u_p \in X_p \setminus \{0\}$ is a minimizer of J_p, then u_p satisfies a problem of the type

$$v''(x) + B|v(x)|^{\frac{2}{p-1}} v(x) = 0, \quad x \in (0,L),\ v'(0) = v'(L) = 0, \tag{2.17}$$

where B is some positive real constant. Also, let us observe that any nontrivial solution of (2.17) belongs to $X_p \setminus \{0\}$. Therefore,

$$\inf_{B \in \mathbb{R}^+} \inf_{v \in S_B} J_p(v) = m_p$$

where, for a given $B \in \mathbb{R}^+$, S_B denotes the set of all nontrivial solutions of (2.17).

Now, if $B \in \mathbb{R}^+$ is a fixed number and v is a nontrivial solution of (2.17), we may assume without loos of generality that $v(0) > 0$. Moreover, since $v \in X_p$, v must change its sign in $(0,L)$. Let x_0 be the first zero point of v in $(0,L)$. Thus, v satisfies the initial value problem

$$w''(x) + B|w(x)|^{\frac{2}{p-1}} w(x) = 0, \quad w(0) = v(0), \quad w'(0) = 0. \tag{2.18}$$

Note that this problem has a unique solution defined in $\mathbb{R}$ (see proposition 2.1. in [4]).

If $x \in (0,x_0)$ is fixed, multiplying both terms of (2.17) by v' and integrating in the interval $[0,x]$ we obtain

$$-\frac{(v'(x))^2}{2} = \frac{B(p-1)}{2p} \left(|v(x)|^{\frac{2p}{p-1}} - |v(0)|^{\frac{2p}{p-1}} \right). \tag{2.19}$$

On the interval $(0,x_0)$ the function v satisfies $v(x)>0$ and $v'(x)\leqslant 0$ (see 2.17) and thus

$$v'(x)=-\left[\frac{B(p-1)}{p}\right]^{1/2}\left[|v(0)|^{\frac{2p}{p-1}}-|v(x)|^{\frac{2p}{p-1}}\right]^{1/2} \tag{2.20}$$

Therefore,

$$\int_0^x \frac{v'(t)}{\left[|v(0)|^{\frac{2p}{p-1}}-|v(t)|^{\frac{2p}{p-1}}\right]^{1/2}}\,dt=-\left[\frac{B(p-1)}{p}\right]^{1/2} x$$

for any $x\in(0,x_0)$. Doing the change of variables $s=\dfrac{v(t)}{v(0)}$, previous relation can be written as

$$-\varphi(1)+\varphi\left(\frac{v(x)}{v(0)}\right)=-v(0)^{\frac{1}{p-1}}\left[\frac{B(p-1)}{p}\right]^{1/2} x.$$

Here $\varphi:[0,1]\to\mathbb{R}$ is the strictly increasing function defined by

$$\varphi(t)=\int_0^t \frac{ds}{\left(1-s^{\frac{2p}{p-1}}\right)^{1/2}}.$$

If $\varphi\,[0,1]=[0,I]$, then we find

$$\frac{v(x)}{v(0)}=\varphi^{-1}\left[I-v(0)^{\frac{1}{p-1}}\left(\frac{B(p-1)}{p}\right)^{1/2} x\right]. \tag{2.21}$$

Moreover, since $v(x_0)=0$, we obtain

$$I-v(0)^{\frac{1}{p-1}}\left(\frac{B(p-1)}{p}\right)^{1/2} x_0=0.$$

Hence,

$$v(0)=\left(\frac{I}{x_0}\left(\frac{p}{B(p-1)}\right)^{1/2}\right)^{p-1}. \tag{2.22}$$

Finally,

$$v(x)=\left(\frac{I}{x_0}\left(\frac{p}{B(p-1)}\right)^{1/2}\right)^{p-1}\varphi^{-1}\left(I-\frac{I}{x_0}x\right),\quad \forall\, x\in[0,x_0]. \tag{2.23}$$

Also, the initial value problem

$$w''(x)+B|w(x)|^{\frac{2}{p-1}}w(x)=0,\quad w(x_0)=0,\quad w'(x_0)=v'(x_0) \tag{2.24}$$

has a unique solution defined in $\mathbb{R}$. Since the function $-v(2x_0-x)$, $x\in(x_0,2x_0)$, is a solution of (2.24), this provides $v(x)$, $\forall x\in[x_0,2x_0]$. Let us note that $v(x)=-v(2x_0-x)$, $\forall\, x\in(x_0,2x_0)$. Now, we can repeat this procedure in the intervals $[nx_0,(n+1)x_0]$, $\forall\, n\in\mathbb{N}$. The conclusion is that if v is a nontrivial solution of (2.17) for some $B\in\mathbb{R}^+$, and x_0 is the first zero point of v in $(0,L)$ then $L=2nx_0$ for some $n\in\mathbb{N}$. Next we calculate $J_p(v)$.

It follows from previous reasonings that

$$J_p(v) = \frac{\int_0^L v'^2}{\left(\int_0^L |v|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}} = \frac{2n\int_0^{x_0} v'^2}{\left(2n\int_0^{x_0} |v|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}}. \tag{2.25}$$

From (2.19) we obtain

$$\int_0^{x_0} (v'(x))^2\,dx = \frac{B(p-1)}{p}\left[-\int_0^{x_0} |v(x)|^{\frac{2p}{p-1}}\,dx + x_0|v(0)|^{\frac{2p}{p-1}}\right] \tag{2.26}$$

and from (2.23) we obtain

$$\int_0^{x_0} |v(x)|^{\frac{2p}{p-1}} = \int_0^{x_0}\left(\frac{I}{x_0}\left(\frac{p}{B(p-1)}\right)^{1/2}\right)^{2p}\left[\varphi^{-1}\left(I-\frac{I}{x_0}x\right)\right]^{\frac{2p}{p-1}}\,dx. \tag{2.27}$$

Doing the change of variables $s=\varphi^{-1}(I(1-\frac{x}{x_0}))$, we have

$$\int_0^{x_0} |v(x)|^{\frac{2p}{p-1}} = \left(\frac{I}{x_0}\left(\frac{p}{B(p-1)}\right)^{1/2}\right)^{2p}\frac{x_0}{I}\int_0^1 s^{\frac{2p}{p-1}}\left(1-s^{\frac{2p}{p-1}}\right)^{-1/2}\,ds. \tag{2.28}$$

Integrating by parts the previous expression with $f(s)=s$, $g'(s)=s^{\frac{p+1}{p-1}}\left(1-s^{\frac{2p}{p-1}}\right)^{-1/2}$, we deduce

$$\int_0^{x_0} |v(x)|^{\frac{2p}{p-1}} = \left(\frac{I}{x_0}\left(\frac{p}{B(p-1)}\right)^{1/2}\right)^{2p}\frac{x_0}{I}\frac{p-1}{2p-1}I \tag{2.29}$$

If we substitute this expression in (2.26) and, moreover, we take into account (2.22) we obtain (think that $L=2nx_0$)

$$\int_0^{x_0} |v'(x)|^2\,dx = \frac{B(p-1)}{p}x_0\left(\frac{I}{x_0}\left(\frac{p}{B(p-1)}\right)^{1/2}\right)^{2p}\frac{p}{2p-1}. \tag{2.30}$$

Now we can substitute (2.29) and (2.30) in (2.25). After some elementary calculations we deduce

$$J_p(v) = \frac{4n^2I^2p}{L^{2-\frac{1}{p}}(p-1)^{1-\frac{1}{p}}(2p-1)^{1/p}}. \tag{2.31}$$

At this point, one may observe two things. First, $J_p(v)$ does not depend on B. Second, all values of $n\in\mathbb{N}$ are possible in (2.31). In fact if $x_0=\frac{L}{2n}$, formula (2.23) defines a nontrivial solution of (2.17). Therefore, the infimum m_p is attained if $n=1$. Finally, doing the change of variables $s^{\frac{p}{p-1}}=\sin t$, we obtain $I=\int_0^1 \frac{ds}{\left(1-s^{\frac{2p}{p-1}}\right)^{1/2}}=\frac{p-1}{p}K$, where $K=\int_0^{\pi/2}(\sin t)^{-1/p}\,dt$. This gives

$$m_p = \frac{4(p-1)^{1+\frac{1}{p}}}{L^{2-\frac{1}{p}}p(2p-1)^{1/p}}\left(\int_0^{\pi/2}(\sin x)^{-1/p}\,dx\right)^2. \qquad \square$$

LEMMA 2.8. *If $1 < p < \infty$ and the functions $u_p, v_p \in X_p \setminus \{0\}$ are minimizers of J_p, then there exists a nonzero constant $c \in \mathbb{R}$ such that $u_p = cv_p$. As a consequence, there is a unique function $a_p \in \Lambda$ such that $\beta_p = I_p(a_p)$. Moreover, the function a_p is given by $a_p = A_p(u_p)|u_p|^{\frac{2}{p-1}}$, where u_p is any minimizer of J_p in $X_p \setminus \{0\}$.*

Proof. It follows from Lemma 2.5 that both functions, u_p and v_p satisfy a problem like (2.17) for possibly different positive constants B_u and B_v. Moreover, if x_u and x_v are, respectively, the first zero points of u and v in $(0,L)$, we must have $L = 2x_u = 2x_v$. Therefore $x_u = x_v$ and from (2.23) we deduce the existence of the constant c.

Now, let $a \in \Lambda$ be such that $\beta_p = I_p(a)$. Let $u \in H^1(0,L)$ be a nontrivial solution of

$$-u''(x) = a(x)u(x), \quad x \in (0,L), \quad u'(0) = u'(L) = 0,$$

Choose $k_0 \in \mathbb{R}$ satisfying $u + k_0 \in X_p$. Then, as in Lemma 2.6, we have

$$\int_0^L (u+k_0)'^2 \leqslant \int_0^L a(u+k_0)^2 \leqslant \|a\|_p \|(u+k_0)^2\|_{p'}$$

$$= m_p \|(u+k_0)^2\|_{p'} = m_p \left(\int_0^L |(u+k_0)|^{\frac{2p}{p-1}} \right)^{\frac{p-1}{p}} \leqslant \int_0^L (u+k_0)'^2.$$

Then, $u + k_0$ is a minimizer of J_p in $X_p \setminus \{0\}$ and we have an equality in Hölder inequality. Therefore there exists a nontrivial constant d such that $a = d|(u+k_0)|^{\frac{2}{p-1}}$.

Now, if $\tilde{a} \in \Lambda$ is such that $\beta_p = I_p(\tilde{a})$, then there exists a nontrivial constant $\tilde{d}$ such that $\tilde{a} = \tilde{d}|(\tilde{u} + \tilde{k}_0)|^{\frac{2}{p-1}}$, where $\tilde{u} \in H^1(0,L)$ is a nontrivial solution of

$$-z''(x) = \tilde{a}(x)z(x), \quad x \in (0,L), \quad z'(0) = z'(L) = 0,$$

and $\tilde{k}_0 \in \mathbb{R}$ satisfies $\tilde{u}+\tilde{k}_0 \in X_p$. Since both functions $u+k_0$ and $\tilde{u}+\tilde{k}_0$ are minimizers of J_p in $X_p \setminus \{0\}$, there exists a positive constant c such that $u+k_0 = c(\tilde{u}+\tilde{k}_0)$. Then $a = dc^{\frac{2}{p-1}}|\tilde{u}+\tilde{k}_0|^{\frac{2}{p-1}}$. Moreover, since $\|a\|_p = \|\tilde{a}\|_p = \beta_p$ we must have $dc^{\frac{2}{p-1}} = \tilde{d}$ and consequently $a = \tilde{a}$.

Finally, u_p is any minimizer of J_p in $X_p \setminus \{0\}$, then we obtain from (2.14) that the function $= A_p(u_p)|u_p|^{\frac{2}{p-1}}$ belongs to Λ. Also the L_p- norm of this function is β_p. This proves the lemma. □

It is trivial to prove that β_p is a continuous function if $p \in (1,\infty)$ and that $\lim_{p\to\infty} \beta_p = \beta_\infty$. To finish the proof of the theorem we need the next two lemmas will be devoted to the study of some qualitative properties of the function β_p.

LEMMA 2.9. *We have* $\lim_{p\to 1^+} \beta_p = \beta_1$.

Proof. Since

$$\beta_p = \frac{4(p-1)^{1+\frac{1}{p}}}{L^{2-\frac{1}{p}}p(2p-1)^{1/p}} \left(\int_0^{\pi/2} (\sin x)^{-1/p}\, dx \right)^2$$

$$= \frac{4(p-1)^{-1+\frac{1}{p}}(p-1)^2}{L^{2-\frac{1}{p}}p(2p-1)^{1/p}} \left(\int_0^{\pi/2} (\sin x)^{-1/p}\, dx \right)^2$$

and

$$\lim_{p\to 1^+} \frac{4}{L^{2-\frac{1}{p}} p(2p-1)^{1/p}} = \frac{4}{L}, \quad \lim_{p\to 1^+} (p-1)^{-1+\frac{1}{p}} = 1,$$

it is sufficient to prove

$$\lim_{p\to 1^+} (p-1) \int_0^{\pi/2} (\sin t)^{-1/p}\, dt = 1. \tag{2.32}$$

To see this, since $\sin t \leqslant t$, $\forall\, t \in (0, \pi/2)$, we have $(\sin t)^{-1/p} \geqslant (t)^{-1/p}$, $\forall\, t \in (0, \pi/2)$. Therefore

$$(p-1) \int_0^{\pi/2} (\sin t)^{-1/p}\, dt \geqslant (p-1) \int_0^{\pi/2} (t)^{-1/p}\, dt = p \left(\frac{\pi}{2}\right)^{1-\frac{1}{p}}.$$

Hence,

$$\liminf_{p\to 1^+} (p-1) \int_0^{\pi/2} (\sin t)^{-1/p}\, dt \geqslant 1. \tag{2.33}$$

On the other hand, as $\lim_{t\to 0} \frac{\sin t}{t} = 1$, we have that for each $\varepsilon \in (0,1)$, there exists $t_0 \in (0, \pi/2)$ such that $\frac{\sin t}{t} \geqslant 1 - \varepsilon$, $\forall\, t \in (0, t_0)$. Therefore,

$$\begin{aligned}
&(p-1) \int_0^{\pi/2} (\sin t)^{-1/p}\, dt \\
&\quad = (p-1) \int_0^{t_0} (\sin t)^{-1/p}\, dt + (p-1) \int_{t_0}^{\pi/2} (\sin t)^{-1/p}\, dt \\
&\quad \leqslant (p-1) \int_0^{t_0} (1-\varepsilon)^{-1/p} t^{-1/p}\, dt + (p-1) \int_{t_0}^{\pi/2} (\sin t_0)^{-1/p}\, dt \\
&\quad = (1-\varepsilon)^{-1/p} p t_0^{1-\frac{1}{p}} + (p-1)(\sin t_0)^{-1/p} \left(\frac{\pi}{2} - t_0\right).
\end{aligned}$$

Hence

$$\limsup_{p\to 1^+} (p-1) \int_0^{\pi/2} (\sin t)^{-1/p}\, dt \leqslant \frac{1}{1-\varepsilon}, \quad \forall \varepsilon \in (0,1)$$

and consequently

$$\limsup_{p\to 1^+} (p-1) \int_0^{\pi/2} (\sin t)^{-1/p}\, dt \leqslant 1. \tag{2.34}$$

Finally, (2.33) and (2.34) prove the lemma. □

LEMMA 2.10. *Assume $L = 1$ and $1 \leqslant p < q < \infty$. Then $\beta_p < \beta_q$. As a trivial consequence, if L is an arbitrary positive number, the mapping $[1, \infty) \to \mathbb{R}$, $p \to L^{-1/p}\beta_p$ is strictly increasing.*

Proof. Since β_p is a continuous function of p, it is sufficient to prove that $\beta_p < \beta_q$ when $1 < p < q < \infty$. Now, if a_p, a_q are the elements of Λ such that $\beta_p = \|a_p\|_p$, $\beta_q = \|a_q\|_q$, then since $\|a_q\|_p \leqslant \|a_q\|_q$, we obtain $\beta_p \leqslant \beta_q$. If $\beta_p = \beta_q$, then

$$\beta_p \leqslant \|a_q\|_p \leqslant \|a_q\|_q = \|a_p\|_p = \beta_p.$$

Therefore $\|a_q\|_p = \|a_q\|_q$. Since $p < q$, we deduce that $|a_q|$ must be a positive constant. But it is easily deduced from Lemma 2.8 that $|a_q|$ can not be a constant.

Now if L is an arbitrary positive number, then it is trivial from the explicit expression of β_p that the mapping $L^{-1/p}\beta_p$ is also strictly increasing. □

As an application of Theorem 2.1 to the linear problem

$$u''(x) + a(x)u(x) = f(x),\ x \in (0,L),\ u'(0) = u'(L) = 0, \tag{2.35}$$

we have the following corollary, which clearly generalizes Theorem 3 in [6].

COROLLARY 2.11. *Let* $a \in L^\infty \setminus \{0\}$, $0 \leqslant \int_0^L a(x)$, *satisfying one of the following conditions:*

1. $\|a\|_1 \leqslant \beta_1$,
2. *There is some* $p \in (1,\infty)$ *such that* $\|a\|_p < \beta_p$ *or* $\|a\|_p = \beta_p$ *and* $a \neq a_p$.
3. $\|a\|_\infty < \beta_\infty$ *or* $\|a\|_\infty = \beta_\infty$ *and* $a \neq a_\infty$.

Then for each $f \in L^\infty(0,L)$, *the boundary value problem* (2.35) *has a unique solution.*

3. The nonlinear problem

In this section we give some new results on the existence and uniqueness of solutions of nonlinear b.v.p. (1.1). To get our purpose, we combine the results obtained in the previous section with the Schauder's fixed point theorem. In fact, once we have the result given by Corollary 2.11, the procedure is standard and may be seen, for example, in [6].

THEOREM 3.1. *Let us consider* (1.1) *where the following requirements are fulfilled:*

1. f *and* f_u *are continuous on* $[0,L] \times \mathbb{R}$.
2. *For some function* $\beta \in L^\infty(0,L)$, *we have* $f_u(x,u) \leqslant \beta(x)$ *on* $[0,L] \times \mathbb{R}$ *and* β *satisfies some of the conditions given in Corollary* 2.11.
3. $0 \leqslant f_u(x,u)$ *in* $[0,L] \times \mathbb{R}$. *Moreover, for each* $u \in C[0,L]$ *one has* $f_u(x,u(x)) \neq 0$, *a.e. on* $[0,L]$ *and* $\int_0^L f(x,0)\,dx = 0$.

Then, problem (1.1) *has a unique solution.*

Proof. We first prove uniqueness. Let u_1 and u_2 be two solutions of (1.1). Then, the function $u = u_1 - u_2$ is a solution of a problem of the type (2.35) with $f \equiv 0$ and $a(x) = \int_0^1 f_u(x, u_2(x) + \theta u(x))\,d\theta$. Applying Corollary 2.11, we obtain $u \equiv 0$.

Next we prove existence. First, we rewrite (1.1) in the equivalent form

$$u''(x) + b(x,u(x))u(x) = -f(x,0), \quad x \in [0,L], \quad u'(0) = u'(L) = 0 \tag{3.1}$$

where the continuous function $b : [0,L] \times \mathbb{R} \to \mathbb{R}$, is defined by $b(x,z) = \int_0^1 f_u(x,\theta z)\,d\theta$. If $X = C^1([0,L],\mathbb{R})$ with the usual norm, i.e., $\|y\|_X = \max_{x\in[0,L]} |y(x)| + \max_{x\in[0,L]} |y'(x)|$,

$\forall\, y \in X,$ we can define the operator $T : X \to X,$ by $Ty = u_y$ where u_y is the unique solution of the linear problem

$$u''(x) + b(x, y(x))u(x) = -f(x,0), \quad x \in [0,L], \quad u'(0) = u'(L) = 0. \tag{3.2}$$

We claim that T is completely continuous and that $T(X)$ is bounded. Then, T has a fixed point which provides a solution of (1.1).

To prove the claim, if $T(X)$ is not bounded, there would exist a sequence $\{y_n\} \subset X$ such that $\|u_{y_n}\|_X \to \infty$. Passing to a subsequence if necessary, we may assume that the sequence of functions $\{b(\cdot, y_n(\cdot))\}$ is weakly convergent in $L^2(0,L)$ to a function β_0 satisfying $0 \leqslant \beta_0(x) \leqslant \beta(x)$, a.e. in $[0,L]$. If $z_n \equiv \dfrac{u_{y_n}}{\|u_{y_n}\|_X}$, passing to a subsequence if necessary, we may assume that $z_n \to z_0$, uniformly in $[0,L]$, where z_0 satisfies $\|z_0\|_X = 1$ and

$$z_0''(x) + \beta_0(x)z_0(x) = 0, \quad x \in [0,L], \quad z_0'(0) = z_0'(L) = 0. \tag{3.3}$$

Moreover, from the hypotheses of the theorem, we have for each $n \in \mathbb{N},$ $\int_0^L b(x, y_n(x))\, u_{y_n}(x)\, dx = -\int_0^L f(x,0)\, dx = 0.$ Therefore, for each $n \in \mathbb{N},$ the function u_{y_n} has a zero in $[0,L]$ and hence so does z_0. Thus, $\beta_0 \in L^\infty(0,L) \setminus \{0\}$. This is a contradiction with Corollary 2.11.

Now, let us prove that the operator T is continuous. To see this, if $\{y_n\} \to y_0$ in the space X and u_{y_n} does not converge to u_{y_0}, passing to a subsequence if necessary, there exists a constant $\delta > 0$ such that $u_{y_n} \notin B_X(u_{y_0}; \delta)$, $\forall\, n \in \mathbb{N}$. Also, taking into account (3.2) and the boundness of operator T, we obtain that the sequence u''_{y_n} is uniformly bounded. Thus, again passing to a subsequence if necessary, we deduce that u_{y_n} converges to some function u_0. But, by the uniqueness of solution for problem (3.2), we must have $u_0 = u_{y_0}$, which is a contradiction.

Finally, by using the Arzela-Ascoli theorem, it is trivial from (3.2) that if $B \subset X$ is bounded, then $T(B)$ is relatively compact in X. $\square$

REMARK 2. Since the change of variables $u(x) = v(x) + z,\ z \in \mathbb{R},$ transforms (1.1) into the problem

$$v''(x) + f(x, v(x) + z) = 0, \quad x \in (0,L), \quad v'(0) = v'(L) = 0,$$

the condition $\int_0^L f(x,0)\, dx = 0$ in the previous Theorem, may be substituted by (1.2).

Previous result generalizes Theorem B in [6]. Also, under the hypothesis (1) of previous Theorem, it is a generalization of Theorem 2 in [8] for the case of ordinary differential equations.

4. Final remarks

REMARK 3. Theorem 2.1 can be slightly improved by considering the positive part a^+ of a function $a \in \Lambda$ (i.e. $a^+(x) = \max\{0, a(x)\}$). Specifically, if we define

$$\beta_p^+ \equiv \inf_{a \in \Lambda} \quad I_p(a^+), \ 1 \leqslant p \leqslant \infty \tag{4.1}$$

it is possible to prove that all the assertions in Theorem 2.1 hold true if we replace β_p by β_p^+. Note that this is a more general result since $I_p(a^+) \leqslant I_p(a), \forall a \in \Lambda$. In order to prove this, it is sufficient to observe that if $a \in \Lambda$, and $u \in H^1(0,L)$ is a nontrivial solution of (2.7), then we have, for each $k \in \mathbb{R}$,

$$\int_0^L (u+k)'^2 \leqslant \int_0^L a(u+k)^2 \leqslant \int_0^L a^+(u+k)^2.$$

Hence, we can apply the same arguments in Lemmas 2.2, 2.4 and 2.6, with a replaced by a^+, and the rest of the proof runs as before. By using this we can obtain a similar result to that given in Corollary 2.11 involving the positive part a^+ of function a.

REMARK 4. We need the positivity of $\int_0^L a$ in order to obtain a positive infimum of $I_p(a)$. In fact if $u \in H^1(0,L)$ is a positive nonconstant solution of (2.7) and we consider $v = \frac{1}{u}$ as test function in the weak formulation, we obtain

$$\int_0^L u'\left(\frac{1}{u}\right)' = \int_0^L au\frac{1}{u},$$

which implies

$$\int_0^L a = -\int_0^L \frac{u'^2}{u^2} < 0.$$

Hence, if we fixed a nonconstant $u_0 \in C^2[0,L]$ such that $u_0'(0) = u_0'(L) = 0$, then, for large $n \in \mathbb{N}$, we have that $u_n = u_0 + n$ is a positive nonconstant solution of (2.7), with $a_n = \frac{-u_0''}{u_0+n}$. Clearly $\|a_n\|_p \to 0$ as $n \to \infty$ for every $1 \leqslant p \leqslant \infty$ and, as we have seen before, $\int_0^L a_n < 0$.

REMARK 5. We can do an analogous study for Dirichlet boundary conditions. That is, consider the linear problem

$$u''(x) + a(x)u(x) = 0, \quad x \in (0,L), \quad u(0) = u(L) = 0, \tag{4.2}$$

where $a \in \bar{\Lambda}$ and $\bar{\Lambda}$ is defined by

$$\bar{\Lambda} = \{a \in L^\infty(0,L) \text{ such that (4.2) has nontrivial solutions }\}. \tag{4.3}$$

In this case, it is possible to prove that

$$\inf_{a \in \bar{\Lambda}} \|a_p\| = \beta_p, \quad 1 \leqslant p \leqslant \infty \tag{4.4}$$

where the constant β_p is the same as in Theorem 2.1. Moreover all the assertions of this theorem hold true if we replace $\bar{u_p}$ by u_p, being $\bar{u_p}$ the unique positive solution of the problem

$$-u''(x) = u(x)^{\frac{p+1}{p-1}}, \quad u(0) = u(L) = 0.$$

(Note that $\bar{u_p}(x) = u_p(L/2 - x)$, $\forall x \in [0, L/2]$ and $\bar{u_p}(x) = u_p(x - L/2)$, $\forall x \in [L/2, L]$.)

To obtain these results, it is sufficient to apply the same arguments as in the Neumann case, with the spaces X_p of Lemmas 2.2, 2.4 and 2.6 replaced by $H_0^1(0,L)$.

We can check that our constants β_p in the case $p = 2, 3$ and $3/2$ are the same that obtained in [7] by different methods.

REMARK 6. One can expect that some results hold true in the case of Neumann boundary value problem for partial differential equations

$$\Delta u(x) + a(x)u(x) = 0, \quad x \in \Omega, \qquad \frac{\partial u(x)}{\partial n} = 0, \quad x \in \partial\Omega \tag{4.5}$$

where Ω is a bounded and regular domain in $\mathbb{R}^N$. But here the role played by the dimension N may be important. For instance, if $N \geqslant 3$ and

$$\Lambda = \left\{ a \in L^\infty(\Omega) \setminus \{0\} : \int_\Omega a(x)\,dx \geqslant 0 \text{ and } (4.5) \text{ has nontrivial solutions} \right\}$$

then it may be proved that $\inf_{a \in \Lambda} \|a\|_p > 0 \Longleftrightarrow p \geqslant \frac{N}{2}$. The details about this problem will be published elsewhere.

Acknowledgement. We thank J. Mawhin for bringing references [2] and [3] to our attention.

REFERENCES

[1] R. C. BROWN, D. B. HINTON, *Lyapunov inequalities and their applications, Survey on Classical Inequalities,* T. M. Rassias, ed. Kluwer, Dordrecht, 2000, 1–25.

[2] G. CROCE, B. DACOROGNA, *On a generalized Wirtinger inequality,* Discrete and Continuous Dynamical Systems, **9**, (2003), 1329–1341.

[3] B. DACAROGNA, W. GANGBO AND N. SUBÍA, *Sur une généralisation de l'inégalité de Wirtinger,* Ann. Inst. Henri Poincaré, Anal. Non Linéaire, **9**, (1992), 29–50.

[4] P. DRÁBEK, *Nonlinear eigenvalue problems and Fredholm altrnative,* In Nonlinear Differential Equations, P. Drábek, P. Krejči and P. Takáč, Editors. Research Notes in Mathematics Series, **404**, Chapman and Hall/CRC, London, 1999, 1–46.

[5] P. HARTMAN, *Ordinary Differential Equations,* John Wiley and Sons Inc., New York-London-Sydney, 1964.

[6] W. HUAIZHONG, L. YONG, *Neumann boundary value problems for second-order ordinary differential equations across resonance,* SIAM J. Control and Optimization, **33**, (1995), 1312–1325.

[7] L. KOTIN, *A generalization of Liapunov's inequality,* J. Math. Anal. Appl., **102**, (1984), 585–598.

[8] J. MAWHIN, J. R. WARD AND M. WILLEM, *Variational methods and semilinear elliptic equations,* Arch. Rational Mech. Anal., **95**, (1986), 269–277.

(Received June 27, 2004)

Departamento de Análisis Matemático
Universidad de Granada
18071 Granada
Spain
e-mail: acanada@ugr.es
e-mail: jmontero@ugr.es
e-mail: svillega@ugr.es